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UPPER BOUNDS ON THE DIAMETER FOR FINSLER MANIFOLDS WITH WEIGHTED RICCI CURVATURE

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Abstract. In this paper we obtain some Cheeger-Gromov-Taylor type compactness theorems for a forward complete and connected Finsler manifold of dimensional $n \geq 2$ via weighted Ricci curvatures. The proofs are based on the index form of a minimal unit speed geodesic segment, Bochner-Weitzenböck formula and Hessian comparison theorem.

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1. INTRODUCTION AND MAIN THEOREMS

In [8], Myers obtained a compactness theorem in Riemannian manifolds. The theorem of Myers concludes that if $\mathrm{Ric} \geq (n-1)K > 0$, then $\mathrm{diam}(M) \leq \pi/\sqrt{K}$. Later, Cheeger-Gromov-Taylor [3] proved that if there exist $p \in M$ and $r_0, \nu > 0$ such that

$$\mathrm{Ric} \geq (n-1)\frac{(\frac{1}{4}+\nu^2)}{r^2} \tag{1.1}$$

holds for all $r(x) \geq r_0 > 0$ where r is distance function defined with respect to a fixed point $p \in M$, i.e., $r(x) = d(x,p)$, then M is compact and the diameter is bounded from above by $\mathrm{diam}_p(M) < r_0 e^{\pi/\nu}$. By using Bakry-Emery Ricci tensor, $\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f$, Soylu [12] attained a generalization of Cheeger-Gromov-Taylor's compactness theorem.

For m-Bakry-Emery Ricci tensor, Wang [14] proved that, if the following inequality

$$\mathrm{Ric}_{f,m} = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{m-n} \geq -(m-1)\frac{K_0}{(1+r)^2} \tag{1.2}$$

holds for all $x \in M$, where $K_0 < -\frac{1}{4}$ and r is distance function defined with respect to a fixed point $p \in M$, then M is compact and the diameter has the upper bound $\mathrm{diam}(M) < 2(e^{2\pi/\overline{K}} - 1)$, where $\overline{K} = \sqrt{-K_0 - \frac{1}{4}}$.

We can find various kinds of generalizations of the Myers theorem in [4, 6, 7, 13, 15].

Finsler geometry is a natural generalization of Riemannian geometry. The validity of the Myers compactness theorem for Finsler manifolds was shown by Shen [11] without any modification. Later, using the weighted Ricci curvature $\mathrm{Ric}_N := \mathrm{Ric} + \dot{S} - \frac{S^2}{N-n} \geq K > 0$, $N \in (n, \infty)$, Ohta [9] obtained a compactness theorem and gave an upper bound for the diameter of n-dimensional Finsler manifolds as $\mathrm{diam}(M) \leq \pi\sqrt{(N-1)/K}$. In [16], Wu establish a generalized Myers theorem under line integral curvature bound for Finsler manifolds. In [2], Anastasiei extended to Finsler manifolds the compactness theorems of Ambrose and Galloway (see [1] and [5], respectively). Yin [18] acquired two Myers-type compactness theorems for a Finsler manifold with a positive weighted Ricci curvature bound and an advisable condition on the distortion or the S-curvature.

Throughout this paper, (M, F) is a connected forward complete n-dimensional smooth Finsler manifold, $r(x) = d(x, p)$ is the forward distance function from $p \in M$ and $d\mu$ is an arbitrary positive $\mathcal{C}^\infty$-measure on M. Here, there is no canonical measure like the volume measure in Riemannian geometry. Thus we begin with an arbitrary measure on M.

We are now ready to give our main results.

Theorem 1. *Let $(M, F, d\mu)$ be a forward complete and connected Finsler manifold of dimension n with arbitrary volume form and let r be the distance function $r(x) = d(x, p)$ with respect to a fixed point $p \in M$. Assume that the weighted Ricci curvature*

$$\mathrm{Ric}_\infty := \mathrm{Ric} + \dot{S} \geq (n-1)\frac{H}{r^2}, \tag{1.3}$$

and the distortion $|\tau| \leq (n-1)k$ for all $x \in M$ such that $r(x) \geq r_0 > 0$, where the constants k and H satisfy the inequalities $k \geq 0$ and $H > 1/4$. Then M is compact and the diameter from the point $p \in M$ satisfies

$$\mathrm{diam}_p(M) \leq r_0 \exp\left(\frac{2}{4H-1}\sqrt{32k^2 + (4H-1)\pi^2 + 16k\sqrt{4k^2 + (4H-1)H\pi^2}}\right). \tag{1.4}$$

The distortion τ is a smooth function on M when M is a Riemannian manifold. Therefore the diameter estimate (1.4) of Theorem 1 coincides with the diameter estimate of Theorem 1.1 in [12].

Theorem 2. *Let $(M, F, d\mu)$ be a forward complete and connected Finsler manifold of dimension n with arbitrary volume form and let r be the distance function $r(x) = d(x, p)$ with respect to a fixed point $p \in M$. Assume that the weighted Ricci curvature*

$$\mathrm{Ric}_N := \mathrm{Ric}_\infty - \frac{S^2}{N-n} \geq (N-1)\frac{H}{r^2} \tag{1.5}$$

for all $N \in (n, \infty)$ and $r(x) \geq r_0 > 0$, where $H > 1/4$. Then M is compact and the diameter from the point $p \in M$ satisfies

$$\mathrm{diam}_p(M) \leq r_0 e^{2\pi/\sqrt{4H-1}}. \tag{1.6}$$

The diameter estimate (1.6) obtained in the above theorem coincides with the result of Cheeger-Gromov-Taylor in [3] obtained for the original Ricci tensor in the Riemannian manifolds.

Theorem 3. *Let $(M,F,d\mu)$ be a forward complete and connected Finsler manifold of dimension n with arbitrary volume form and let r be the distance function $r(x)=d(x,p)$ with respect to a fixed point $p\in M$. Suppose that the weighted Ricci curvature*

$$\mathrm{Ric}_N := \mathrm{Ric}_\infty - \frac{S^2}{N-n} \geq (N-1)\frac{H}{(1+r)^2} \tag{1.7}$$

for all $x\in M$ and $N\in(n,\infty)$, where $H>1/4$. Then M is compact and the diameter satisfies

$$\mathrm{diam}(M) \leq (1+\lambda)(e^{2\pi/\sqrt{4H-1}}-1), \tag{1.8}$$

where λ is the reversibility.

We review below some basic informations about the Finsler manifolds to be used in the proofs of main theorems.

2. A brief review of Finsler geometry

Let (M,F) be a Finsler n-manifold with Finsler metric $F:TM\to[0,\infty)$. Let $\pi:TM\to M$ be the natural projection and (x,y) be a point of TM such that $x\in M$ and $y\in T_xM$. A *Finsler metric* is a $\mathscr{C}^\infty$-Finsler structure of M with the following properties:

1. F is $\mathscr{C}^\infty$ on $TM\setminus 0$ (Regularity),
2. $F(x,\lambda y)=\lambda F(x,y)$ for all $\lambda>0$ (Positive homogeneity),
3. The $n\times n$ Hessian matrix

$$g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$$

is positive-definite at every point of $TM\setminus 0$ (Strong convexity).

The *Chern curvature* R^V for vectors fields $X,Y,Z\in T_xM\setminus 0$ is defined by

$$R^V(X,Y)Z := \nabla^V_X\nabla^V_Y Z - \nabla^V_Y\nabla^V_X Z - \nabla^V_{[X,Y]}Z, \tag{2.1}$$

and the *flag curvature* is defined as follows:

$$K(V,W) := \frac{g_V(R^V(V,W)W,V)}{g_V(V,V)g_V(W,W)-g_V(V,W)^2}, \tag{2.2}$$

where $V,W\in T_xM\setminus 0$ are linearly independent vectors. Then the *Ricci curvature* of V (as the trace of the flag curvature) is defined by

$$\mathrm{Ric}(V) := \sum_{i=1}^{n-1} K(V,E_i), \tag{2.3}$$

where $\{E_1, E_2, ..., E_{n-1}, V/F(V)\}$ is an orthonormal basis of T_xM with respect to g_V.

Let $d\mu = \sigma(x)dx^1dx^2...dx^n$ be the volume form on M. For a vector $V \in T_xM \setminus 0$,

$$\tau(x,V) := \ln \frac{\sqrt{\det(g_{ij}(x,V))}}{\sigma(x)} \tag{2.4}$$

is a scalar function on $T_xM \setminus 0$ which is called the *distortion* of $(M, F, d\mu)$. We say that the distortion τ is a $\mathcal{C}^\infty$-function, if M is a Riemannian manifold. Setting

$$S(x,V) := \frac{d}{dt}\big(\tau(\gamma(t),\dot{\gamma}(t))\big)|_{t=0}, \tag{2.5}$$

where γ is the geodesic with $\gamma(0) = x$, $\dot{\gamma}(0) = V$. $S(x,\lambda V) = \lambda S(x,V)$ for all $\lambda > 0$. S is a scalar function on $T_xM \setminus 0$ which is called the *S-curvature*. From the definition, it seems that the S-curvature measures the rate of change in the distortion along geodesics in the direction $V \in T_xM$.

For all $N \in (n,\infty)$, we define the *weighted Ricci curvature* of $(M, F, d\mu)$ as follows (see [9]):

$$\begin{cases} \mathrm{Ric}_N(V) := \mathrm{Ric}(V) + \dot{S}(V) - \frac{S(V)^2}{N-n}, \\ \mathrm{Ric}_\infty(V) := \mathrm{Ric}(V) + \dot{S}(V), \\ \mathrm{Ric}_n(V) \ := \begin{cases} \mathrm{Ric} + \dot{S}(V), & \text{if } S(V) = 0 \\ -\infty & \text{otherwise.} \end{cases} \end{cases}$$

Also $\mathrm{Ric}_N(cV) := c^2\mathrm{Ric}_N(V)$ for $c > 0$.

We say that (M, F) is *forward complete* if each geodesic $\gamma : [0,\ell] \to M$ is extended to a geodesic on $[0,\infty)$, in other words, if exponential map is defined on whole TM. Then the Hopf-Rinow theorem gives that every pair of points in M can be joined by a minimal geodesic.

The *Legendre transformation* $\mathfrak{L} : TM \to T^*M$ is defined by

$$\mathfrak{L}(W) := \begin{cases} g_W(W,.), & W \neq 0, \\ 0 & W = 0. \end{cases}$$

For a smooth function $h : M \to \mathbb{R}$, the *gradient vector* of h at $x \in M$ is defined as $\nabla h(x) := \mathfrak{L}^{-1}(dh)$.

Given a smooth vector field $Z = Z^i\partial/\partial x^i$ on M, the *divergence* of Z with respect to an arbitrary volume form $d\mu = e^\varphi dx^1dx^2...dx^n$ is defined by

$$\mathrm{div}Z := \sum_{i=1}^{n}\left(\frac{\partial Z^i}{\partial x^i} + Z^i\frac{\partial\varphi}{\partial x^i}\right). \tag{2.6}$$

Then we define the *Finsler-Laplacian* of h by $\Delta h := \mathrm{div}(\nabla h) = \mathrm{div}(\mathfrak{L}^{-1}(dh))$.

The following lemma is useful to prove Theorem 3 (see [17]).

Lemma 1. *Let $(M, F, d\mu)$ be a Finsler n-manifold, and $h : M \to \mathbb{R}$ a smooth function on M. Then on $U = \{x \in M : \nabla h|_x \neq 0\}$ we have*

$$\Delta h = \sum_i H(h)(E_i, E_i) - S(\nabla h) := \mathrm{tr}_{\nabla h} H(h) - S(\nabla h), \tag{2.7}$$

where $E_1, E_2, ..., E_n$ is a local $g_{\nabla h}$-orthonormal frame on U.

Finally, define *reversibility* $\lambda := \lambda(M, F)$ as follows:

$$\lambda := \sup_{x \in M, y \in TM \setminus 0} \frac{F(x, -y)}{F(x, y)}. \tag{2.8}$$

Obviously, $\lambda \in [1, \infty]$, and $\lambda = 1$ if and only if (M, F) is reversible.

3. The proofs of the theorems

Let $(M, F, d\mu)$ be a Finsler manifold of dimensional n and $r(x) = d(x, p)$ be a distance function with respect to a fixed point $p \in M$. It is well known that r is only smooth on $M - (C_p \cup \{p\})$ where C_p is the cut locus of the point $p \in M$. We assume that γ is a minimal unit speed geodesic segment. We have $\nabla r = \dot{\gamma}$ in the adapted coordinates with respect to the r, and also have $F(\nabla r) = 1$ (see [11]). On the other hand, using the Finsler metric we obtain a weighted Riemannian metric $g_{\nabla r}$. Thus we can apply the Riemannian calculation for $g_{\nabla r}$ (on $M - (C_p \cup \{p\})$).

In order to prove the Theorem 1 and Theorem 2, we use the index form of a minimal unit speed geodesic, and to prove Theorem 3, we use Bochner-Weitzenböck formula and Hessian comparison theorem in Finsler geometry.

Proof of Theorem 1. Let $q \in M$ be a point and let σ be a minimal unit speed geodesic segment from p to q of length ℓ such that $\sigma(0) = p$, $\sigma(\ell) = q$ and $\ell > r_0 > 0$. Since the inequality $\ell > r_0$ holds, ℓ can be parametrized by $\mu > 0$ such that

$$\ell = r_0 e^{\mu\pi} > r_0. \tag{3.1}$$

By virtue of any subsegment of a minimal unit speed geodesic segment is also a minimal unit speed geodesic segment, we have the minimal unit speed geodesic segment γ defined by $\gamma(t) = \sigma|_{[r_0, \ell]}(t)$ where $\gamma : [r_0, \ell] \to M$ and $\gamma(r_0) = \sigma(r_0) = \tilde{q}$, $\gamma(\ell) = \sigma(\ell) = q$. Let $\{E_1 = \dot{\gamma}, E_2, \ldots, E_n\}$ be a parallel $g_{\nabla r}$-orthonormal frame along γ and let $f \in \mathcal{C}^\infty([r_0, \ell])$ be a real-valued smooth function such that $f(r_0) = f(\ell) = 0$. Then we have

$$\mathrm{I}(fE_i, fE_i) = \int_{r_0}^{\ell} \Big(g_{\nabla r}(\dot{f} E_i, \dot{f} E_i) - g_{\nabla r}(R^{\nabla r}(fE_i, \nabla r)\nabla r, fE_i) \Big) dt. \tag{3.2}$$

It is obvious that (3.2) yields, by $g_{\nabla r}(R^{\nabla r}(\nabla r, \nabla r)\nabla r, \nabla r) = 0$ and the assumption (1.3) given in Theorem 1,

$$\begin{aligned}\sum_{i=2}^{n} \mathrm{I}(fE_i, fE_i) &= \int_{r_0}^{\ell} \Big((n-1)\dot{f}^2 - f^2 \mathrm{Ric}(\nabla r)\Big)dt \\ &= \int_{r_0}^{\ell} \Big((n-1)\dot{f}^2 - f^2 \mathrm{Ric}_{\infty}(\nabla r) + f^2 \dot{S}(\nabla r)\Big)dt \\ &\leq \int_{r_0}^{\ell} \Big((n-1)(\dot{f}^2 - \frac{Hf^2}{r^2}) + f^2 \dot{S}(\nabla r)\Big)dt. \end{aligned} \tag{3.3}$$

Here, the term $f^2 \dot{S}(\nabla r)$ equals to

$$\begin{aligned} f^2 \dot{S}(\nabla r) &= -2f\dot{f}S(\nabla r) + \frac{d}{dt}(f^2 S(\nabla r)) = -2f\dot{f}\frac{d\tau}{dt} + \frac{d}{dt}(f^2 S(\nabla r)) \\ &= 2\tau \frac{d}{dt}(f\dot{f}) - 2\frac{d}{dt}(\tau f \dot{f}) + \frac{d}{dt}\big(f^2 S(\nabla r)\big). \end{aligned} \tag{3.4}$$

Integrating both sides of (3.4) and using the assumption $|\tau| \leq (n-1)k$, we obtain

$$\int_{r_0}^{\ell} \big(f^2 \dot{S}(\nabla r)\big)dt = 2\int_{r_0}^{\ell} \tau \frac{d}{dt}(f\dot{f})dt \leq 2(n-1)k \int_{r_0}^{\ell} \Big|\frac{d}{dt}(f\dot{f})\Big|dt, \tag{3.5}$$

because of $f(r_0) = f(\ell) = 0$. By use of (3.5), the inequality (3.3) becomes

$$\sum_{i=2}^{n} \mathrm{I}(fE_i, fE_i) \leq \int_{r_0}^{\ell} (n-1)\Big(\dot{f}^2 - \frac{Hf^2}{r^2}\Big)dt + 2(n-1)k \int_{r_0}^{\ell} \Big|\frac{d}{dt}(f\dot{f})\Big|dt. \tag{3.6}$$

Set

$$f(t) = \mu r_0 \sqrt{r(\gamma(t))}\sin\Big(\frac{1}{\mu}\ln\frac{r(\gamma(t))}{r_0}\Big). \tag{3.7}$$

Therefore we have

$$\begin{aligned} \frac{1}{r_0^2(n-1)} \sum_{i=2}^{n} \mathrm{I}(fE_i, fE_i) \leq &-\frac{1}{4}\int_{r_0}^{\ell} \frac{(4H-1)\mu^2}{r}\sin^2\Big(\frac{1}{\mu}\ln\frac{r}{r_0}\Big)dr \\ &+ \int_{r_0}^{\ell} \frac{1}{r}\Bigg(\cos^2\Big(\frac{1}{\mu}\ln\frac{r}{r_0}\Big) + \frac{\mu}{2}\sin\Big(\frac{2}{\mu}\ln\frac{r}{r_0}\Big)\Bigg)dr \\ &+ 2k\int_{r_0}^{\ell} \frac{1}{r}\Big|\frac{\mu}{2}\sin\Big(\frac{2}{\mu}\ln\frac{r}{r_0}\Big) + \cos\Big(\frac{2}{\mu}\ln\frac{r}{r_0}\Big)\Big|dr. \end{aligned} \tag{3.8}$$

In (3.8), considering the change variable $u = \ln\frac{r}{r_0}$, by $\ell = r_0 e^{\mu\pi}$, we obtain

$$\frac{1}{r_0^2(n-1)} \sum_{i=2}^{n} \mathrm{I}(fE_i, fE_i) \leq -\frac{1}{4}\int_0^{\mu\pi} (4H-1)\mu^2 \sin^2\Big(\frac{1}{\mu}u\Big)du$$

$$+\int_0^{\mu\pi}\left(\cos^2(\frac{1}{\mu}u)+\frac{\mu}{2}\sin(\frac{2}{\mu}u)\right)du$$
$$+2k\int_0^{\mu\pi}\left|\frac{\mu}{2}\sin(\frac{2}{\mu}u)+\cos(\frac{2}{\mu}u)\right|du, \tag{3.9}$$

from which

$$\frac{1}{r_0^2(n-1)}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\leq\frac{\mu}{8}\big(4\pi-(4H-1)\pi\mu^2+16k\sqrt{\mu^2+4}\big). \tag{3.10}$$

In the right hand side of (3.10), if the inequality

$$4\pi-(4H-1)\pi\mu^2+16k\sqrt{\mu^2+4}<0 \tag{3.11}$$

holds, then the index form I is not positive semi-definite. This is a contradiction. Hence, we must take

$$4\pi-(4H-1)\pi\mu^2+16k\sqrt{\mu^2+4}\geq 0. \tag{3.12}$$

Thus

$$\mu\leq\frac{2}{(4H-1)\pi}\sqrt{32k^2+(4H-1)\pi^2+16k\sqrt{4k^2+(4H-1)H\pi^2}}. \tag{3.13}$$

Using the parametrization $\ell=r_0e^{\mu\pi}$ given in (3.1), we find

$$\ell=r_0e^{\mu\pi}\leq r_0\exp\left(\frac{2}{4H-1}\sqrt{32k^2+(4H-1)\pi^2+16k\sqrt{4k^2+(4H-1)H\pi^2}}\right). \tag{3.14}$$

Thus, M is compact and the diameter of M has the upper bound (1.4).

Proof of Theorem 2. By similar arguments given in the proof of Theorem 1, we have

$$\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)=\int_{r_0}^{\ell}\Big((n-1)\dot{f}^2-f^2\mathrm{Ric}(\nabla r)\Big)dt. \tag{3.15}$$

Using the assumption (1.5) in the above integral expression, we get

$$\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\leq\int_{r_0}^{\ell}\left((n-1)\dot{f}^2-(N-1)\frac{Hf^2}{r^2}\right)dt$$
$$+\int_{r_0}^{\ell}\left(f^2\dot{S}(\nabla r)-f^2\frac{(S(\nabla r))^2}{N-n}\right)dt. \tag{3.16}$$

In the inequality (3.16), the term $f^2\dot{S}(\nabla r)$ equals to

$$f^2\dot{S}(\nabla r)=-2f\dot{f}S(\nabla r)+\frac{d}{dt}(f^2S(\nabla r)). \tag{3.17}$$

Integrating both sides of (3.17), we obtain

$$\int_{r_0}^{\ell} f^2 \dot{S}(\nabla r)dt = \int_{r_0}^{\ell} -2f\dot{f}S(\nabla r)dt, \tag{3.18}$$

by $f(r_0) = f(\ell) = 0$. If we take $P = -\dot{f}$ and $T = fS(\nabla r)$, then the Cauchy-Schwarz inequality

$$\int_{r_0}^{\ell} PT\,dt = \int_{r_0}^{\ell} -f\dot{f}S(\nabla r)dt \le \Big(\int_{r_0}^{\ell} \dot{f}^2 dt\Big)^{1/2}\Big(\int_{r_0}^{\ell} f^2(S(\nabla r))^2 dt\Big)^{1/2}. \tag{3.19}$$

Because of the facts

$$A = (N-n)\int_{r_0}^{\ell} \dot{f}^2 dt \ge 0 \text{ and } B = \frac{1}{N-n}\int_{r_0}^{\ell} f^2(S(\nabla r))^2 dt \ge 0, \tag{3.20}$$

where $N \in (n,\infty)$, we have the inequality $\sqrt{AB} \le \frac{1}{2}(A+B)$, i.e.,

$$\Big(\int_{r_0}^{\ell} \dot{f}^2 dt\Big)^{1/2}\Big(\int_{r_0}^{\ell} f^2(S(\nabla r))^2 dt\Big)^{1/2} \le \int_{r_0}^{\ell} \frac{1}{2}(N-n)\dot{f}^2 dt + \int_{r_0}^{\ell} f^2\frac{(S(\nabla r))^2}{2(N-n)}dt. \tag{3.21}$$

Using (3.21) in (3.19), we find

$$\int_{r_0}^{\ell} -f\dot{f}S(\nabla r)dt \le \int_{r_0}^{\ell}\Big(\frac{1}{2}(N-n)\dot{f}^2 + f^2\frac{(S(\nabla r))^2}{2(N-n)}\Big)dt. \tag{3.22}$$

Therefore we have

$$\int_{r_0}^{\ell} f^2\dot{S}(\nabla r)dt = \int_{r_0}^{\ell} -2f\dot{f}S(\nabla r)dt \le \int_{r_0}^{\ell}\Big((N-n)\dot{f}^2 + f^2\frac{(S(\nabla r))^2}{N-n}\Big)dt. \tag{3.23}$$

Inserting (3.23) into (3.16), we obtain

$$\sum_{i=2}^{n} \mathrm{I}(fE_i, fE_i) \le (N-1)\int_{r_0}^{\ell}\Big(\dot{f}^2 - \frac{Hf^2}{r^2}\Big)dt. \tag{3.24}$$

In the inequality (3.24), let us consider the choice

$$f(t) = \mu r_0\sqrt{r(\gamma(t))}\sin\Big(\frac{1}{\mu}\ln\frac{r(\gamma(t))}{r_0}\Big). \tag{3.25}$$

Thereby the inequality (3.24) yields

$$\frac{1}{N-1}\sum_{i=2}^{n} \mathrm{I}(fE_i, fE_i) \le \int_{r_0}^{\ell} \frac{r_0^2}{r}\Big(\cos^2\big(\frac{1}{\mu}\ln\frac{r}{r_0}\big) + \frac{\mu}{2}\sin\big(\frac{2}{\mu}\ln\frac{r}{r_0}\big)\Big)dr$$

$$-\frac{1}{4}\int_{r_0}^{\ell}\frac{r_0^2}{r}(4H-1)\mu^2\sin^2(\frac{1}{\mu}\ln\frac{r}{r_0})dr. \tag{3.26}$$

In the above inequality, considering the change variable $u=\ln\frac{r}{r_0}$, , by use of $\ell = r_0e^{\mu\pi}$, we get

$$\frac{1}{N-1}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\le\int_0^{\mu\pi}r_0^2\left(\cos^2(\frac{1}{\mu}u)+\frac{\mu}{2}\sin(\frac{2}{\mu}u)\right)du$$
$$-\frac{1}{4}\int_0^{\mu\pi}r_0^2(4H-1)\mu^2\sin^2(\frac{1}{\mu}u)du, \tag{3.27}$$

which implies

$$\frac{1}{N-1}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\le r_0^2\frac{\mu\pi}{8}\left(4-(4H-1)\mu^2\right). \tag{3.28}$$

In the right hand side of (3.28), if the inequality

$$4-(4H-1)\mu^2<0 \tag{3.29}$$

holds, then we conclude that the index form I is not positive semi-definite. But, since γ is minimal geodesic, this is a contradiction. Hence, we must take

$$4-(4H-1)\mu^2\ge 0. \tag{3.30}$$

Thus we obtain

$$\mu\le\frac{2}{\sqrt{4H-1}}. \tag{3.31}$$

Using the parametrization $\ell = r_0e^{\mu\pi}$, we find

$$\ell = r_0e^{\mu\pi}\le r_0e^{2\pi/\sqrt{4H-1}}. \tag{3.32}$$

Thus, M is compact and the diameter of M has the upper bound (1.6).

Proof of Theorem 3. We know that $r(x)=d(x,p)$ is a distance function from a fixed point $p\in M$ and it is smooth on $M-\left(C_p\cup\{p\}\right)$. Also it satisfies $F(\nabla r)=1$. In Finsler geometry, recall that the Bochner-Weitzenböck formula [10] for a smooth function $u\in\mathcal{C}^\infty(M)$

$$0=\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right)=\mathrm{Ric}_\infty(\nabla u)+D(\Delta u)(\nabla u)+\|\nabla^2u\|^2_{HS(\nabla u)}. \tag{3.33}$$

From the Bochner formula applied to distance function r and by Lemma 1, we have, on $M-\left(C_p\cup\{p\}\right)$,

$$0=\mathrm{Ric}_\infty(\nabla r)+D(\Delta r)(\nabla r)+\|\nabla^2r\|^2_{HS(\nabla r)}$$
$$\ge\mathrm{Ric}_\infty(\nabla r)+g_{\nabla r}(\nabla^{\nabla r}\Delta r,\nabla r)+\frac{(\Delta r+S(\nabla r))^2}{n-1}. \tag{3.34}$$

By virtue of the inequality $(a \mp b)^2 \geq \frac{1}{\beta+1}a^2 - \frac{1}{\beta}b^2$ holding for all real numbers a, b and positive real number β, we have

$$\frac{\left(\Delta r + S(\nabla r)\right)^2}{n-1} \geq \frac{(\Delta r)^2}{(n-1)(\beta+1)} - \frac{(S(\nabla r))^2}{(n-1)\beta}. \tag{3.35}$$

In the case where $N > n$, taking $\beta = \frac{N-n}{n-1} > 0$, (3.34) yields

$$0 \geq \mathrm{Ric}_\infty(\nabla r) + g_{\nabla r}(\nabla^{\nabla r}\Delta r, \nabla r) + \frac{(\Delta r)^2}{N-1} - \frac{(S(\nabla r))^2}{N-n}. \tag{3.36}$$

Applying the assumption (1.7) given in Theorem 3 to (3.36), we find

$$0 \geq \partial_r(\Delta r) + \frac{(\Delta r)^2}{N-1} + (N-1)\frac{H}{(1+r)^2}. \tag{3.37}$$

The above inequality can be rewritten as

$$0 \geq \partial_r(\frac{\Delta r}{N-1}) + \left(\frac{\Delta r}{N-1}\right)^2 + \frac{H}{(1+r)^2}. \tag{3.38}$$

We know from the Hessian comparison theorem in [17], if there is a local vector field X on an open set U of $p \in M$ with $g_{\nabla r}(X, X) = 1$, $g_{\nabla r}(\nabla r, X) = 0$, then $H(r)(X, X) \sim \frac{1}{r}$ as $r \to 0^+$. Hence, using the Lemma 1, we have

$$\lim_{r\to 0^+} r(\frac{1}{N-1}\Delta r) = \lim_{r\to 0^+} r\left(\frac{1}{N-1}\left(\mathrm{tr}_{\nabla r} H(r) - S(\nabla r)\right)\right) = \frac{n-1}{N-1} < 1, \tag{3.39}$$

where $N > n$. By (3.38) and (3.39), we obtain, on $M - \left(C_p \cup \{p\}\right)$,

$$\frac{1}{N-1}\Delta r \leq \frac{1}{2(1+r)}\left(1 + \sqrt{4H-1}\cot(\frac{\sqrt{4H-1}}{2}\ln(1+r))\right), \tag{3.40}$$

where $H > 1/4$. Indeed, the function

$$Y(r) = \frac{1}{2(1+r)}\left(1 + \sqrt{4H-1}\cot(\frac{\sqrt{4H-1}}{2}\ln(1+r))\right) \tag{3.41}$$

is a solution of the Riccati differential equation

$$Y'(r) + (Y(r))^2 + \frac{H}{(1+r)^2} = 0. \tag{3.42}$$

Because of $\lim_{r\to 0^+} rY(r) = 1$ and (3.39), we have

$$\lim_{r\to 0^+} r(\frac{1}{N-1}\Delta r) \leq \lim_{r\to 0^+} rY(r). \tag{3.43}$$

Thus, for a sufficiently small positive constant $\varepsilon \in (0, T)$ the inequality

$$\frac{1}{N-1}\Delta r(\varepsilon) \leq Y(\varepsilon) \tag{3.44}$$

is ensured. In that case, the Riccati comparison theorem gives the inequality

$$\frac{1}{N-1}\Delta r(t) \leq Y(t) \tag{3.45}$$

for every $t \in [\varepsilon, T)$.

Let $q \in M$ be any point, and let σ be a minimal unit speed geodesic segment from p to q. Suppose that the inequality

$$d(p,q) > e^{2\pi/\sqrt{4H-1}} - 1 \tag{3.46}$$

is satisfied. Then, since σ is a minimal unit speed geodesic segment from p to q, we have the fact that the point $\sigma(e^{2\pi/\sqrt{4H-1}} - 1)$ is outside the cut locus of $p \in M$, i.e.,

$$\sigma(e^{2\pi/\sqrt{4H-1}} - 1) \in M - \big(C_p \cup \{p\}\big). \tag{3.47}$$

Therefore the distance function r is smooth at this point. Namely, at this point, left hand side of (3.40) is a constant. However, the right side of (3.40) tends to $-\infty$ as $r \to (e^{2\pi/\sqrt{4H-1}} - 1)^-$, i.e.,

$$\lim_{r \to (e^{2\pi/\sqrt{4H-1}}-1)^-} \frac{1}{2(1+r)}\Big(1 + \sqrt{4H-1}\cot\big(\frac{\sqrt{4H-1}}{2}\ln(1+r)\big)\Big) = -\infty. \tag{3.48}$$

This is a contradiction. Hence, (3.46) does not hold. It must be

$$d(p,q) \leq e^{2\pi/\sqrt{4H-1}} - 1. \tag{3.49}$$

Therefore M is compact. Let λ be the reversibility. For any points $p', q' \in M$, due to the triangle inequality and the inequality (3.49), we obtain

$$d(p',q') \leq d(p',p) + d(p,q') \leq \lambda d(p,p') + d(p,q'), \tag{3.50}$$

and so

$$d(p',q') \leq (1+\lambda)(e^{2\pi/\sqrt{4H-1}} - 1). \tag{3.51}$$

This completes the proof of theorem.

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