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ON MULTIPLE MATHIEU $(\boldsymbol{a},\boldsymbol{\lambda})$-SERIES

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Abstract. In this paper we introduce the multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series. We obtain two integral representations for multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series applying Ivanov's and then Pogány's variant of multiple Euler-Maclaurin summation formula. Then, a bilateral bounding inequality is derived by virtue of the achieved integral expressions. Finally, the special case of multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series, the multiple Mathieu $\boldsymbol{a}$-series has been investigated.

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1. Introduction

The generalization of the classical Mathieu series [3]

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}\qquad (r>0) \tag{1.1}$$

has been introduced by Cerone and Lenard [1]:

$$\mathcal{S}(r,\mu,\alpha,\beta,\boldsymbol{a})=\sum_{n=1}^{\infty}\frac{a_n^{\alpha}}{(a_n^{\beta}+r^2)^{\mu}}\qquad (r,\mu,\alpha,\beta,\boldsymbol{a}=(a_n)>0). \tag{1.2}$$

The series (1.2) is in the focus of interest by numerous authors, such as Pogány [5, 6], Qi [11, 12], Srivastava and Tomovski [13, 14]. However, according to our best knowledge, the only work on the multidimensional generalization of the series (1.2) is the paper [10] by Pogány and Tomovski, where they introduce a generalized multiple Mathieu series of the form

$$S_p^r(\boldsymbol{s},\boldsymbol{q},\rho)=\sum_{\boldsymbol{n}\in\mathbb{N}^r}\frac{2\boldsymbol{n}^{|\boldsymbol{s}|}}{(\langle\boldsymbol{n}^{\boldsymbol{q}},\boldsymbol{n}^{\boldsymbol{q}}\rangle+\rho)^{p+1}},$$

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where $\boldsymbol{n}^{\boldsymbol{q}} := (n_1^{q_1}, \dots, n_r^{q_r})$, $\boldsymbol{n}^{\boldsymbol{\alpha}|\boldsymbol{s}|} := n_1^{\alpha_1 s_1} \cdots n_r^{\alpha_r s_r}$, $\boldsymbol{s}, \boldsymbol{q}$ have positive coordinates, i. e., $s_l, q_l > 0$, $l = 1, \dots, r$, while $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ stands for inner product in $\mathbb{R}^r$. For $r = 1$, the above series, obviously, reduces to the classical Mathieu series (1.1).

Pogány and Tomovski found two integral representations of the multiple series $\mathcal{S}_p^r(\boldsymbol{s}, \boldsymbol{q}, \rho)$ (see Theorem 1, Eq. (9) and Theorem 3, Eq. (19) in [10]). They also derived a bilateral bounding inequality [10, Theorem 2] and established two other bounds [10, Theorems 4 and 5].

2. Summation formula for finite multiple sums

The well-known Euler–Maclaurin summation formula has the form

$$\begin{aligned} \sum_{n=k}^{l} f(n) = \int_k^l f(x)\,\mathrm{d}x + \frac{1}{2}\big(f(k) + f(l)\big) \\ + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}\Big(f^{(2j-1)}(l) - f^{(2j-1)}(k)\Big) \\ - \int_k^l \frac{B_{2m}(x)}{(2m)!} f^{(2m)}(x)\,\mathrm{d}x \qquad (m \in \mathbb{N}), \end{aligned} \tag{2.1}$$

where $B_p(x) = (x + B)^p$, $0 \le x < 1$, stands for the Bernoulli polynomial of order $p \in \mathbb{N}$ and $B^k = B_k$ are the Bernoulli numbers. One can rewrite it in a condensed form

$$\sum_{n=k+1}^{l} a_n = \int_k^l \big(a(x) + \{x\}a'(x)\big)\,\mathrm{d}x \equiv \int_k^l \mathfrak{d} a(x)\,\mathrm{d}x, \tag{2.2}$$

for $a \in C^1[k,l]$, $a_n = a(n)$, $k, l \in \mathbb{Z}$, $k < l$, where

$$\mathfrak{d} := 1 + \{x\}\frac{\partial}{\partial x}$$

and $\{x\}$ stands for the fractional part of a real number x (see (3) in [9] and (6.5) in [8]).

For the multidimensional bounded summation domain D, we use the summation formulas derived by Müller [4], Ivanov [2], and another type of formula due to Pogány [7].

Now the role of the Bernoulli polynomials $B_p(x)$ in (2.1) is played by the so-called *basic functions* ("Grundfunktion," see [4]) $G(x_1, \dots, x_r)$, which satisfy the following conditions:

(1) G is a 1-periodic function in all variables;
(2) On the lattice $\mathbb{Z}^r$, the function G satisfies the equation

$$\Delta G + \lambda G = \sum_{\boldsymbol{k}:\, 4\pi^2 \boldsymbol{k}^2 = \lambda} \mathrm{e}^{2\pi i \langle \boldsymbol{k}, x \rangle}, \tag{2.3}$$

where Δ is the Laplace operator and the summation in (2.3) is carried out over all $\boldsymbol{k} = (k_1,\dots,k_r) \in \mathbb{Z}^r$ such that $4\pi^2\boldsymbol{k}^2 = 4\pi^2\langle \boldsymbol{k},\boldsymbol{k}\rangle = \lambda$.

Let us introduce the following notation: $\boldsymbol{n} = (n_1,\dots,n_r) \in \mathbb{N}_0^r$, where $\mathbb{N}_0 = \{0,1,2,\dots\}$; $\boldsymbol{x} = (x_1,\dots,x_r) \in \mathbb{R}^r$, $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\cdots\mathrm{d}x_r$, $L = L\big(\frac{\partial}{\partial x}\big)$ is a linear differential operator with real constant coefficients, which is a polynomial in $\frac{\partial}{\partial x_j}$.*

Let us put

$$H(\boldsymbol{x}) = \sum_{\boldsymbol{n}\colon L(2\pi i\boldsymbol{n})=0} \mathrm{e}^{2\pi i\langle \boldsymbol{n},\boldsymbol{x}\rangle}. \tag{2.4}$$

Let G be a basic function of the operator L if it has period 1 with respect to each variable and satisfies the equality

$$LG = \sum_{\boldsymbol{n}\in\mathbb{Z}^r} \mathrm{e}^{2\pi i\langle \boldsymbol{n},\boldsymbol{x}\rangle}.$$

It follows that

$$G(\boldsymbol{x}) = {\sum_{\boldsymbol{n}\in\mathbb{Z}^r}}' \frac{\mathrm{e}^{2\pi i\langle \boldsymbol{n},\boldsymbol{x}\rangle}}{L(2\pi i\boldsymbol{n})}, \tag{2.5}$$

where $'$ denotes the absence of the term with zero denominator in the sum.

Let the boundary ∂D of D be smooth. Then, by Green's formula, we have

$$\int_{\bar{D}} (uLv - vMu)\,\mathrm{d}\boldsymbol{x} = \int_{\partial D} P(u,v)\,\mathrm{d}\boldsymbol{s}, \tag{2.6}$$

where $\bar{D} = D + \partial D$, M is a conjugate of L, and $P(u,v)$ is a polynomial with respect to u and v and their partial derivatives.

Theorem 1 ([2, Theorem 1])**.** *Let us assume that the boundary ∂D of D is smooth and does not contain integer points. Let L be a linear differential operator of order p with constant coefficients and $f \in C^p(\bar{D})$, where $\bar{D} = D + \partial D$. Then*

$$\sum_{\boldsymbol{n}\in D} f(\boldsymbol{n}) = \int_D f(\boldsymbol{x})H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial D} P(f,G)\,\mathrm{d}\boldsymbol{s} + \int_D GM(f)\,\mathrm{d}\boldsymbol{x}, \tag{2.7}$$

where M is a conjugate of L, whereas H and G are defined by (2.4) *and* (2.5).

The following result of [7] is a multidimensional generalization of (2.2).

Theorem 2 ([7])**.** *Assume that $a : \mathbb{R}_+^r \to \mathbb{R}_+$ is a function satisfying the condition of differentiability*

$$\frac{\partial^r a}{\partial x_1\cdots\partial x_r} \in C\bigg(\prod_{j=1}^{r}[0,n_j]\bigg),$$

and, for any $\boldsymbol{j} = (j_1,\dots,j_r)$ with $0 \le j_l \le n_l$, $l = 1,2,\dots,r$, put $a_{\boldsymbol{j}} := a(j_1,\dots,j_r)$.

---

*In what follows, the expression $L(2\pi i\boldsymbol{n})$ stands for the value of that polynomial where, instead of $\frac{\partial}{\partial x_j}$, one puts $2\pi i n_j$.

Then

$$\begin{aligned}\sum_{l=1}^{r}\sum_{j_l=0}^{n_l} a_{\boldsymbol{j}} &= a_{\boldsymbol{0}} + \sum_{l=1}^{r}\int_0^{n_l} \eth_l\, a(x_l)\,\mathrm{d}x_l \\ &\quad + \sum_{1\le j<k\le r}\int_0^{n_j}\int_0^{n_k} \eth_j \eth_k\, a(x_j,x_k)\,\mathrm{d}x_j\,\mathrm{d}x_k + \cdots \\ &\quad + \int_0^{n_1}\int_0^{n_2}\cdots\int_0^{n_r} \eth_1\eth_2\cdots\eth_r\, a(x_1,x_2,\ldots,x_r)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_r,\end{aligned} \tag{2.8}$$

where $a_{\boldsymbol{0}} \equiv a(0,\ldots,0)$, $a(x_{j_1},\ldots,x_{j_k}) = a(\boldsymbol{x})|_{x_m=0,\, m\in\{1,\ldots,r\}\setminus\{j_1,\ldots,j_k\}}$, *and*

$$\eth_j := 1 + \{x_j\}\frac{\partial}{\partial x_j}.$$

3. Multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series

Let $a,\lambda\colon \mathbb{R}_+^r \to \mathbb{R}_+$, where $r > 1$, be certain functions, $\boldsymbol{a}$ and $\boldsymbol{\lambda}$ be their restrictions to $\mathbb{N}_0^r$, i.e., $a|_{\mathbb{N}_0^r} = (a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}_0^r} = (a(n_1,\ldots,n_r))_{\boldsymbol{n}\in\mathbb{N}_0^r}$, $\lambda|_{\mathbb{N}_0^r} = (\lambda_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}_0^r} = (\lambda(n_1,\ldots,n_r))_{\boldsymbol{n}\in\mathbb{N}_0^r}$, and $\mu > 0$ and $\rho > 0$ be parameters. Then the series

$$\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^r} \frac{a_{\boldsymbol{n}}}{(\lambda_{\boldsymbol{n}}+\rho)^\mu} \tag{3.1}$$

is called a *multiple Mathieu* $(\boldsymbol{a},\boldsymbol{\lambda})$*-series*.

We are interested in deriving an integral representation of the series (3.1). Let us begin with transforming $\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)$ via the Gamma function. We have

$$\begin{aligned}\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho) &= \frac{1}{\Gamma(\mu)}\sum_{\boldsymbol{n}\in\mathbb{N}_0^r} a_{\boldsymbol{n}}\int_0^\infty \mathrm{e}^{-(\lambda_{\boldsymbol{n}}+\rho)x} x^{\mu-1}\,\mathrm{d}x \\ &= \frac{1}{\Gamma(\mu)}\int_0^\infty \mathrm{e}^{-\rho x} x^{\mu-1}\Bigg(\sum_{\boldsymbol{n}\in\mathbb{N}_0^r} a_{\boldsymbol{n}}\mathrm{e}^{-\lambda_{\boldsymbol{n}} x}\Bigg)\mathrm{d}x.\end{aligned}$$

The inner r-fold Dirichlet series

$$\mathcal{D}_{\boldsymbol{a},\boldsymbol{\lambda}}(x) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^r} a_{\boldsymbol{n}}\mathrm{e}^{-\lambda_{\boldsymbol{n}} x},$$

by virtue of the Laplace integral formula, can be rewritten in the form

$$\mathcal{D}_{\boldsymbol{a},\boldsymbol{\lambda}}(x) = x\int_0^\infty \mathrm{e}^{-xt}\Bigg(\sum_{\boldsymbol{n}:\lambda_{\boldsymbol{n}}\le t} a_{\boldsymbol{n}}(t)\Bigg)\mathrm{d}t = x\int_0^\infty \mathrm{e}^{-xt} A_{\boldsymbol{n}}(t)\,\mathrm{d}t.$$

Putting all back into the expression for $\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)$, we obtain

$$\begin{aligned}\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho) &= \frac{1}{\Gamma(\mu)}\int_0^\infty \mathrm{e}^{-\rho x}x^\mu \int_0^\infty \mathrm{e}^{-xt}A_{\boldsymbol{n}}(t)\,\mathrm{d}t \\ &= \frac{1}{\Gamma(\mu)}\int_0^\infty \left(\int_0^\infty \mathrm{e}^{-(\rho+t)x}x^\mu\,\mathrm{d}x\right)A_{\boldsymbol{n}}(t)\,\mathrm{d}t \\ &= \mu\int_0^\infty \frac{A_{\boldsymbol{n}}(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t.\end{aligned}$$

Now we have to calculate the sum $A_{\boldsymbol{n}}(t)$. Let $\mathfrak{D}=\{\boldsymbol{n}:\lambda_{\boldsymbol{n}}\le t,\ t>0\}$ and let $\widehat{\lambda}_1=\widehat{\lambda}_1(t),\dots,\widehat{\lambda}_r=\widehat{\lambda}_r(t)\in\mathbb{N}_0$ be such that

$$\lambda_{\boldsymbol{n}}=\lambda(\widehat{\lambda}_1,\dots,\widehat{\lambda}_r)\le t. \tag{3.2}$$

The summation domain is the r-dimensional parallelepiped $\mathfrak{D}=\prod_{j=1}^r[0,\widehat{\lambda}_j]$ and, obviously, $\dim\partial\mathfrak{D}=r-1$. Let us set $\mathrm{d}\boldsymbol{s}_{r-1}=\mathrm{d}s_1\cdots\mathrm{d}s_{r-1}$. Let L be an elliptic differential operator, M be its conjugate, P be the polynomial defined by (2.6), and H and G be given by (2.4) and (2.5), respectively. By Ivanov's formula (2.7), we obtain

$$A_{\boldsymbol{n}}(t)=\int_{\mathfrak{D}}a(\boldsymbol{x})H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{\partial\mathfrak{D}}P(a,G)\,\mathrm{d}\boldsymbol{s}_{r-1}+\int_{\mathfrak{D}}GM(a)\,\mathrm{d}\boldsymbol{x}.$$

Putting this expression for $A_{\boldsymbol{n}}(t)$ back to $\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)$, we immediately arrive at the following statement.

Theorem 3. *Let $a,\lambda\colon\mathbb{R}_+^r\to\mathbb{R}_+$, $a_{\boldsymbol{n}}=a(n_1,\dots,n_r)$, $\lambda_{\boldsymbol{n}}=\lambda(n_1,\dots,n_r)$, $\boldsymbol{n}\in\mathbb{N}_0^r$, $\mu,\rho>0$, and $r\in\mathbb{N}_2$. Let $t>0$ and let $\widehat{\lambda}_1=\widehat{\lambda}_1(t),\dots,\widehat{\lambda}_r=\widehat{\lambda}_r(t)\in\mathbb{N}_0$ be such that (3.2) holds. Then the multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series admits the integral representation*

$$\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)=\mu\int_0^\infty\frac{\int_{\mathfrak{D}}a(\boldsymbol{x})H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{\partial\mathfrak{D}}P(a,G)\,\mathrm{d}\boldsymbol{s}_{r-1}+\int_{\mathfrak{D}}GM(a)\,\mathrm{d}\boldsymbol{x}}{(\rho+t)^{\mu+1}}\,\mathrm{d}t,$$

where $\mathfrak{D}=\prod_{j=1}^r[0,\widehat{\lambda}_j]$.

If we use formula (2.8) to calculate $A_{\boldsymbol{n}}(t)$ in $\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)$, we deduce

Theorem 4. *Let $a,\lambda\colon\mathbb{R}_+^r\to\mathbb{R}_+$, $a_{\boldsymbol{n}}=a(n_1,\dots,n_r)$, $\lambda_{\boldsymbol{n}}=\lambda(n_1,\dots,n_r)$, $\boldsymbol{n}\in\mathbb{N}_0^r$, $\mu>0,\rho>0$, $r>0$, and $\widehat{\lambda}_1=\widehat{\lambda}_1(t),\dots,\widehat{\lambda}_r=\widehat{\lambda}_r(t)\in\mathbb{N}_0$ such that*

$$\lambda_{\boldsymbol{n}}=\lambda(\widehat{\lambda}_1,\dots,\widehat{\lambda}_r)\le t.$$

Then multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series has the following integral representation

$$\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)=\rho^{-\mu}a_{\boldsymbol{0}}+\mu\int_0^\infty\left(\sum_{l=1}^r\int_0^{\widehat{\lambda}_l}\mathfrak{d}_l\,a(x_l)\,\mathrm{d}x_l\right.$$

$$+ \sum_{1\le j<k\le r} \int_0^{\widehat{\lambda}_j} \int_0^{\widehat{\lambda}_k} \eth_j \eth_k\, a(x_j, x_k)\, \mathrm{d}x_j\, \mathrm{d}x_k + \cdots$$
$$+ \int_0^{\widehat{\lambda}_1} \int_0^{\widehat{\lambda}_2} \cdots \int_0^{\widehat{\lambda}_r} \eth_1 \cdots \eth_r\, a(x_1, \ldots, x_r)\, \mathrm{d}x_1 \cdots \mathrm{d}x_r \Big) \frac{\mathrm{d}t}{(\rho+t)^{\mu+1}} .$$

4. Bilateral bounding inequality

We estimate $A_{\boldsymbol{n}}(t)$ given by Pogány's formula (2.8). We will focus here on the case where $r = 2$, since the higher order terms are far more complicated and the procedure is identical to that applicable for $r = 2$.

Let $a\colon \mathbb{R}_+^2 \to \mathbb{R}$ be monotone in both variables. Since, for $j = 1, 2$,

$$\begin{aligned} \eth_j a &\in \Big[a, a + \frac{\partial a}{\partial x_j}\Big) && (\text{if } a \uparrow \text{ in } x_j); \\ \eth_j a &\in \Big(a + \frac{\partial a}{\partial x_j}, a\Big] && (\text{if } a \downarrow \text{ in } x_j), \end{aligned} \tag{4.1}$$

successively applying the operators $\eth_1$ and $\eth_2$ in (2.8), we estimate

$$\begin{aligned} \boldsymbol{A}_2 &= \sum_{j=0}^{n_1} \sum_{k=0}^{n_2} a_{jk} \\ &= a_{\boldsymbol{0}} + \sum_{j=1}^{2} \int_0^{n_j} \eth_j\, a(x_j)\, \mathrm{d}x_j + \int_0^{n_1} \int_0^{n_2} \eth_j \eth_k\, a(x_j, x_k)\, \mathrm{d}x_j\, \mathrm{d}x_k . \end{aligned}$$

Assume that $a \uparrow$. Then

$$\begin{aligned} A_{\boldsymbol{2}}(t) \le a_{\boldsymbol{0}} &+ \sum_{j=1}^{2} \int_0^{n_j} \big(a(x_j) + a'(x_j)\big)\, \mathrm{d}x_j \\ &+ \int_0^{n_1} \int_0^{n_2} \Big(a(x_1, x_2) + \frac{\partial a(x_1, x_2)}{\partial x_1} \\ &\qquad + \frac{\partial a(x_1, x_2)}{\partial x_2} + \frac{\partial 2 a(x_1, x_2)}{\partial x_1 \partial x_2}\Big)\, \mathrm{d}x_1\, \mathrm{d}x_2 =: \boldsymbol{R}_2 . \end{aligned}$$

Now we have

$$\begin{aligned}
\boldsymbol{R}_2 &= a_{\mathbf{0}} + \sum_{j=1}^{2}\left(\int_0^{n_j} a(x_j)\,\mathrm{d}x_j + a(n_j) - a_{\mathbf{0}}\right) + \int_0^{n_1}\int_0^{n_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad + \int_0^{n_1}\big(a(x_1,n_2)-a(x_1,0)\big)\,\mathrm{d}x_1 + \int_0^{n_2}\big(a(n_1,x_2)-a(0,x_2)\big)\,\mathrm{d}x_2 \\
&\quad + a(n_1,n_2) - a(n_1,0) - a(0,n_2) + a_{\mathbf{0}} \\
&= a_{\mathbf{0}} + \sum_{j=1}^{2}\left(\int_0^{n_j} a(x_j)\,\mathrm{d}x_j + a(n_j) - a_{\mathbf{0}}\right) + \int_0^{n_1}\int_0^{n_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad + \int_0^{n_1}\big(a(x_1,n_2)-a(x_1)\big)\,\mathrm{d}x_1 + \int_0^{n_2}\big(a(n_1,x_2)-a(x_2)\big)\,\mathrm{d}x_2 \\
&\quad + a(n_1,n_2) - a(n_1) - a(n_2) + a_{\mathbf{0}} \\
&= a(n_1,n_2) + \int_0^{n_1} a(x_1,n_2)\,\mathrm{d}x_1 + \int_0^{n_2} a(n_1,x_2)\,\mathrm{d}x_2 \\
&\quad + \int_0^{n_1}\int_0^{n_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,.
\end{aligned} \tag{4.2}$$

Similarly, by virtue of (4.1), we have

$$\begin{aligned}
\boldsymbol{A}_2 \geq a_{\mathbf{0}} &+ \int_0^{n_1} a(x_1)\,\mathrm{d}x_1 + \int_0^{n_2} a(x_2)\,\mathrm{d}x_2 + \\
&+ \int_0^{n_1}\int_0^{n_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 =: \boldsymbol{L}_2\,.
\end{aligned} \tag{4.3}$$

This proves the following result.

Theorem 5. *Let $a\colon \mathbb{R}_+^2 \to \mathbb{R}$ and $a \in C^2\big([0,n_1]\times[0,n_2]\big)$, monotone in both variables. Then for the two-dimensional Euler–Maclaurin summation formula (2.8) the following bilateral bounds are valid*

$$\boldsymbol{L}_2 < \sum_{j=0}^{n_1}\sum_{k=0}^{n_2} a_{jk} \leq \boldsymbol{R}_2\,, \quad (a\uparrow),$$

$$\boldsymbol{R}_2 \leq \sum_{j=0}^{n_1}\sum_{k=0}^{n_2} a_{jk} < \boldsymbol{L}_2\,, \quad (a\downarrow),$$

where $a(\boldsymbol{x})|_{\mathbb{N}_0^2} = \boldsymbol{a} = (a_{jk})_{\mathbb{N}_0^2}$ and bounds $\boldsymbol{L}_2$, $\boldsymbol{R}_2$ are given in (4.3), (4.2).

If we apply the above theorem to integral representation of multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series given in Theorem 4, for $r=2$ with simple exchange $\widehat{\lambda}_1 \to n_1$, $\widehat{\lambda}_2 \to n_2$ we obtain

Theorem 6. *Let* $a, \lambda\colon \mathbb{R}_+^2 \to \mathbb{R}$, $a, \lambda \in C^2([0,\widehat{\lambda}_1]\times[0,\widehat{\lambda}_2])$, *where* $\widehat{\lambda}_j = \widehat{\lambda}_j(t) \in \mathbb{N}_0$, $j = 1,2$, *for some fixed* $t > 0$, *such that* $\lambda(\widehat{\lambda}_1,\widehat{\lambda}_2) \le t$ *and* a *be monotone in both variables. Then the following bilateral inequalities hold*

$$\mu\int_0^\infty \frac{\widehat{\boldsymbol{L}}_2(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t < \mathscr{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho) \le \mu\int_0^\infty \frac{\widehat{\boldsymbol{R}}_2(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t, \quad (a\uparrow),$$

$$\mu\int_0^\infty \frac{\widehat{\boldsymbol{R}}_2(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t \le \mathscr{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho) < \mu\int_0^\infty \frac{\widehat{\boldsymbol{L}}_2(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t, \quad (a\downarrow),$$

where $\widehat{\boldsymbol{X}}_2(t) = \boldsymbol{X}_2\big(\widehat{\lambda}_1,\widehat{\lambda}_2\big)$, $\boldsymbol{X} \in \{\boldsymbol{R},\boldsymbol{L}\}$, *that is*

$$\begin{aligned}\widehat{\boldsymbol{R}}_2(t) &= a\big(\widehat{\lambda}_1(t),\widehat{\lambda}_2(t)\big) + \int_0^{\widehat{\lambda}_1(t)} a\big(x_1,\widehat{\lambda}_2(t)\big)\,\mathrm{d}x_1 \\ &\quad + \int_0^{\widehat{\lambda}_2(t)} a\big(\widehat{\lambda}_1(t),x_2\big)\,\mathrm{d}x_2 + \int_0^{\widehat{\lambda}_1(t)}\int_0^{\widehat{\lambda}_2(t)} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ \widehat{\boldsymbol{L}}_2(t) &= a_{\boldsymbol{0}} + \int_0^{\widehat{\lambda}_1(t)} a(x_1)\,\mathrm{d}x_1 + \\ &\quad + \int_0^{\widehat{\lambda}_2(t)} a(x_2)\,\mathrm{d}x_2 + \int_0^{\widehat{\lambda}_1(t)}\int_0^{\widehat{\lambda}_2(t)} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

Remark 1. Since $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ are solutions of the Diophantine equation $\lambda(\widehat{\lambda}_1,\widehat{\lambda}_2) \le t$, they are not uniquely determined, nor are the functions $\widehat{\boldsymbol{L}}_2$, $\widehat{\boldsymbol{R}}_2$.

5. MULTIPLE MATHIEU $\boldsymbol{a}$-SERIES

If we take $\lambda \equiv a$, it is straightforward to see that from multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series we easily obtain the series

$$\mathscr{S}_\mu(\boldsymbol{a};\rho) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^r} \frac{a_{\boldsymbol{n}}}{(a_{\boldsymbol{n}}+\rho)^\mu}$$

which we call *multiple Mathieu* $\boldsymbol{a}$*-series*. Now from integral representations for multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series we directly get the following results.

Corollary 1 (of Theorem 3)**.** *Let* $a\colon \mathbb{R}_+^r \to \mathbb{R}_+$, $a_{\boldsymbol{n}} = a(n_1,\dots,n_r)$, $\boldsymbol{n} \in \mathbb{N}_0^r$, $\mu,\rho > 0$, *and* $r \in \mathbb{N}_2$. *Let* $t > 0$ *and* $\widehat{a}_1 = \widehat{a}_1(t),\dots,\widehat{a}_r = \widehat{a}_r(t) \in \mathbb{N}_0$ *such that*

$$a_{\boldsymbol{n}} = a(\widehat{a}_1,\dots,\widehat{a}_r) \le t.$$

If $\mathfrak{D} = \prod_{j=1}^r [0,\widehat{a}_j]$ *then multiple Mathieu a-series have the following integral representation*

$$\mathscr{S}_\mu(a;\rho) = \mu\int_0^\infty \frac{\int_{\mathfrak{D}} a(\boldsymbol{x})H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial\mathfrak{D}} P(a,G)\,\mathrm{d}\boldsymbol{s}_{r-1} + \int_{\mathfrak{D}} GM(a)\,\mathrm{d}\boldsymbol{x}}{(\rho+t)^{\mu+1}}\,\mathrm{d}t,$$

with L, H and G keeping the heretofore meaning.

Corollary 2 (of Theorem 4)**.** *Let $a\colon \mathbb{R}_+^r \to \mathbb{R}_+$, $a_{\boldsymbol{n}} = a(n_1,\dots,n_r)$, $\boldsymbol{n} \in \mathbb{N}_0^r$, $\mu > 0$, $\rho > 0$, $r > 0$, and $\widehat{a}_1 = \widehat{a}_1(t),\dots,\widehat{a}_r = \widehat{a}_r(t) \in \mathbb{N}_0$ such that*

$$a_{\boldsymbol{n}} = a(\widehat{a}_1,\dots,\widehat{a}_r) \le t.$$

Then multiple Mathieu $\boldsymbol{a}$-series possesses the integral representation

$$\begin{aligned}\mathcal{S}_\mu(\boldsymbol{a};\rho) = \rho^{-\mu} a_{\boldsymbol{0}} + \mu \int_0^\infty \Bigg(&\sum_{l=1}^r \int_0^{\hat{a}_l} \partial_l\, a(x_l)\,\mathrm{d}x_l \\ &+ \sum_{1\le j<k\le r} \int_0^{\hat{a}_j}\int_0^{\hat{a}_k} \partial_j \partial_k\, a(x_j,x_k)\,\mathrm{d}x_j\,\mathrm{d}x_k + \cdots \\ &+ \int_0^{\hat{a}_1}\cdots\int_0^{\hat{a}_r} \partial_1\cdots\partial_r\, a(x_1,\dots,x_r)\,\mathrm{d}x_1\cdots\mathrm{d}x_r \Bigg) \frac{1}{(\rho+t)^{\mu+1}}\,\mathrm{d}t.\end{aligned}$$

Corollary 3 (of Theorem 5)**.** *Let $a\colon \mathbb{R}_+^2 \to \mathbb{R}$, $a \in C^2\big([0,\widehat{a}_1]\times[0,\widehat{a}_2]\big)$, $\mu > 0$, $\rho > 0$, and $\widehat{a}_1,\widehat{a}_2 \in \mathbb{N}_0$ such that*

$$a_{\boldsymbol{n}} = a(\widehat{a}_1,\widehat{a}_2) \le t.$$

Then the following bilateral inequalities hold

$$\begin{aligned}\mu\int_0^\infty \frac{\boldsymbol{L}_2^a(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t < \mathcal{S}_\mu(\boldsymbol{a};\rho) \le \mu\int_0^\infty \frac{\boldsymbol{R}_2^a(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t, \quad (a\uparrow),\\ \mu\int_0^\infty \frac{\boldsymbol{R}_2^a(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t \le \mathcal{S}_\mu(\boldsymbol{a};\rho) < \mu\int_0^\infty \frac{\boldsymbol{L}_2^a(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t, \quad (a\downarrow),\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{L}_2^a(t) &= a_{\boldsymbol{0}} + \sum_{j=1}^2 \int_0^{\hat{a}_j} a(x_j)\,\mathrm{d}x_j + + \int_0^{\hat{a}_1}\int_0^{\hat{a}_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,,\\ \boldsymbol{R}_2^a(t) &= a(\hat{a}_1,\hat{a}_2) + \int_0^{\hat{a}_1} a(x_1,\hat{a}_2)\,\mathrm{d}x_1 + \int_0^{\hat{a}_2} a(\hat{a}_1,x_2)\,\mathrm{d}x_2\\ &\quad + \int_0^{\hat{a}_1}\int_0^{\hat{a}_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,.\end{aligned}$$

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