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SOME NEW INTEGRAL INEQUALITIES FOR N-TIMES DIFFERENTIABLE R-CONVEX AND R-CONCAVE FUNCTIONS

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Abstract. In this work, by using an integral identity together with both the Hölder and the Power-Mean integral inequality we establish several new inequalities for n-time differentiable r-convex and concave functions.

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1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Many articles have been written by a number of mathematicians on convex functions and inequalities for their different classes, using, for example, the last articles [3, 8–15] and the references in these papers.

$f : [a,b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality (see [6] for more information). Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (*e.g.*, [3, 4, 15]). In [20], the first author obtained a new refinement of the Hermite-Hadamard inequality for convex functions. The Hermite-Hadamard inequality was generalized in [17] to an r-convex positive function which is defined on an interval $[a,b]$.

Definition 1. A positive function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called r-convex function on $[a,b]$, if for each the $x, y \in [a,b]$ and $t \in [0,1]$,

$$f(tx+(1-t)y) \leq \begin{cases} [tf^r(x)+(1-t)f^r(y)]^{\frac{1}{r}}, & r \neq 0, \\ [f(x)]^t[f(y)]^{1-t}, & r = 0. \end{cases}$$

If the equality is reversed, then the function f is said to be r-concave.

It is obvious 0-convex functions are simply log-convex functions, 1-convex functions are ordinary convex functions and -1-convex functions are arithmetically harmonically convex. One should note that if f is r-convex on $[a,b]$, then the function f^r is a convex function for $r > 0$ and f^r is a concave function for $r < 0$. We note that if f and g are convex and g is increasing, then gof is convex; moreover, since $f = exp(log\ f)$, it follows that a log-convex function is convex.

The definition of r-convexity naturally complements the concept of r-concavity, in which the inequality is reversed [18] and which plays an important role in statistics.

It is easily seen that if f is r-convex on $[a,b]$,

$$f^r\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f^r(x)\,dx \leq \frac{f^r(a)+f^r(b)}{2}, \quad r>0 \tag{1.1}$$

$$f^r\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a}\int_a^b f^r(x)\,dx \geq \frac{f^r(a)+f^r(b)}{2}, \quad r<0 \tag{1.2}$$

Some refinements of the Hadamard inequality for r-convex functions could be found in [2,7,16,19,21]. In [1], Bessenyei studied Hermite-Hadamard-type inequalities for generalized 3-convex functions. In [16], the authors showed that if f is r-convex in $[a,b]$ and $0 < r \leq 1$, then

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{r}{r+1}\left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}. \tag{1.3}$$

Theorem 1 ([5]). *Suppose that f is a positive r-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq L_r(f(a), f(b)).$$

If f is a positive r-concave function, then the inequality is reversed, where

$$L_r(f(a), f(b)) = \begin{cases} \frac{r}{r+1}\frac{f^{r+1}(a)-f^{r+1}(b)}{f^r(a)-f^r(b)}, & r \neq 0,-1, & f(a) \neq f(b) \\ \frac{f(a)-f(b)}{lnf(a)-lnf(b)}, & r = 0, & f(a) \neq f(b) \\ f(a)\,f(b)\frac{lnf(a)-lnf(b)}{f(a)-f(b)}, & r=-1, & f(a) \neq f(b) \\ f(a), & & f(a) = f(b). \end{cases}$$

Theorem 2. *Let $f:[a,b]\to(0,\infty)$ be r-convex function and $r\geq 1$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx\leq\left[\frac{f^r(a)+f^r(b)}{2}\right]^{\frac{1}{r}}.$$

Lemma 1. *Let $a\geq 0,\ b\geq 0$. Then $(a+b)^\lambda\leq a^\lambda+b^\lambda,\ 0<\lambda\leq 1$.*

Let $0<a<b$, throughout this paper we will use

$$A(a,b)=\frac{a+b}{2},\ \ G(a,b)=\sqrt{ab}$$

$$L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ \ a\neq b,\ p\in\mathbb{R},\ \ p\neq -1,0$$

for the arithmetic, geometric, generalized logarithmic mean, respectively. Also for shortness we will use the following notation:

$$I(a,b,n,f)=\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx$$

where an empty sum is understood to be nil.

2. Main results

We will use the following Lemma for obtain our main results.

Lemma 2 ([14]). *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be n-times differentiable mapping on I° for $n\in\mathbb{N}$ and $f^{(n)}\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$, we have the identity*

$$I(a,b,n,f)=\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)\,dx.$$

where an empty sum is understood to be nil.

Theorem 3. *For $n\in\mathbb{N}$; let $f:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I°, $r>0$ and $a,b\in I^\circ$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, then the following inequality holds:*

$$|I(a,b,n,f)|\leq\frac{b-a}{n!}L_{np}^n(a,b)L_{\frac{1}{r}}^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr},\left|f^{(n)}(b)\right|^{qr}\right)$$

Proof. If $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$ and $r>0$, using Lemma 2, the Hölder integral inequality and

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q$$

$$\leq \left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr}+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}},$$

we have

$$\begin{aligned}
&|I(a,b,n,f)| \\
&\leq \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx \\
&\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^{q}dx\right)^{\frac{1}{q}} \\
&\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr}+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}}dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_{|f^{(n)}(a)|^{qr}}^{|f^{(n)}(b)|^{qr}} u^{\frac{1}{r}}\frac{b-a}{\left|f^{(n)}(b)\right|^{qr}-\left|f^{(n)}(a)\right|^{qr}}du\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)\left(\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^{qr\left(\frac{1}{r}+1\right)}-\left|f^{(n)}(a)\right|^{qr\left(\frac{1}{r}+1\right)}}{\left(\frac{1}{r}+1\right)\left(\left|f^{(n)}(b)\right|^{qr}-\left|f^{(n)}(a)\right|^{qr}\right)}\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)L_{np}^{n}(a,b)\left(\frac{\left|f^{(n)}(b)\right|^{qr\left(\frac{1}{r}+1\right)}-\left|f^{(n)}(a)\right|^{qr\left(\frac{1}{r}+1\right)}}{\left(\frac{1}{r}+1\right)\left(\left|f^{(n)}(b)\right|^{qr}-\left|f^{(n)}(a)\right|^{qr}\right)}\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)L_{np}^{n}(a,b)L_{\frac{1}{r}}^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr},\left|f^{(n)}(b)\right|^{qr}\right).
\end{aligned}$$

This completes the proof of theorem. □

Remark 1. The results obtained in this paper reduces to the results of [14] in case of $r=1$.

Corollary 1. *Under the conditions Theorem 3 for $n=1$ we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b)L_{\frac{1}{r}}^{\frac{1}{qr}}\left(\left|f'(a)\right|^{qr},\left|f'(b)\right|^{qr}\right).$$

Proposition 1. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\geq 1$, $r\geq 1$ we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq L_p(a,b)L_{\frac{1}{r}}^{\frac{1}{qr}}\left(a^{mr},b^{mr}\right).$$

Proof. Under the assumption of the Proposition, let $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. Then $|f'(x)|^q = x^m$ is r-convex on $(0,\infty)$ and the result follows directly from Corollary 1. □

Remark 2. Under the assumption of the Proposition 2.1, If $r = 1$, $m = 1$, then the results obtained in this paper reduces to the results of [14].

Theorem 4. *For $n \in \mathbb{N}$; let $f : I \subset (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I°, $r > 0$ and $a,b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q \geq 1$ is r-convex function on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
&|I(a,b,n,f)| \\
\leq &\begin{cases} \frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[C_1\left|f^{(n)}(b)\right|^q + C_2\left|f^{(n)}(a)\right|^q\right]^{\frac{1}{q}}, & r \geq 1 \\ \frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[C_1^r\left|f^{(n)}(b)\right|^{qr} + C_1^r\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{qr}}, & r \leq 1 \end{cases}
\end{aligned}
$$

where

$$
C_1 = C_1(a,b,r,n) = \int_a^b x^n (x-a)^{\frac{1}{r}} dx, C_2 = C_2(a,b,r,n) = \int_a^b x^n (b-x)^{\frac{1}{r}} dx.
$$

Proof. From Lemma 2 and Power-Mean integral inequality, we get

$$
\begin{aligned}
&|I(a,b,n,f)| \\
\leq &\frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx \\
\leq &\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
\leq &\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr} + \frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}} dx\right)^{\frac{1}{q}}.
\end{aligned}
$$

Here, using Lemma 1 we obtain respectively,

For $r \geq 1$

$$
\begin{aligned}
&|I(a,b,n,f)| \\
\leq &\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}} \times \left(\int_a^b x^n \left\{\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr}\right]^{\frac{1}{r}} + \left[\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}}\right\} dx\right)^{\frac{1}{q}} \\
= &\frac{1}{n!}\left(\frac{1}{b-a}\right)^{\frac{1}{qr}}\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$\times\left(\int_a^b x^n (x-a)^{\frac{1}{r}} \left|f^{(n)}(b)\right|^q dx + \int_a^b x^n (b-x)^{\frac{1}{r}} \left|f^{(n)}(a)\right|^q dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\frac{1}{b-a}\right)^{\frac{1}{qr}} (b-a)^{1-\frac{1}{q}} \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{1-\frac{1}{q}}$$

$$\times\left[C_1(a,b,r,n)\left|f^{(n)}(b)\right|^q + C_2(a,b,q,n)\left|f^{(n)}(a)\right|^q\right]^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[C_1\left|f^{(n)}(b)\right|^q + C_2\left|f^{(n)}(a)\right|^q\right]^{\frac{1}{q}},$$

For $r \leq 1$, using Minkowski inequality, we have

$$|I(a,b,n,f)|$$

$$\leq\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\frac{1}{b-a}\right)^{\frac{1}{qr}}$$

$$\times\left(\int_a^b \left[x^{nr}(x-a)\left|f^{(n)}(b)\right|^{qr} + x^{nr}(b-x)\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}} dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left\{\left[\int_a^b x^n (x-a)^{\frac{1}{r}}\left|f^{(n)}(b)\right|^q dx\right]^r + \left[\int_a^b x^n (b-x)^{\frac{1}{r}}\left|f^{(n)}(a)\right|^q dx\right]^r\right\}^{\frac{1}{r}}\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[C_1^r\left|f^{(n)}(b)\right|^{qr} + C_2^r\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{qr}}.$$

This completes the proof of theorem. □

Corollary 2. *Under the conditions Theorem 4 for $n=1$ we have the following inequalities:*

$$|J(a,b,f)|$$

$$\leq\begin{cases} A^{1-\frac{1}{q}}(a,b)\left\{\frac{r(b-a)}{2r+1}\left[|f'(b)|^q - |f'(a)|^q\right] + \frac{r\left[a|f'(b)|^q + b|f'(a)|^q\right]}{r+1}\right\}^{\frac{1}{q}}, & r \geq 1 \\ A^{1-\frac{1}{q}}(a,b)\left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r |f'(b)|^{qr} + \left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r |f'(a)|^{qr}\right\}^{\frac{1}{qr}}, & r \leq 1 \end{cases}$$

where $J(a,b,f) = \frac{I(a,b,1,f)}{b-a}$.

Proposition 2. *Let $a,b \in (0,\infty)$ with $a<b$, $q\geq 1$ and $m\geq 1$, we have the following inequalities:*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq \begin{cases} A^{1-\frac{1}{q}}(a,b)\left[\frac{2rA(a^{m+1},b^{m+1})}{2r+1}+\frac{2r^2G^2(a,b)A(a^{m-1},b^{m-1})}{(r+1)(2r+1)}\right]^{\frac{1}{q}}, & r\geq 1 \\ A^{1-\frac{1}{q}}(a,b)\left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r b^{mr}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r a^{mr}\right\}^{\frac{1}{qr}}, & r\leq 1. \end{cases}$$

Proof. The result follows directly from Corollary 2 for function $f(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}$, $x\in(0,\infty)$. □

Corollary 3. *Using Proposition 2 for $m=1$, we have following inequalities:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b) \leq \begin{cases} A^{1-\frac{1}{q}}(a,b)\left[\frac{2r}{2r+1}A\left(a^2,b^2\right)+\frac{2r^2}{(r+1)(2r+1)}G^2(a,b)\right]^{\frac{1}{q}}, & r\geq 1 \\ A^{1-\frac{1}{q}}(a,b)\left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r b^{r}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r a^{r}\right\}^{\frac{1}{qr}}, & r\leq 1. \end{cases}$$

Corollary 4. *Using Proposition 2 for $q=1$, we have following inequalities:*

$$L_{m+1}^{m+1}(a,b) \leq \begin{cases} \frac{2rA(a^{m+1},b^{m+1})}{2r+1}+\frac{2r^2G^2(a,b)A(a^{m-1},b^{m-1})}{(r+1)(2r+1)}, & r\geq 1 \\ \left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r b^{mr}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r a^{mr}\right\}^{\frac{1}{r}}, & r\leq 1. \end{cases}$$

Corollary 5. *Using Corollary 4. for $m=1$, we have following inequalities:*

$$L_2^2(a,b) \leq \begin{cases} \frac{2r}{2r+1}A\left(a^2,b^2\right)+\frac{2r^2}{(r+1)(2r+1)}G^2(a,b), & r\geq 1 \\ \left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r b^{r}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r a^{r}\right\}^{\frac{1}{r}}, & r\leq 1. \end{cases}$$

Corollary 6. *Under the conditions Theorem 4 for $q=1$ we have the following inequalities:*

$$|I(a,b,n,f)| \leq \begin{cases} \frac{1}{n!}(b-a)^{-\frac{1}{r}}\left[C_1\left|f^{(n)}(b)\right|+C_2\left|f^{(n)}(a)\right|\right], & r\geq 1 \\ \frac{(b-a)^{-\frac{1}{r}}}{n!}\left[C_1^r\left|f^{(n)}(b)\right|^r+C_2^r\left|f^{(n)}(a)\right|^r\right]^{\frac{1}{r}}, & r\leq 1 \end{cases}$$

Theorem 5. *For $n\in\mathbb{N}$; let $f:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I°, $r>0$ and $a,b\in I^\circ$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, then the following inequality holds:*

$$|I(a,b,n,f)| \leq \begin{cases} \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\left|f^{(n)}(b)\right|^q D_1+\left|f^{(n)}(a)\right|^q D_2\right)^{\frac{1}{q}}, & r\geq 1 \\ \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\left|f^{(n)}(b)\right|^{qr} D_1^r+\left|f^{(n)}(a)\right|^{qr} D_2^r\right)^{\frac{1}{qr}}, & r\leq 1 \end{cases}$$

where

$$D_1 = D_1(a,b,r,n,q) = \int_a^b x^{nq}(x-a)^{\frac{1}{r}}dx$$

$$D_2 = D_2(a,b,r,n,q) = \int_a^b x^{nq}(b-x)^{\frac{1}{r}}dx.$$

Proof. Since $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, using Lemma 2 and the Hölder integral inequality, we have the following inequality:

$$\begin{aligned}
&|I(a,b,n,f)| \\
&\leq \frac{1}{n!}\int_a^b 1.x^n\left|f^{(n)}(x)\right|dx \\
&\leq \frac{1}{n!}\left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\leq \frac{1}{n!}\left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr}+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}}dx\right)^{\frac{1}{q}}.
\end{aligned}$$

Here, using Lemma 1 we obtain respectively,

For $r \geq 1$,

$$\begin{aligned}
&|I(a,b,n,f)| \\
&\leq \frac{1}{n!}\left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1}{r}}\left|f^{(n)}(b)\right|^q+\left(\frac{b-x}{b-a}\right)^{\frac{1}{r}}\left|f^{(n)}(a)\right|^q\right]dx\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^{-\frac{1}{qr}}}{n!}(b-a)^{\frac{1}{p}}\left(\left|f^{(n)}(b)\right|^q D_1+\left|f^{(n)}(a)\right|^q D_2\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\left|f^{(n)}(b)\right|^q D_1+\left|f^{(n)}(a)\right|^q D_2\right)^{\frac{1}{q}},
\end{aligned}$$

For $r \leq 1$, using Minkowski inequality, we have

$$\begin{aligned}
&|I(a,b,n,f)| \\
&\leq \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\int_a^b\left[x^{nqr}(x-a)\left|f^{(n)}(b)\right|^{qr}+x^{nqr}(b-x)\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}}\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}} \\
&\quad\times\left\{\left(\left|f^{(n)}(b)\right|^q\int_a^b x^{nq}(x-a)^{\frac{1}{r}}dx\right)^r+\left(\left|f^{(n)}(a)\right|^q\int_a^b x^{nq}(b-x)^{\frac{1}{r}}dx\right)^r\right\}^{\frac{1}{qr}} \\
&= \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\left|f^{(n)}(b)\right|^{qr}D_1^r+\left|f^{(n)}(a)\right|^{qr}D_2^r\right)^{\frac{1}{qr}}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 7. *Under the conditions Theorem 5 for $n=1$ we have the following inequalities:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\begin{cases}(b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(|f'(b)|^q D_1+|f'(a)|^q D_2\right)^{\frac{1}{q}}, & r\geq 1\\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(|f'(b)|^{qr} D_1^r+|f'(a)|^{qr} D_2^r\right)^{\frac{1}{qr}}, & r\leq 1.\end{cases}$$

Proposition 3. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\geq 1$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b)\leq\begin{cases}(b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(b^m D_1+a^m D_2\right)^{\frac{1}{q}}, & r\geq 1\\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(b^{mr} D_1^r+a^{mr} D_2^r\right)^{\frac{1}{q}}, & r\leq 1.\end{cases}$$

Proof. The result follows directly from Corollary 7 for $f(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}$, $x\in(0,\infty)$. □

Corollary 8. *For $m=1$ from Proposition 3, we obtain the following inequality:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b)\leq\begin{cases}(b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(bD_1+aD_2\right)^{\frac{1}{q}}, & r\geq 1\\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(b^r D_1^r+a^r D_2^r\right)^{\frac{1}{q}}, & r\leq 1.\end{cases}$$

Theorem 6. *For $n\in\mathbb{N}$; let $f:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° (interior of I), $r>0$ and $a,b\in I^\circ$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, then the following inequalities holds:*

$$|I(a,b,n,f)|$$
$$\leq\begin{cases}2^{\frac{1}{qr}}\frac{b-a}{n!}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}L_{np}^n(a,b)\,A^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr},\left|f^{(n)}(b)\right|^{qr}\right), & 0<r\leq 1,\\ \frac{b-a}{n!}L_{np}^n(a,b)\,A^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr},\left|f^{(n)}(b)\right|^{qr}\right), & r\geq 0,\\ \frac{1}{n!}(b-a)\,L_{np}^n(a,b)\left(L_r\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}}, & r>0,\end{cases}$$

Proof. For $0<r\leq 1$, since $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, with respect to Hermite-Hadamard inequality we have

$$\int_a^b\left|f^{(n)}(x)\right|^q dx\leq(b-a)\frac{r}{r+1}\left[\left|f^{(n)}(a)\right|^{qr}+\left|f^{(n)}(b)\right|^{qr}\right]^{\frac{1}{r}}.$$

Using Lemma 2 and the Hölder integral inequality we have

$$|I(a,b,n,f)|$$

$$\le \frac{1}{n!}\int_a^b x^n \left| f^{(n)}(x)\right| dx$$
$$\le \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left| f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$\le \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left((b-a)\frac{r}{r+1}\left[\left| f^{(n)}(a)\right|^{qr}+\left| f^{(n)}(b)\right|^{qr}\right]^{\frac{1}{r}}\right)^{\frac{1}{q}}$$
$$=2^{\frac{1}{qr}}\frac{b-a}{n!}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}\left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}}\left[\frac{\left| f^{(n)}(a)\right|^{qr}+\left| f^{(n)}(b)\right|^{qr}}{2}\right]^{\frac{1}{qr}}$$
$$=2^{\frac{1}{qr}}\frac{b-a}{n!}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}L_{np}^n(a,b)\,A^{\frac{1}{qr}}\left(\left| f^{(n)}(a)\right|^{qr},\left| f^{(n)}(b)\right|^{qr}\right).$$

For $r \ge 1$, since $\left| f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, with respect to Theorem 2 we get

$$\int_a^b \left| f^{(n)}(x)\right|^q dx \le (b-a)\left[\frac{\left| f^{(n)}(a)\right|^{qr}+\left| f^{(n)}(b)\right|^{qr}}{2}\right]^{\frac{1}{r}}.$$

Using Lemma 2 and the Hölder integral inequality we have

$$|I(a,b,n,f)| \le \frac{1}{n!}\int_a^b x^n \left| f^{(n)}(x)\right| dx$$
$$\le \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left| f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$=\frac{b-a}{n!}\left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}}\left[\frac{\left| f^{(n)}(a)\right|^{qr}+\left| f^{(n)}(b)\right|^{qr}}{2}\right]^{\frac{1}{qr}}$$
$$=\frac{b-a}{n!}L_{np}^n(a,b)\,A^{\frac{1}{qr}}\left(\left| f^{(n)}(a)\right|^{qr},\left| f^{(n)}(b)\right|^{qr}\right).$$

For $r>0$, using Lemma 2, Theorem 1 and the Hölder integral inequality we have

$$|I(a,b,n,f)| \le \frac{1}{n!}\int_a^b x^n \left| f^{(n)}(x)\right| dx$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left((b-a)L_r\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)L_{np}^{n}(a,b)\left(L_r\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}}.$$

This completes the proof of theorem. □

Corollary 9. *Under the conditions Theorem 6 for $n=1$ we have the following inequalities:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \begin{cases} 2^{\frac{1}{qr}}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}L_p(a,b)\,A^{\frac{1}{qr}}\left(|f'(a)|^{qr},|f'(b)|^{qr}\right), 0<r\leq 1 \\ L_p(a,b)\,A^{\frac{1}{qr}}\left(|f'(a)|^{qr},|f'(b)|^{qr}\right), \quad r\geq 1 \\ L_p(a,b)\left(L_r\left(|f'(a)|^q,|f'(b)|^q\right)\right)^{\frac{1}{q}}, \quad r>0. \end{cases}$$

Proposition 4. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in[0,1]$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b)\leq \begin{cases} 2^{\frac{1}{qr}}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}L_p(a,b)\,A^{\frac{1}{qr}}\left(a^{mr},b^{mr}\right), & 0<r\leq 1 \\ L_p(a,b)\,A^{\frac{1}{qr}}\left(a^{mr},b^{mr}\right), & r\geq 1 \\ L_p(a,b)\left(L_r\left(a^m,b^m\right)\right)^{\frac{1}{q}}, & r>0 \end{cases}$$

Under the assumption of the Proposition, let $f(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}$, $x\in(0,\infty)$. Then $|f'(x)|^q=x^m$ is r-convex on $(0,\infty)$ and the result follows directly from Corollary 9.

Corollary 10. *For $m=1$ from Proposition 4, we obtain the following inequalities:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b)\leq \begin{cases} 2^{\frac{1}{qr}}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}L_p(a,b)\,A^{\frac{1}{qr}}\left(a^{r},b^{r}\right), & 0<r\leq 1 \\ L_p(a,b)\,A^{\frac{1}{qr}}\left(a^{r},b^{r}\right), & r\geq 1 \\ L_p(a,b)\left(L_r(a,b)\right)^{\frac{1}{q}}, & r>0 \end{cases}$$

Theorem 7. *For $n\in\mathbb{N}$; let $f:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I°, $r>0$ and $a,b\in I^\circ$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^{\frac{q}{r}}$ for $q>1$ is r-convex function on $[a,b]$, then the following inequality holds:*

$$|I(a,b,n,f)|\leq \frac{1}{n!}(b-a)L_{np}^{n}(a,b)\,A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right).$$

Proof. If $\left|f^{(n)}\right|^{\frac{q}{r}}$ for $q>1$ is r-convex function on $[a,b]$ and $r>0$, using (1.1) inequality, Lemma 2 and the Hölder integral inequality respectively, we have

$$\int_a^b \left|f^{(n)}(x)\right|^q dx = \int_a^b \left(\left|f^{(n)}(x)\right|^{\frac{q}{r}}\right)^r dx \le (b-a)\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{2}$$

and

$$\begin{aligned}
|I(a,b,n,f)| &\le \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx \\
&\le \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}\left(\frac{b^{np+1}-a^{np+1}}{np+1}\right)^{\frac{1}{p}}\left((b-a)\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{2}\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)L_{np}^n(a,b)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right).
\end{aligned}$$

This completes the proof of theorem. □

Corollary 11. *Under the conditions Theorem 7 for $n=1$ we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le L_p(a,b)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right).$$

Proposition 5. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in[0,1]$, we have*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le L_p(a,b)A^{\frac{1}{q}}\left(a^{mr}, b^{mr}\right).$$

Proof. Under the assumption of the Proposition, let $f(x)=\frac{q}{mr+q}x^{\frac{mr}{q}+1}$, $x\in(0,\infty)$. Then $|f'(x)|^{\frac{q}{r}}=x^m$ is r-convex on $(0,\infty)$ and the result follows directly from Corollary 11. □

Corollary 12. *For $m=1$ from Proposition 5, we obtain the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le L_p(a,b)A^{\frac{1}{q}}\left(a^r, b^r\right).$$

Theorem 8. *For $n\in\mathbb{N}$; let $f:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I°, $r>0$ and $a,b\in I^\circ$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^{\frac{q}{r}}$ for $q>1$ is*

r-concave function on $[a,b]$, then the following inequality holds:

$$\left|I\left(a,b,n,f\right)\right| \leq \frac{b-a}{n!}L_{np}^{n}\left(a,b\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.$$

Proof. If $\left|f^{(n)}\right|^{\frac{q}{r}}$ for $q>1$ is r-concave function on $[a,b]$ and $r>0$, using Lemma 2, the Hölder integral inequality and

$$\int_a^b \left|f^{(n)}(x)\right|^q dx = \int_a^b \left(\left|f^{(n)}(x)\right|^{\frac{q}{r}}\right)^r dx \leq (b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q$$

we have

$$\begin{aligned}
\left|I\left(a,b,n,f\right)\right| &\leq \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx \\
&\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left((b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)L_{np}^{n}(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 13. *Under the conditions Theorem 8 for $n=1$ we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b)\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Proposition 6. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in[0,1]$, we have*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b)A^{\frac{mr}{q}}(a,\ b).$$

Proof. Under the assumption of the Proposition, let $f(x)=\frac{q}{mr+q}x^{\frac{mr}{q}+1}$, $x\in(0,\infty)$. Then $|f'(x)|^{\frac{q}{r}}=x^m$ is r-concave on $(0,\infty)$ and the result follows directly from Corollary 13. □

Corollary 14. *For $m=1$ from Proposition 6, we obtain the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b)A^{\frac{r}{q}}(a,\ b).$$

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