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A SMALLER COVER FOR CLOSED UNIT CURVES

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Abstract. Forty years ago Schaer and Wetzel showed that a $\frac{1}{\pi} \times \frac{1}{2\pi}\sqrt{\pi^2 - 4}$ rectangle, whose area is about 0.12274, is the smallest rectangle that is a cover for the family of all closed unit arcs. More recently Füredi and Wetzel showed that one corner of this rectangle can be clipped to form a pentagonal cover having area 0.11224 for this family of curves. Here we show that then the opposite corner can be clipped to form a hexagonal cover of area less than 0.11023 for this same family. This irregular hexagon is the smallest cover currently known for this family of arcs.

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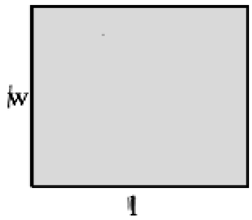
Keywords: covering of unit arcs, covering by convex sets, worm problem

1. Introduction

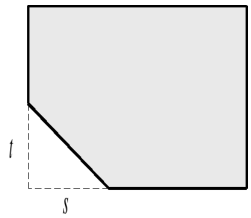
Forty years ago Schaer and Wetzel [7] (and independently Chakarian and Klamkin [2]) showed that the smallest rectangular region $\mathcal{R}$ that is a cover for the family $\mathcal{F}_0$ of all closed unit arcs in the Euclidean plane, that is to say, contains a congruent (i.e., an isometric) copy of each closed unit arc, is $l = \frac{1}{\pi}$ by $w = \sqrt{\frac{1}{4} - \frac{1}{\pi^2}}$ (Figure 1a.) This rectangle has area about 0.12274.

The underlying problem here is a variant for closed unit arcs of a well-known unsolved problem posed in 1966 by Leo Moser (see [5, pp. 211, 218-219]): find the area of the smallest (convex) region in the plane that contains a congruent (i.e., isometric) copy of every arc of unit length. The existence of a convex cover of least area α_2 for the family $\mathcal{F}_0$ of all closed unit arcs follows from standard compactness arguments, but its uniqueness is not known. Problems of these kinds are closely related to the well-known Lebesgue Universal Cover problem (see [3, §D15]), and they seem equally intractable.

Both [7] and [2] made the elementary observation that the least area α_2 must be greater than the smallest convex hull of a circle of unit circumference and a unit line segment, a set having area 0.09632. Using geometric methods Füredi and Wetzel [4] recently raised the lower bound to 0.09670, and they showed that the pentagonal region formed by suitably clipping one corner of $\mathcal{R}$ (Figure 1b) is a cover for $\mathcal{F}_0$ with area less than 0.11222. They achieved a further slight reduction in the upper bound



a. Cover $\mathcal{R}$.
Area < 0.12274.



b. Clipped $\mathcal{R}$.
Area < 0.11222.

FIGURE 1. Two covers for $\mathcal{F}_0$.

by replacing a portion of the clipping line segment by a short elliptic arc, forming a curvilinear convex hexagonal cover having area less than 0.11213. These are the best published bounds for α_2 at present:

$$0.09670 < \alpha_2 < 0.11213.$$

2. A SMALLER COVER

In recent years computational methods have increasingly been employed to attack geometric questions. For example, Brass and his student Sharifi [1] used a grid search numerical method to improve the known lower bound for the Lebesgue Universal Cover problem. In this article we employ numerical convex optimization to reduce the known upper bound for α_2 by nearly 1.7 per cent. More precisely, we establish the following theorem.

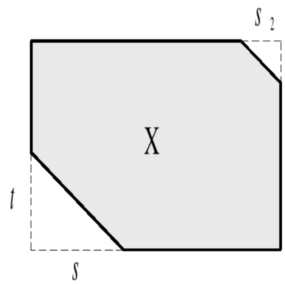
Theorem 1. *Let* $s = 0.142\,0171$, $t = 0.148\,1552$, *and* $s_2 = 0.061\,7141$, *and using these data, let* $\mathfrak{X}$ *be the rectangular region* $\mathcal{R}$ *clipped by two parallel line segments as pictured in Figure 2. Then* $\mathfrak{X}$ *is a cover for the family* $\mathcal{F}_0$ *of all closed curves of unit length, and its area is about* $0.110\,2299$.

The area of $\mathfrak{X}$ is $lw - \frac{1}{2}st\left(1 + (s_2/s)^2\right) \approx 0.110\,2299$. Showing that $\mathfrak{X}$ is a cover for $\mathcal{F}_0$ is the objective of this article. The value of s_2 is chosen so that every closed arc that cannot be covered by $\mathfrak{X}$ must be longer than 1.00001 numerically.

3. PLACING A CLOSED ARC IN THE RECTANGLE

From [8], it suffices for $\mathfrak{X}$ to cover all simple closed unit arcs. From [6], every closed unit arc is contained in the convex hull of another convex closed unit arc. Since $\mathfrak{X}$ is convex, we may assume that we are dealing with convex closed unit arcs. Since every convex closed unit arc has diameter at least $1/\pi$, we may place such arc in the rectangle $\mathcal{R}$ so that it touches the upper and lower boundaries [7]

For convenient, let $\theta = \arctan m$ where $m = t/s$. Let $p_1, p_2, \ldots, p_8$ be the (chosen) lower left, bottom, lower right, right, upper right, top, upper left, left points touched



The cover $\mathfrak{X}$.
Area < 0.11023.

FIGURE 2. A smaller cover for $\mathscr{F}_0$.

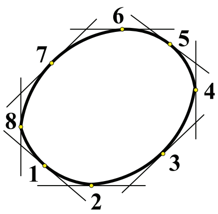


FIGURE 3. All touched points $p_1, p_2, ..., p_8$.

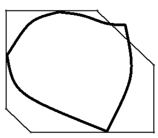


FIGURE 4. The situation OUTL5.

by support lines (coming) from angles $-90^o - \theta, -90^o, ..., 180^o$ (with slopes $-m, 0, m, \infty, -m, 0, m, \infty$). The points p_i appear in counter clockwise order since the closed arc is simple. Note that 2 consecutive points, including p_1 and p_8, may coincide (see Figure 3). For $i = 1, ..., 8$, let (x_i, y_i) be the coordinate of p_i. Let x_L and x_R be the left and right x-coordinate of a concerned rectangle, called ***box***. As we move the box to the right, we end up with $x_L = x_8$. For the opposite direction, we have $x_R = x_4$.

When the box is moving to the right (the arc is relatively moving to the left), the situation that p_5 is in the big corner is called OUTL5 (see Figure 4), and the situation that p_7 is in the small corner is called outL7 (see Figure 5). When the box is moving to the left (the arc is relatively moving to the right), the situation that p_7 is in the big corner is called OUTR7 (see Figure 7), and the situation that p_5 is in the small corner

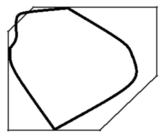


FIGURE 5. The situation outL7.

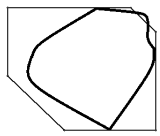


FIGURE 6. The situation outR5.

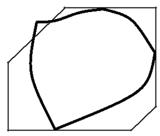


FIGURE 7. The situation OUTR7.

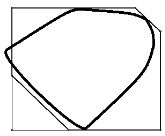


FIGURE 8. The situation that p_5 is on the border of the small corner and the arc leaves the box.

is called outR5 (see Figure 6). When the box is moving to the situation that p_5 is on the border of the small corner, the arc may leave the box (see Figure 8).

4. THE MAIN THEOREM

To prove Theorem 1, we start by suppose that a convex closed unit arc γ cannot be covered. We will finally find a contradiction, mostly by that γ is longer than 1.00001, using numerical optimization.

When we translate the box horizontally, consider a big corner and the opposite small corner, the arc is either passing through the big one or the small one. When the arc is placed to the left, the points p_3 and p_5 may visit the big corners. Similarly for the right justification, the point p_1 and p_7 may visit the big corners. We then divide into cases as follows (see Figure 9).

Case 1: OUTL3 and OUTL5.

Case 2: OUTL5 and (not OUTL3, then) outL7.

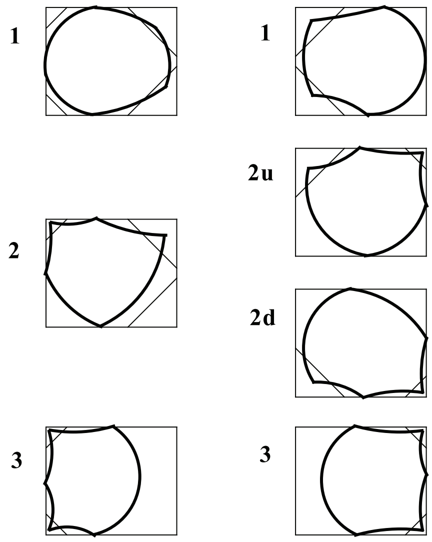


FIGURE 9. All cases.

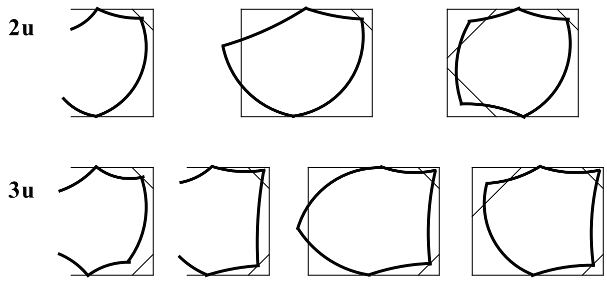


FIGURE 10. Translations in subcases ending with .2u and .3u.

Case 3: (not OUTL3 nor OUTL5, then) outL1 and outL7.

Now we are going into each big case where we will encounter many subcases.

Case 1: OUTL3 and OUTL5. Now consider the right justification. Due to the symmetry of case 1, we have the following subcases.

Subcase 1.1: OUTR7 and OUTR1. From [4], $l(\gamma) \geq l(p_1 p_2 ... p_8) > 1$.

Subcase 1.2u: (2up) OUTR7 and (not OUTR1, then) outR5. Now we translate the box so that p_5 is on the border of the small corner (see Figure 10). Consequently $x_R = \frac{w-t_2-y_5}{-m} + x_5$, called $\mathbf{x_{R5}}$. Then the arc leaves these covers on the left side. Suppose the arc leave the box on the left side, i.e. $x_8 < x_L$, from numerical work 1.2uL, its length is at least 1.001. Hence the arc does not leave the box and then it leaves the cover from corners. Thus it is OUT7 and OUT1. Note that we may regard OUTL5 as OUT5 with $x_L = x_8$ and regard OUTR7 as OUT7 with $x_R = x_4$. Here

OUT7 and OUT1 are with specific x_R above. From numerical work 1.2uC, the length is at least 1.02231.

Subcase 1.3: (not OUTR7 nor OUTR1, then) outR3 and outR5. Now we translate the box so that p_5 is on the border of the small corner and (without of loss of generality) p_3 does not visit the small corner. Suppose the arc does not leave the box, it leaves the cover through corners. Now it is OUTL7 and OUTL1 with this x_R. Now translate the box so that p_3 is on the border of the small corner where $x_R = \frac{t_2 - y_3}{m} + x_3$, called $\mathbf{x_{R3}}$. Here p_5 visits the small corner and the arc does not leave the left of the box. Hence it is OUT7 with this x_R. From numerical work 1.3C, the length is at least 1.00852. Now suppose the arc leaves the box when $x_R = x_{R5}$. We translate the box so that p_3 is on the border of the small corner. If the arc does not leave the box, from numerical work 1.3LC, the length is at least 1.00994. If the arc leaves the box, from numerical work 1.3LL, the length is at least 1.04008.

Case 2: OUTL5 and (not OUTL3, then) outL7. We start by translating so that p_7 is on the border of the left corner. Here $x_L = \frac{w - t_2 - y_7}{m} + x_7$, called $\mathbf{x_{L7}}$. Subcases are according to right justification.

Subcase 2.2u: (2up) OUTR7 and (not OUTR1, then) outR5. Now we translate so that $x_R = x_{R5}$. If both translations keep the arc in side the boxes, we have OUT3 and OUT5 when $x_L = x_{L7}$ and have OUT1 and OUT7 when $x_R = x_{R5}$. From numerical work 2C.2uC, the length is at least 1.0093. If the arc leaves the box when $x_R = x_{R5}$, we have OUT1 and OUT7. (2C.2uL) The length is at least 1.01069. If the arc leaves the box when $x_L = x_{L7}$ and when $x_R = x_{R5}$, (2R.2uL) the length is at least 1.03344.

Subcase 2.2d: (2down) OUTR1 and (not OUTR7, then) outR3. From 2.2d, the length is at least 1.00584.

Subcase 2.3: (not OUTR7 nor OUTR1, then) outR3 and outR5. When we translate so that $x_R = x_{R5}$. Now we consider whether p_3 is in the small corner.

Subsubcase 2.3u: p_3 is not in the small corner (see Figure 10). We will translate so that $x_L = x_{L7}$ and so that $x_R = x_{R5}$. If the arc are in both boxes, we have OUT3 and OUT5 when $x_L = x_{L7}$. (2C.3uC) The length is at least 1.00596. If the arc leaves the box when $x_R = x_{R5}$, (2.3uL) the length is at least 1.00318. If the arc leaves the box when $x_L = x_{L7}$, (2R.3u) the length is at least 1.00392.

Subsubcase 2.3d: p_3 is in the small corner. So we may think that when we translate so that $x_R = x_{R3}$, we have p_5 not in the small corner. Similarly to the previous subsubcase, from 2C.3dC, 2.3dL and 2R.3d, we have the length is at least 1.00854, 1.00504 and 1.00392 respectively.

Case 3: (not OUTL3 nor OUTL5, then) outL1 and outL7. We now have to consider only when the right justification causes (not OUTR7 nor OUTR1, then) outR3 and outR5. We will divide into subcase similar to the subcase 2.3 but with fewer subcases due to symmetry.

Subcase 3.3u: From 3C.3uC (OUT3 and OUT5 when x_L=x_{L7}, OUT1 and OUT7 when $x_R = x_{R5}$) and 3.3uL ($x_8 < x_4 - l$ when $x_R = x_{R5}$), the length is at least 1.00001 and 1.05382 respectively.

Subcase 3.3d: From 3C.3dC and 3.3dL, the length is at least 1.00001 and 1.05367 respectively.

In every case, the length is greater than 1 (or numerically greater than 1.00001). This is a contradiction. Therefore $\mathfrak{X}$ is a cover for closed unit arcs. The theorem is proved completely.

5. Further improvements

Repeating all the numerical process with threshold smaller than 1.00001 would yield a smaller cover.

We tried to make a significant improvement by subdividing the subsubcases according to numerical works 3C.3uC and 3C.3dC using the same method in subcase 1.3. The result is negative.

As the cover has to accommodate the straight back-and-forth segment of length $\frac{1}{2}$, it is clearly that there is no way to improve to another smaller cover by cutting the third corner (clipping 3 corners).

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Appendix: Numerical work

Each numerical minimization used here is called a ***convex programming*** where we minimize a convex function over a convex domain. It is theoritically confirmed that the minimum may be obtained numerically with high accuracy.

We numerically compute those minimums using Wolfram Mathematica. The Mathematica notebook file can be found at

www.math.sc.chula.ac.th/ wacharin/optimization/closed_arcs and at **https://www.researchgate.net/publication/322266021_A_smaller_cover_for_closed_unit_curves**.

The explation can be found at **http://arxiv.org/abs/1801.08405** [9] and at **https://www.researchgate.net/publication/322266021_A_smaller_cover_for_closed_unit_curves** with DOI 10.13140/RG.2.2.24191.51364.

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