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CERTAIN COMBINATORIC CONVOLUTION SUMS ARISING FROM BERNOULLI AND EULER POLYNOMIALS

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Abstract. In this study, we introduce the absolute Möbius divisor function $U(n)$. According to some numerical computational evidence, we consider integer pairs $(n, n+1)$ satisfying; $\varphi(n) = \varphi(n+1) = U(n) = U(n+1)$. Furthermore, we give some examples and proofs for our results.

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1. INTRODUCTION

Let $\mathbb{N}$ denote the set of positive integers. A positive integer n has a unique prime factorization $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$, where $\omega(n)$ is the number of distinct prime factors of n and each prime factor being counted only once. The divisor function $\sigma(n) = \sum_{d|n} d$ and the Euler totient function $\varphi(n)$ are widely studied in the field of elementary number theory. It is well-known [12, p. 22–23] that

$$n = \sum_{d|n} d \text{ and } \varphi(n) = \sum_{d|n} \mu(d) \cdot \frac{n}{d}.$$

Here, $\mu(d)$ is the Möbius μ-function. We investigate the absolute Möbius divisor function U, given by $U(n) = |\sum_{d|n} \mu(d)d|$. If n is a square-free(resp., not square-free) integer then $U(n) = \varphi(n)$(resp., $U(n) \neq \varphi(n)$). For $n \in \mathbb{N}$, let us define the function $\omega_0(n)$ is the number of odd primes factors of n. That is,

$$\omega_0(n) = \begin{cases} \omega(n), & if\ n\ is\ odd, \\ \omega(n) - 1, & if\ n\ is\ even. \end{cases}$$

Ratat [11], who asked for which values of n the equation $\varphi(n) = \varphi(n+1)$ holds and gave $n = 1, 3, 15, 104$ for examples. In 1918, answering to Ratat's question, Goormaghtigh [6] gave $n = 164, 194, 255, 495$.

Klee [8], Moser [10], Lal and Gillard [9], Ballew, Case and Higgins [3], Baillie [1,2] and Erdös, Pomerance and Sarkozy [5] studied for solutions to $\varphi(n)=\varphi(n+k)$. On the other hand, there is an unsolved problem on the divisor function, which asks that if $\sigma(n)=\sigma(n+1)$ infinitely often (see [7, p. 103], [13, p. 166]).

It should be mentioned that the Möbius conjecture associated with the Möbius function have been studied by Bayad and Goubi [4].

Throughout the paper, $p, q_1, \ldots, q_6$ are distinct odd primes with $q_1 < \ldots < q_6$ unless otherwise specified hereafter.

The aim of this article is to study an equation

$$\varphi(n)=\varphi(n+1)=U(n)=U(n+1). \tag{1.1}$$

More precisely, we prove the following theorem.

Theorem 1. *($\omega_0(n), \omega_0(n+1)) = (1,t)$ or $(t,1)$ with $t \le 6$)*
Let

$$\begin{cases} Case\ 1) & n=p, & n+1=2\prod_{i=1}^{t} q_i & or \\ Case\ 2) & n=2\prod_{i=1}^{t} q_i, & n+1=p & or \\ Case\ 3) & n=2p, & n+1=\prod_{i=1}^{t} q_i & or \\ Case\ 4) & n=\prod_{i=1}^{t} q_i, & n+1=2p. & \end{cases} \tag{1.2}$$

Then there exist two pairs of $(n,n+1)=(194,195)$ and $(n,n+1)=(5186,5187)$ satisfying Eq. (1.1) and (1.2).

Remark 1. Let $p, q, p_1, \ldots, p_t$ be distinct odd primes. Euler's totient function $\varphi(n)$ plays a key role in the RSA encryption. Given $N = pq$ with p and q distinct odd primes. If $n = 2p$(resp., $p_1 \cdots p_t$) and $n+1 = p_1 \cdots p_t$(resp., $2p$) then the computation time of $2(p+1)q$ is more shorter than it of pq. Furthermore, we find $N = pq$ derived from $2(p+1)q$.

Remark 2. Table 18 and Table 19 in Appendix give us examples of values of $U(n)=U(n+1)$ and $\varphi(n)=\varphi(n+1)=U(n)=U(n+1)$. We conjectured that $12 \mid U(n)=U(n+1)$ except $n=1$. In fact, if $n \neq 1$, we note that $12 \mid \varphi(n) = \varphi(n+1)=U(n)=U(n+1)$ and $(\omega(n),\omega(n+1))=(1,t)$ *or* $(t,1)$ with $t \le 6$ by Theorem 1.

2. LEMMAS FOR THEOREM 1 ((1,t) OR (t,1) WITH $t \le 6$)

To prove Theorem 1, we need following lemmas.

Lemma 1. *($(\omega_0(n),\omega_0(n+1))=(1,1)$) Let p and q_1 be distinct odd primes and let*

$$\begin{cases} n=p, & n+1=2q_1 \quad or \\ n=2q_1, & n+1=p. \end{cases} \tag{2.1}$$

Then there does not exist a pair of positive integers $(n,n+1)$ satisfying Eq. (1.1) and (2.1).

Proof. Assume that $U(n) = U(n+1)$. It is easily check that $p = q_1$. This is a contradiction for $p \neq q_1$. This completes the proof of Lemma 1. □

Remark 3. If the both p and $2p+1$ are primes, then p is a Sophie Germain prime (or safe prime). Then there does not exist a Sophie Germain prime p satisfying $U(2p) = U(q_1) = \varphi(2p) = \varphi(q_1)$ by Lemma 1.

Lemma 2. *Let $q_1, q_2, \ldots, q_r$ be distinct odd primes with $q_1 < q_2 < \ldots < q_r$ and $r \geq 2$. If $S = \{1, 2, \ldots, r\}$ and $S'' \subset S' \subset S$ then*

$$\prod_{j \in S'} \left(1 - q_j^{-1}\right) < \prod_{j \in S''} \left(1 - q_j^{-1}\right) \tag{2.2}$$

and

$$\prod_{i=1}^{t} q_i^{-1} > \prod_{j=1}^{t+j} q_j^{-1} \quad (j \geq 1). \tag{2.3}$$

Furthermore, if $i \leq i'$ with $i, i' \in S$, then

$$\prod_{i=1}^{t} (1 - q_i^{-1}) > \prod_{i'=1}^{t} (1 - q_{i'}^{-1}) \tag{2.4}$$

and

$$\prod_{i=1}^{t} q_i^{-1} \geq \prod_{i'=1}^{t+j} q_{i'}^{-1}. \tag{2.5}$$

Proof. It is trivial. □

The following lemma makes to reduce conditions of (1.2).

Lemma 3. *Let $p+1 = 2q_1 \cdots q_s$ or $p-1 = 2q_1 \cdots q_s$ be positive integers, where $p, q_1, \ldots, q_s$ are distinct odd prime integers with $s \geq 2$. Then there does not exist $n = p$ $(resp., n = 2q_1 \cdots q_s)$ and $n+1 = 2q_1 \cdots q_s$ $(resp., n+1 = p)$ satisfying $\varphi(n) = \varphi(n+1) = U(n) = U(n+1)$.*

Proof. Assume $U(n) = U(n+1)$. Then, we get $p - 1 = (q_1 - 1) \cdots (q_s - 1)$ and

$$\begin{cases} \frac{p-1}{p+1} = \frac{1}{2}\left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_s}\right) \quad or \\ 1 = \frac{1}{2}\left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_s}\right). \end{cases} \tag{2.6}$$

We note that

$$\frac{1}{2}\left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_s}\right) < \frac{1}{2} \tag{2.7}$$

and

$$1 > \frac{p-1}{p+1} = 1 - \frac{2}{p+1} = 1 - \frac{2}{q_1 \cdots q_s} \geq 1 - \frac{2}{3 \cdot 5} > \frac{1}{2} \quad \text{by (2.5)}. \tag{2.8}$$

Thus the proof is completed by (2.6), (2.7) and (2.8). □

Lemma 4. *($(\omega_0(n),\omega_0(n+1)) = (1,2)$ or $(2,1)$) Let p, q_1, q_2 be distinct odd prime integers with $q_1 < q_2$ and let*

$$\begin{cases} Case\ 1) & n = p, & n+1 = 2q_1q_2 & or \\ Case\ 2) & n = 2q_1q_2, & n+1 = p & or \\ Case\ 3) & n = 2p & n+1 = q_1q_2 & or \\ Case\ 4) & n = q_1q_2, & n+1 = 2p. \end{cases} \tag{2.9}$$

Then we can not find positive integers n and $n+1$ satisfying (1.1) and (2.9).

Proof. Eq. (1.1) has no positive integers of n and $n+1$ in Case 1) and Case 2) by Lemma 3. Thus, we consider Cases 3) and Case 4).

Case 3) Put $n = 2p$ and $n+1 = q_1q_2$ in (1.1). This yields that

$$p-1 = U(2p) = U(q_1q_2) = (q_1-1)(q_2-1) \text{ and } (q_1-2)(q_2-2) = -1.$$

This is completed the Case 3).

Case 4) Finally, we consider Eq. (1.1) with $n = q_1q_2$ and $n+1 = 2p$. Similarly, we get $(q_1-2)(q_2-2) = -1$ and $q_1 = q_2 = 3$. Thus, for this case, we cannot find distinct odd prime integers q_1 and q_2 satisfying Eq. (1.1). Therefore, we prove Lemma 4. □

Remark 4. Using Mathematica 11.0, we find odd primes p, q_1, q_2 satisfying $(q_1-1)(q_2-1) = (p-1)$ with $p < 200$.

But these odd primes p, q_1 and q_2 in the Table 1 do not satisfy $n = 2p$ and $n+1 = q_1q_2$.

Lemma 5. *($(\omega_0(n),\omega_0(n+1)) = (1,3)$ or $(3,1)$) Let p, q_1, q_2 and q_3 be distinct odd prime integers with $q_1 < q_2 < q_3$ and let*

$$\begin{cases} Case\ 1) & n = p, & n+1 = 2q_1q_2q_3 & or \\ Case\ 2) & n = 2q_1q_2q_3, & n+1 = p & or \\ Case\ 3) & n = 2p, & n+1 = q_1q_2q_3 & or \\ Case\ 4) & n = q_1q_2q_3, & n+1 = 2p. \end{cases} \tag{2.10}$$

Then there exists an unique pair of $(n, n+1) = (194, 195)$ satisfying (1.1) and (2.10).

Proof. To prove Lemma 5, we need to check Case 3) and Case 4) only by Lemma 3.

Recall the identity

$$(p-1) = (q_1-1)(q_2-1)(q_3-1). \tag{2.11}$$

We consider two cases step by step.

Case 3) Let $n = 2p$ and $n+1 = q_1q_2q_3$. Using (2.11), we get

$$\left(1 - \frac{3}{q_1q_2q_3}\right) = 2\left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right)\left(1 - \frac{1}{q_3}\right). \tag{2.12}$$

If $q_1 \geq 5$, then R.H.S.(the right hand side) of (2.12) is

$$2(1-\frac{1}{q_1})(1-\frac{1}{q_2})(1-\frac{1}{q_3}) \geq 2(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11}) = \frac{96}{77} > 1$$

and L.H.S.(the left hand side) of (2.12) is $\left(1-\frac{3}{q_1q_2q_3}\right) < 1$. Thus we cannot find distinct odd primes q_1, q_2, q_3 satisfying Eq. (2.12) with $q_1 \geq 5$.

Assume $q_1 = 3$. Then we have

$$p-1 = 2(q_2-1)(q_3-1) \text{ and } 2p+1 = 3q_2q_3. \tag{2.13}$$

By (2.13), it is easy to see that

q_2-4	q_3-4	(q_2,q_3)	$2p+1$	p	prime
1	9	$(5,13)$	195	97	O

Therefore, an unique solution of this case is $(n,n+1) = (194,195)$.

Case 4) Let $n = q_1q_2q_3$ and $n+1 = 2p$. From (2.11), we deduce

$$\left(1-\frac{1}{q_1q_2q_3}\right) = 2\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)\left(1-\frac{1}{q_3}\right). \tag{2.14}$$

Similarly, the same method of (2.13) in Case 3), we have two inequalities $2\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)\left(1-\frac{1}{q_3}\right) > 1$ and $1-\frac{1}{q_1q_2q_3} < 1$ with $q_1 \geq 5$. Thus, we consider the only prime $q_1 = 3$. An equation (2.14) yields $(q_2-4)(q_3-4) = 11$ and $(q_2,q_3) = (5,15)$. But q_3 is not a prime integer.

This proves the lemma. □

Remark 5. Using Mathematica 11.0, we find the number of odd prime pairs (p,q_1,q_2,q_3) satisfying $U(2p) = p-1 = (q_1-1)(q_2-1)(q_3-1) = U(q_1q_2q_3)$ with $p < 400$ as follows in the Table 2.

Lemma 6. *$((\omega_0(n),\omega_0(n+1)) = (1,4)$ or $(4,1))$ Let p,q_1,q_2,q_3,q_4 be distinct odd prime integers with $q_1 < q_2 < q_3 < q_4$ and let*

$$\begin{cases} \text{Case 3)} & n = 2p \quad\quad n+1 = q_1q_2q_3q_4 \quad or \\ \text{Case 4)} & n = q_1q_2q_3q_4, \quad\quad n+1 = 2p. \end{cases} \tag{2.15}$$

Then there exists an unique pairs of $(n,n+1) = (5186,5197)$ satisfying (1.1) and (2.15).

Proof. The proof is similar to Lemma 5.

□

Remark 6. Let $n = 2p$ and $n+1 = \prod_{i=1}^{l} p_i$ (or $n-1 = \prod_{i=1}^{l} p_i$), where p_i are distinct odd prime integers with $p_1 < p_2 < \cdots < p_l$. Assume $\varphi(n) = \varphi(n+1) =$

$U(n) = U(n+1)$. We note that

$$2p+1 = \prod_{i=1}^{l} p_i \quad (\text{or } 2p-1 = \prod_{i=1}^{l} p_i) \text{ and } 2(p-1) = 2\prod_{i=1}^{l}(p_i - 1). \tag{2.16}$$

By Eq. (2.16) we get

$$\left(1 - \frac{3}{\prod_{i=1}^{l} p_i}\right) = 2\prod_{i=1}^{l}\left(1 - \frac{1}{p_i}\right) \; or \; \left(1 - \frac{1}{\prod_{i=1}^{l} p_i}\right) = 2\prod_{i=1}^{l}\left(1 - \frac{1}{p_i}\right)$$

and

$$2\prod_{i=1}^{l}\left(1 - \frac{1}{p_i}\right) + \frac{3}{\prod_{i=1}^{l} p_i} \geq 2\prod_{i=1}^{l}\left(1 - \frac{1}{p[i]}\right) + \frac{3}{\prod_{i=1}^{l} p[i]}. \tag{2.17}$$

Here, $p[i]$ is the i-th prime integer. Inequality (2.17) yields on the Table 3.

For example, if $q_1 = 37$ then $n+1$ (or $n-1$) $\geq$ 75347738233715510682119691 21548009585444856281368482589916445090657521119430259527166840110198 38699393817326126389986232066116624296065771602290209012057820739178 72287545170087475416051810829082770128252239632108848112410020471053 53707182094449546678344114135023520667353779163828815640309227081776 53264817868019846139296555645562400572533870903274086461776912104807 86948103160891681120747752822120795047461510487499681318260518950750 72046046180977008378417515269482748132721747320752111324046327166749 59534769799425231633065885082282111753064922202067920112201792625821 2559931694286547280935039182762405680807691871507 68754091 and the digit of $n+1$(or $n-1$) is greater than 627 with $l \geq 229$.

Lemma 7. *Let $p, q_1, q_2, q_3, q_4, q_5$ be distinct odd prime integers with $q_1 < q_2 < q_3 < q_4 < q_5$ and let*

$$\begin{cases} \mathit{Case}\ 3) & n = 2p, \quad n+1 = q_1q_2q_3q_4q_5 \quad or \\ \mathit{Case}\ 4) & n = q_1q_2q_3q_4q_5 \quad n+1 = 2p. \end{cases} \tag{2.18}$$

Then there does not exist n and $n+1$ satisfying (1.1) *and* (2.18).

Proof. The proof is similar to Lemma 4. □

As a result, from Lemma 1, Lemma 4, Lemma 5, Lemma 6 and Lemma 7, there exist two pairs of $(n, n+1) = (194, 195)$ and $(n, n+1) = (5186, 5187)$ satisfying Eq. (1.1) and (1.2).

3. LEMMAS FOR THEOREM 1 ($(1,6)$ OR $(6,1)$)

To prove Theorem 1, we need following lemmas.

Lemma 8. *Let $q_2,\ldots,q_6$ be distinct odd primes integers greater than 3 and let $S=\{q_2,q_3,q_4,q_5,q_6\}$, $S'=\{q_i\in S\,|q_i\equiv -1 \pmod 3)\}$ and $S''=S-S'$. If $\#S'\equiv 1 \pmod 2$ then*

$$3\left(\prod_{i=2}^{6} q_i-1\right)\neq 4\prod_{i=2}^{6}(q_i-1). \tag{3.1}$$

Proof. If $\#S'=1$ or 3 then

$$\begin{cases}\prod\limits_{i=2}^{6} q_i-1\equiv -1-1\equiv -2 \pmod 3,\\ \quad \frac{4}{3}\prod\limits_{i=2}^{6}(q_i-1)\equiv 0 \pmod 3.\end{cases} \tag{3.2}$$

If $\#S'=5$ then

$$\begin{cases}3\left(\prod\limits_{i=2}^{6}(q_i-1)\right)\equiv 0 (\bmod 3),\\ \quad 4\prod\limits_{i=2}^{6}(q_i-1)\not\equiv 0 (\bmod 3).\end{cases} \tag{3.3}$$

From Eq. (3.2) and Eq. (3.3), our claim follows. □

Remark 7. Let $ayz-b(y+z)+d=0$ with $a\in\mathbb{N}$ and $b,d\in\mathbb{Z}$. Here, $\mathbb{Z}$ denotes the set of ring of integers. More precisely, we can write it as

$$(ay-b)(az-b)=-ad+b^2. \tag{3.4}$$

If Eq. (3.4) have (at least one) positive integer solutions (y,z) then there exist (at least one) positive integers d_i satisfying

$$\begin{cases}\quad d_i\,\big|-ad+b^2,\\ d_i+b\equiv 0 \pmod a,\\ \frac{-ad+b^2}{d_i}\equiv 0 \pmod a.\end{cases} \tag{3.5}$$

We can focus our attention to identity, namely, Eq. (1.1) with $\omega_0(n)=1$ $(resp., \omega_0(n)=6)$ and $\omega_0(n+1)=6$ $(resp., \omega_0(n+1)=1)$.

Lemma 9. *Let $p,q_1,\ldots,q_6$ be distinct odd primes and let*

$$\begin{cases}\textit{Case } 3) \quad n=2p, \quad n+1=q_1\ldots q_6 \quad or\\ \textit{Case } 4) \quad n=q_1\ldots q_6, \quad n+1=2p\end{cases} \tag{3.6}$$

with $q_1<\ldots<q_6$. Then, we cannot find 7-tuples $(p,q_1,\ldots,q_6)$ primes satisfying Eq. (3.6).

Proof. Since the proof is very similar, we consider only Case 3) and Case 4) by Lemma 3. By the Table 3, we can obtain $q_1 = 3$.

First, we consider the Case 3), that is,

$$\begin{cases} n = 2p, \\ n+1 = 2p+1 = 3q_2 \dots q_6, \end{cases} \tag{3.7}$$

where $q_1, \dots, q_6$ are distinct odd primes with $3 < q_2 < \dots < q_6$.

From Eq. (1.1) we deduce that

$$3\left(\prod_{i=2}^{6} q_i - 1\right) = 4\prod_{i=2}^{6}(q_i - 1) \tag{3.8}$$

and

$$1 = \frac{4}{3}\prod_{i=2}^{6}\left(1 - \frac{1}{q_i}\right) + \prod_{i=2}^{6}\frac{1}{(q_i - 1)}. \tag{3.9}$$

Let $f(p[i]) = \frac{4}{3}\prod_{j=i}^{i+4}\left(1 - \frac{1}{p[i]}\right) + \prod_{j=i}^{i+4}\frac{1}{p[i]} - 1$. Here, $p[i]$ is the i-th prime integer. That is, $p[1] = 2$, $p[2] = 3, \dots$

Since $f(p[5]) = -\frac{2858}{62491} < 0$ and $f(p[6]) = \frac{37796}{2800733} > 0$, the set of possible primes satisfying (3.8) are $\{5, 7, 11\}$. We note that we can write Eq. (3.8) as

$$15\prod_{i=3}^{6} q_i - 3 = 16\prod_{i=3}^{6}(q_i - 1), \tag{3.10}$$

$$7\prod_{i=3}^{6} q_i - 1 = 8\prod_{i=3}^{6}(q_i - 1), \tag{3.11}$$

$$33\prod_{i=3}^{6} q_i - 3 = 40\prod_{i=3}^{6}(q_i - 1). \tag{3.12}$$

In Eq. (3.10), if $q_3 \equiv 1 \pmod 5$, then $L.H.S \not\equiv R.H.S \pmod 5$ and so,

$$q_3 \not\equiv 1 \pmod 5. \tag{3.13}$$

Furthermore, we get

q_3	$3 \cdot 5 \cdot q_3$	$16(q_3 - 1)$
13	195	192
17	255	256

(3.14)

Let $f_1(p[i]) = \frac{16}{15}\prod_{j=i}^{i+3}\left(1 - \frac{1}{p[i]}\right) + \prod_{j=i}^{i+3}\frac{1}{(p[i])} - 1$. Then, we observe that

$$\begin{cases} f_1(p[16]) = -\frac{202324}{6390045} < 0, \\ f_1(p[17]) = \frac{144986}{85602215} > 0. \end{cases} \tag{3.15}$$

Possible prime integers for q_3 are 17, 19, 23, 29, 31, 37, 41, 43, 47, 53 by (3.13), (3.14) and (3.15).

Case 3-a) $(q_1,q_2,q_3) = (3,5,17)$. Similarly, we consider

$$3\times 5\times 17\times q_4q_5q_6 - 3 = 16^2(q_4-1)(q_5-1)(q_6-1). \tag{3.16}$$

Let $f_2(p[i]) = \frac{256}{255}\prod_{j=i}^{i+2}\left(1-\frac{1}{p[i]}\right) + \frac{1}{85}\prod_{j=i}^{i+2}\frac{1}{p[i]} - 1$. Then, we observe that

$$f_2(p[134]) < 0 \text{ and } f_2(p[135]) > 0. \tag{3.17}$$

It is easily checked that

$$255q_4 < 256(q_4-1) \text{ and } q_4 > 256. \tag{3.18}$$

On the other hand, if $q_4 \equiv 1 \pmod 5$ or $q_4 \equiv 1 \pmod{17}$, then Eq. (3.16) has no solution. Thus, the set of possible prime integers for q_4 is $P_1 := \{257, 263, 269, 277, 283, 293, 313, 317, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 419, 433, 439, 449, 457, 463, 467, 479, 487, 499, 503, 509, 523, 547, 557, 563, 569, 577, 587, 593, 599, 607, 617, 619, 643, 653, 659, 673, 677, 683, 709, 719, 727, 733, 739, 743, 757\}$.

Let $f_3(x) := -255xyz + 3 + 256(x-1)(y-1)(z-1)$. Inserting $x \in P_1$ into $f_3(x)$, we have Table 4. Eight Diophantine equations $65539 - 65536(y+z) + yz = 0$, $67075 - 67072(y+z) + 7yz = 0$, $70659 - 70656y - 70656z + 21yz = 0$, $90115 - 90112(y+z) + 97yz = 0$, $101379 - 101376y - 101376z + 141yz = 0$, $119299 - 119296(y+z) + 211yz = 0$, $139779 - 139776y - 139776z + 291yz = 0$, $183811 - 183808(y+z) + 463yz = 0$ have integer solutions (y,z) using (3.5). But solutions (y,z) of these Diophantine equations have not pairs of prime integer solutions.

Therefore, Eq. (3.16) has no solution.

Case 3-b) $(q_1,q_2,q_3) = (3,5,19)$. Consider

$$3\times 5\times 19q_4q_5q_6 - 3 = 16\times 18(q_4-1)(q_5-1)(q_6-1)$$

and

$$95q_4q_5q_6 - 1 = 96(q_4-1)(q_5-1)(q_6-1). \tag{3.19}$$

An inequality $95q_4 < 96(q_4-1)$ deduces the lower bound for q_4, that is,

$$q_4 > 96. \tag{3.20}$$

Let $f_4(p[i]) := \frac{96}{95}\prod_{j=i}^{i+2}\left(1-\frac{1}{p[i]}\right) + \frac{1}{95}\prod_{j=i}^{i+2}\frac{1}{p[i]} - 1$.

Inequalities $f_4(p[60]) < 0$ and $f_4(p[61]) > 0$ deduce the upper bound for q_4, that is,

$$q_4 \le p[60] = 281. \tag{3.21}$$

If $q_4 \equiv 1 \pmod 5$ or $q_4 \equiv 1 \pmod{19}$, then Eq. (3.19) has no solution. Thus, the set of possible prime integers for q_4 is $P_2 := \{97, 103, 107, 109, 113, 127, 137, 139, 149, 157, 163, 167, 173, 179, 193, 197, 199, 223, 227, 233, 239, 257, 263, 269, 277\}$.

Let $f_5(x) := -95xyz + 3 + 96(x-1)(y-1)(z-1)$. We derive Table 5. Four Diophantine equations $99217 - 9216y - 9216z + yz = 0$, $9793 - 9792y - 9792z + 7yz = 0$, $10369 - 10368y - 10368z + 13yz = 0$, $13249 - 13248y - 13248z + 43yz = 0$ have integer solutions (y,z) by (3.5), but these solutions are not pairs of prime integers. Therefore, Eq. (3.19) has no solution.

Case 3-c) $(q_1,q_2,q_3) = (3,5,23)$. Consider

$$3 \times 5 \times 23 q_4 q_5 q_6 - 3 = 16 \times 22 (q_4-1)(q_5-1)(q_6-1). \tag{3.22}$$

Since $345q_4 < 352(q_4-1)$ the lower bound of possible prime q_4 is 53.

Let $f_6(p[i]) := \frac{352}{345} \prod_{j=i}^{i+2} \left(1 - \frac{1}{p[i]}\right) + \frac{1}{115} \prod_{j=i}^{i+2} \frac{1}{p[i]} - 1.$

The upper bound of possible prime q_4 is 139 since $f_6(p[34]) < 0$ and $f_6(p[35]) > 0$. Similarly, if $q_4 \equiv 1 \pmod 5$ or $q_4 \equiv 1 \pmod{23}$ has no solution. Thus, the set of possible primes for q_4 is $\{53, 59, 67, 73, 79, 83, 89, 97, 103,$
$107, 109, 113, 127, 137\}$.

Let $f_7(x) := -345xyz + 352(x-1)(y-1)(z-1) + 3$.

We derive Table 6. According to the Table 6, we have no pairs of prime integer solutions (y,z). Therefore, Eq. (3.22) has no solution.

Case 3-d) $(q_1,q_2,q_3) = (3,5,29)$. Consider

$$435 q_4 q_5 q_6 - 3 = 448(q_4-1)(q_5-1)(q_6-1). \tag{3.23}$$

Since $435q_4 < 448(q_4-1)$ the lower bound of possible prime q_4 is 37.

Let $f_8(p[i]) := \frac{448}{435} \prod_{j=i}^{i+2} \left(1 - \frac{1}{p[i]}\right) + \frac{1}{145} \prod_{j=i}^{i+2} \frac{1}{p[i]} - 1.$

The upper bound of possible prime q_4 is 97 since $f_8(p[25]) < 0$ and $f_8(p[26]) > 0$. If $q_4 \equiv 1 \pmod 5$ and $q_4 \equiv 1 \pmod{29}$ then Eq. (3.23) has no solution.

Let $f_9(x) := -435xyz + 448(x-1)(y-1)(z-1) + 3$.

Similarly, we get Table 7. According to the Table 7, there are no pairs of integer solutions $(y,z) \in \mathbb{Z} \times \mathbb{Z}$.

Case 3-e) $(q_1,q_2,q_3) = (3,5,37)$. In (3.8), put $(q_1,q_2,q_3) = (3,5,37)$. Then

$$185 q_4 q_5 q_6 - 1 = 192(q_4-1)(q_5-1)(q_6-1). \tag{3.24}$$

Here,

$$q_4 > 37, \ \ q_4 \not\equiv 1 \pmod 5 \ \text{ and } \ q_4 \not\equiv 1 \pmod{37}. \tag{3.25}$$

Let $f_{10}(p[i]) := \frac{192}{1855} \prod_{j=i}^{i+2} \left(1 - \frac{1}{p[i]}\right) + \frac{1}{185} \prod_{j=i}^{i+2} \frac{1}{p[i]} - 1$. Clearly,

$$f_{10}(p[21]) < 0 \ \text{ and } \ f_{10}(p[22]) > 0. \tag{3.26}$$

Let $f_{11}(x) := -185xyz + 192(x-1)(y-1)(z-1) + 1$. Combining (3.24), (3.25) and (3.26), we have Table 8 for the set of possible primes q_4 and $f_{11}(q_4)$.

Similarly, we get Table 8. Using the methods of Case 3-d) these Diophantine equations have no $(y,z) \in \mathbb{Z}\times\mathbb{Z}$.

Case 3-f) $(q_1,q_2,q_3) = (3,5,43)$ or $(3,5,47)$ or $(3,5,53)$. In (3.8), put $q_3 = 43$ or 47 or 53. Then

$$\begin{aligned} 215q_4q_5q_6 - 1 &= 224(q_4-1)(q_5-1)(q_6-1) \text{ or} \\ 705q_4q_5q_6 - 3 &= 736(q_4-1)(q_5-1)(q_6-1) \text{ or} \\ 795q_4q_5q_6 - 3 &= 832(q_4-1)(q_5-1)(q_6-1). \end{aligned}$$

Let $f_{12}(x) := -215xyz + 224(x-1)(y-1)(z-1)+1$, $f_{13}(x) := -705xyz + 736(x-1)(y-1)(z-1)+3$ and $f_{14}(x) := -795xyz+832(x-1)(y-1)(z-1)+3$. Similarly, using the same method of Case 3-d), we get Table 9. It is easily checked that seven Diophantine equations have no integer solutions.

Next, we consider Eq. (3.11) with $q_1 = 3$ and $q_2 = 7$.

Let $f_{15}(p[i]) := \frac{8}{7}\prod_{j=i}^{i+3}\left(1-\frac{1}{p[i]}\right) + \frac{1}{7}\prod_{j=i}^{i+3}\frac{1}{p[i]} - 1$.

Since $f_{15}(p[9]) = -\frac{33102}{5355343} < 0$ and $f_{15}(p[10]) = \frac{130320}{9546481} > 0$, the set of possible primes for q_3 is $\{11,13,17,19,23\}$. Next, we consider (3.11) with $q_3 = 11,13,17,19,23$ step by step.

Case 3-g) $(q_1,q_2,q_3) = (3,7,11)$. In (3.11), put $(q_1,q_2,q_3) = (3,7,11)$. That is, we consider

$$77q_4q_5q_6 - 1 = 80(q_4-1)(q_5-1)(q_6-1). \tag{3.27}$$

Similarly, we obtain the lower bound of possible prime integers q_4 is 29 and the upper bound of possible prime integers q_4 is 73. If $q_4 \equiv 1 \pmod 7$ or $q_4 \equiv 1 \pmod{11}$ then Eq. (3.27) has no solution. Thus, the set of possible primes of q_4 is $\{31,37,41,47,53,61,73\}$. Let $f_{16}(x) := -77xyz + 80(x-1)(y-1)(z-1)+1$.

Similarly, using the same method of Case 3-d), we get Table 10. Two Diophantine equations $2881-2880(y+z)+31yz = 0$ and $3201-3200(y+z)+43yz = 0$ have integer solutions (y,z) but not pairs of prime integers. Therefore, it is easily checked that eight Diophantine equations have no pairs of prime integer solutions (y,z).

Case 3-h) $(q_1,q_2,q_3) = (3,7,13)$. In (3.11), put $(q_1,q_2,q_3) = (3,7,13)$. Then

$$91q_4q_5q_6 - 1 = 96(q_4-1)(q_5-1)(q_6-1). \tag{3.28}$$

Let $f_{17}(x) := -91xyz + 96(x-1)(y-1)(z-1)+1$.

Similarly, we have Table 11. If we use (3.5), then five Diophantine equations have no $(y,z) \in \mathbb{Z}\times\mathbb{Z}$. Thus, Eq. (3.28) has no solution.

Case 3-i) $(q_1,q_2,q_3) = (3,7,17)$ or $(3,7,19)$. Given two equations

$$\begin{aligned} 119q_4q_5q_6 - 1 &= 128(q_4-1)(q_5-1)(q_6-1) \text{ and} \\ 133q_4q_5q_6 - 1 &= 144(q_4-1)(q_5-1)(q_6-1), \end{aligned}$$

we consider two Diophantine equations $f_{18}(x) := -119xyz+128(x-1)(y-1)(z-1)+1$ and $f_{19}(x) := -133xyz+144(x-1)(y-1)(z-1)+1$.

Similarly, we have Table 12. Six Diophantine equations have no $(y,z) \in \mathbb{Z}\times\mathbb{Z}$.

Case 3-j) $(q_1,q_2,q_3) = (3,7,23)$. In (3.11), put $(q_1,q_2,q_3) = (3,7,23)$. Then

$$161q_4q_5q_6 - 1 = 176(q_4-1)(q_5-1)(q_6-1). \tag{3.29}$$

The lower bound and upper bound of possible primes q_4 is 29. But, $29 \equiv 1 \pmod 7$. So, we cannot find pairs of prime integers of (3.29).

Next, we consider Eq. (3.12) with $q_1 = 3$ and $q_2 = 11$, that is,

$$33q_3q_4q_5q_6 - 3 = 40(q_3-1)(q_4-1)(q_5-1)(q_6-1). \tag{3.30}$$

Let $f_{20}(p[i]) := \frac{40}{33}\prod_{j=i}^{i+3}\left(1-\frac{1}{p[i]}\right) + \frac{1}{11}\prod_{j=i}^{i+3}\frac{1}{p[i]} - 1.$

Since $f_{20}(p[7]) < 0$ and $f_{20}(p[8]) > 0$, the set of possible primes q_3 is $\{13,17\}$. Thus, we solve the equation (3.12) one by one.

Case 3-k) $(q_1,q_2,q_3) = (3,11,13)$ and $(3,11,17)$. Let $f_{21}(x) := -143xyz + 160(x-1)(y-1)(z-1)+1$ and $f_{22}(x) := -561xyz + 640(x-1)(y-1)(z-1)+3$.

Then we get Table 13. Three Diophantine equations have no pairs of integer solutions.

Therefore, we cannot find 7-tuples of prime integers $(p, q_1, ..., q_6)$ for Case 3) by Case 3-a)$\sim$Case 3-k).

Second, we consider the Case 4), that is, $n = 3q_2\cdots q_6$ and $n+1 = 2p$.

From (1.1), we observe that

$$3\prod_{i=2}^{6} q_i - 1 = 4\prod_{i=2}^{6}(q_i - 1). \tag{3.31}$$

It is easy to see that

$$3\prod_{i=2}^{6} q_i - 1 \not\equiv 4\prod_{i=2}^{6}(q_i-1) \pmod 3 \text{ with } q_2 \equiv 1 \pmod 3. \tag{3.32}$$

Let $f_{23}(p[i]) := \frac{4}{3}\prod_{j=i}^{i+4}\left(1-\frac{1}{p[j]}\right) + \frac{1}{3}\prod_{j=i}^{i+4}\frac{1}{p[j]} - 1$. Since $f_{23}(p[5]) = -\frac{145760}{3187041}$ and $f_{23}(p[6]) = \frac{8722}{646323}$, we have the set of possible primes q_2 satisfying (3.31) is $\{5,11\}$.

Let $q_1 = 3$ and $q_2 = 5$. Then (3.31) becomes

$$15\prod_{i=3}^{6} q_i - 1 = 16\prod_{i=3}^{6}(q_i - 1). \tag{3.33}$$

Similar to (3.14) and (3.15), we derive that the lower bound and upper bound of possible prime integers q_3 are 17 and 53. The equation (3.33) deduces $q_3 \not\equiv 1 \pmod 3$ and $q_3 \not\equiv 1 \pmod 5$. Therefore, the set of possible prime integers q_3 satisfying (3.33) is $\{17,23,29,47,53\}$.

Case 4-a) $(q_1,q_2,q_3)=(3,5,17)$. In (3.31), put $(q_1,q_2,q_3)=(3,5,17)$. Then

$$255q_4q_5q_6-1=256(q_4-1)(q_5-1)(q_6-1). \tag{3.34}$$

Using the same method in Case 3-a), we have the same lower bound and upper bound of possible primes q_4 in Case 3-a). Furthermore, we get $q_4\not\equiv 1 \pmod 3$, $q_4\not\equiv 1 \pmod 5$ and $q_4\not\equiv 1 \pmod{17}$ by (3.34).

Let $f_{24}(x)=f_3(x)-2=-255q_4q_5q_6+1+256(q_4-1)(q_5-1)(q_6-1)$. We have the following Table 14. Two Diophantine equations $65537-65536(y+z)+yz=0$ and $90113-90112(y+z)+97yz=0$ have pairs of integer solutions (y,z) satisfying $f_{24}(x)=0$ but these are not pairs of primes.

Therefore, Eq. (3.34) has no solution.

Case 4-b) $(q_1,q_2,q_3)=(3,5,23)$. In (3.31), put $(q_1,q_2,q_3)=(3,5,23)$. Then

$$345q_4q_5q_6-1=352(q_4-1)(q_5-1)(q_6-1). \tag{3.35}$$

It is easy to see that the lower bound and upper bound of possible primes q_4 are same numbers in Case 3-c). Furthermore, we get $q_4\not\equiv 1 \pmod 3$, $q_4\not\equiv 1 \pmod 5$ and $q_4\not\equiv 1 \pmod{23}$ by (3.35).

Thus, the set of possible primes for q_4 is $\{53,59,83,89,107,113,137\}$.

Let $f_{25}(x)=f_7(x)-2:=-345xyz+352(x-1)(y-1)(z-1)+1$.

We derive Table 15. Diophantine equations have no pairs of integer solutions (y,z).

Case 4-c) $(q_1,q_2,q_3)=(3,5,29)$. In (3.31), put $(q_1,q_2,q_3)=(3,5,29)$. Then

$$435q_4q_5q_6-1=448(q_4-1)(q_5-1)(q_6-1). \tag{3.36}$$

It is easy to see that the lower bound and upper bound of possible primes q_4 are same numbers in Case 3-d). Furthermore, we get $q_4\not\equiv 1 \pmod 3$, $q_4\not\equiv 1 \pmod 5$ and $q_4\not\equiv 1 \pmod{29}$ by (3.36).

Let $f_{26}(x)=f_9(x)-2:=-435xyz+448(x-1)(y-1)(z-1)+1$. Similarly, we get Table 16. There is no integer solution $(y,z)\in\mathbb{Z}\times\mathbb{Z}$.

Case 4-d) $(q_1,q_2,q_3)=(3,5,47)$ and $(3,5,53)$. Let $f_{27}(x):=-705xyz+736(x-1)(y-1)(z-1)+1$ and $f_{28}(x):=-795xyz+832(x-1)(y-1)(z-1)+1$.

Then we get Table 17. Three Diophantine equations have no pairs of integer solutions. Next, we consider the equation (3.12) with $q_1=3$ and $q_2=11$. It is easily checked that $q_3\not\equiv 1 \pmod 3$ and $q_3\not\equiv 1 \pmod{11}$ by (3.12). So, we only consider a prime $q_3=17$.

Case 4-e) $(q_1,q_2,q_3)=(3,11,17)$. Consider

$$561q_4q_5q_6-1=640(q_4-1)(q_5-1)(q_6-1). \tag{3.37}$$

Similarly, the possible prime q_4 satisfying (3.37) is 19. Let $f_{29}(x):=-561q_4q_5q_6+640(q_4-1)(q_5-1)(q_6-1)+1$.

Then the Diophantine equation $f_{29}(19)=11523-11520y-11520z+861yz=0$ has no pairs of integer solutions (y,z).

Therefore, we complete the proof of Lemma 9. □

As a result, from Lemma 9, there does not exist satisfying Eq. (1.1) and (1.2).

Proof of Theorem 1. From Lemma 1, Lemma 4, Lemma 5, Lemma 6, Lemma 7 and Lemma 9, we complete the proof of Theorem 1. □

APPENDIX

TABLE 1. Primes p, q and r

p	(p,q_1,q_2)	$(2p+1,q_1,q_2)$	p	$(p,q_1,r1)$	$(2p+1,q_1,q_2)$
13	(13,3,7)	(27,21)	97	(97,7,17)	(195,119)
37	(37,3,19)	(75,57)	109	(109,7,19)	(219,133)
41	(41,5,11)	(83,55)	113	(113,5,29)	(227,145)
61	(61,3,31)	(123,93)	157	(157,3,79)	(315,237)
61	(61,7,11)	(123,77)	181	(181,7,31)	(363,217)
73	(73,3,37)	(147,111)	181	(181,11,19)	(363,209)
73	(73,5,19)	(147,95)	193	(193,3,97)	(387,291)
73	(73,7,13)	(147,91)	193	(193,13,17)	(387,221)
89	(89,5,23)	(179,115)			

TABLE 2. Primes p, q_1, q_2 and q_3

p	(q_1,q_2,q_3)	$2p$	$q_1q_2q_3-1$	$q_1q_2q_3+1$	$U(n)=U(n+1)$ $=\varphi(n)=\varphi(n+1)$
97	(3,5,13)	194	194	196	O
193	(3,7,17)	386	356	358	×
241	(3,5,31)	482	464	466	×
241	(3,11,13)	482	428	430	×
241	(5,7,11)	482	384	386	×
337	(3,5,43)	674	644	646	×
337	(3,7,29)	674	608	610	×

TABLE 3. Bound of prime p_1 with respect to l

l	$possible\ prime\ p_1$
$l \geq 3$	$p_1 = 3$
$l \geq 7$	$p_1 = 3, 5$
$l \geq 15$	$p_1 = 3, 5, 7$
$l \geq 27$	$p_1 = 3, 5, 7, 11$
$l \geq 41$	$p_1 = 3, 5, 7, 11, 13$
$l \geq 62$	$p_1 = 3, 5, 7, 11, 13, 17$
$l \geq 85$	$p_1 = 3, 5, 7, 11, 13, 17, 19$
$l \geq 115$	$p_1 = 3, 5, 7, 11, 13, 17, 19, 23$
$l \geq 150$	$p_1 = 3, 5, 7, 11, 13, 17, 19, 23, 29$
$l \geq 186$	$p_1 = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$
$l \geq 229$	$p_1 = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37$

TABLE 4. Diophantine equations for Lemma 9 Case 3-a

x	$f_3(x)$	x	$f_3(x)$
257	$65539 - 65536(y+z) + yz$	263	$67075 - 67072(y+z) + 7yz$
269	$68611 - 68608(y+z) + 13yz$	277	$70659 - 70656(y+z) + 21yz$
283	$72195 - 72192(y+z) + 27yz$	293	$74755 - 74752(y+z) + 37yz$
313	$79875 - 79872(y+z) + 57yz$	317	$80899 - 80896(y+z) + 61yz$
337	$86019 - 86016(y+z) + 81yz$	347	$88579 - 88576(y+z) + 91yz$
349	$89091 - 89088(y+z) + 93yz$	353	$90115 - 90112(y+z) + 97yz$
359	$91651 - 91648(y+z) + 103yz$	367	$93699 - 93696(y+z) + 111yz$
373	$95235 - 95232(y+z) + 117yz$	379	$96771 - 96768(y+z) + 123yz$
383	$97795 - 97792(y+z) + 127yz$	389	$99331 - 99328(y+z) + 133yz$
397	$101379 - 101376(y+z) + 141yz$	419	$107011 - 107008(y+z) + 163yz$
433	$110595 - 110592(y+z) + 177yz$	439	$112131 - 112128(y+z) + 183yz$
449	$114691 - 114688(y+z) + 193yz$	457	$116739 - 116736(y+z) + 201yz$
463	$118275 - 118272(y+z) + 207yz$	467	$119299 - 119296(y+z) + 211yz$
479	$122371 - 122368(y+z) + 223yz$	487	$124419 - 124416(y+z) + 231yz$
499	$127491 - 127488(y+z) + 243yz$	503	$128515 - 128512(y+z) + 247yz$
509	$130051 - 130048(y+z) + 253yz$	523	$133635 - 133632(y+z) + 267yz$
547	$139779 - 139776(y+z) + 291yz$	557	$142339 - 142336(y+z) + 301yz$
563	$143875 - 143872(y+z) + 307yz$	569	$145411 - 145408(y+z) + 313yz$
577	$147459 - 147456(y+z) + 321yz$	587	$150019 - 150016(y+z) + 331yz$
593	$151555 - 151552(y+z) + 337yz$	599	$153091 - 153088(y+z) + 343yz$
607	$155139 - 155136(y+z) + 351yz$	617	$157699 - 157696(y+z) + 361yz$
619	$158211 - 158208(y+z) + 363yz$	643	$164355 - 164352(y+z) + 387yz$
653	$166915 - 166912(y+z) + 397yz$	659	$168451 - 168448(y+z) + 403yz$
673	$172035 - 172032(y+z) + 417yz$	677	$173059 - 173056(y+z) + 421yz$
683	$174595 - 174592(y+z) + 427yz$	709	$181251 - 181248(y+z) + 453yz$
719	$183811 - 183808(y+z) + 463yz$	727	$185859 - 185856(y+z) + 471yz$
733	$187395 - 187392(y+z) + 477yz$	739	$188931 - 188928(y+z) + 483yz$
743	$189955 - 189952(y+z) + 487yz$	757	$193539 - 193536(y+z) + 501yz$

TABLE 5. Diophantine equations for Lemma 9 Case 3-b

x	$f_5(x)$	x	$f_5(x)$
97	$9217-9216(y+z)+yz$	103	$9793-9792(y+z)+7yz$
107	$10177-10176(y+z)+11yz$	109	$10369-10368(y+z)+13yz$
113	$10753-10752(y+z)+17yz$	127	$12097-12096(y+z)+31yz$
137	$13057-13056(y+z)+41yz$	139	$13249-13248(y+z)+43yz$
149	$14209-14208(y+z)+53yz$	157	$14977-14976(y+z)+61yz$
163	$15553-15552(y+z)+67yz$	167	$15937-15936(y+z)+71yz$
173	$16513-16512(y+z)+77yz$	179	$17089-17088(y+z)+83yz$
193	$18433-18432(y+z)+97yz$	197	$18817-18816(y+z)+101yz$
199	$19009-19008(y+z)+103yz$	223	$21313-21312(y+z)+127yz$
227	$21697-21696(y+z)+131yz$	233	$22273-22272(y+z)+137yz$
239	$22849-22848(y+z)+143yz$	257	$24577-24576(y+z)z+161yz$
263	$25153-25152(y+z)+167yz$	269	$25729-25728(y+z)+173yz$
277	$26497-26496(y+z)+181yz$		

TABLE 6. Diophantine equations for Lemma 9 Case 3-c

x	$f_7(x)$	x	$f_7(x)$
53	$18307-18304(y+z)+19yz$	59	$20419-20416(y+z)+61yz$
67	$23235-23232(y+z)+117yz$	73	$25347-25344(y+z)+159yz$
79	$27459-27456(y+z)+201yz$	83	$28867-28864(y+z)+229yz$
89	$30979-30976(y+z)+271yz$	97	$33795-33792(y+z)+327yz$
109	$38019-38016(y+z)+411yz$	113	$39427-39424(y+z)+439yz$
127	$44355-44352(y+z)+537yz$	137	$47875-47872(y+z)+607yz$

TABLE 7. Diophantine equations for Lemma 9 Case 3-d

x	$f_9(x)$	x	$f_9(x)$
37	$16131-16128(y+z)+33yz$	43	$18819-18816(y+z)+111yz$
47	$20611-20608(y+z)+163yz$	53	$23299-23296(y+z)+241yz$
67	$29571-29568(y+z)+423yz$	73	$32259-32256(y+z)+501yz$
79	$34947-34944(y+z)+579yz$	83	$36739-36736(y+z)+631yz$
89	$39427-39424(y+z)+709yz$	97	$43011-43008(y+z)+813yz$

TABLE 8. Diophantine equations for Lemma 9 Case 3-e

x	$f_{11}(x)$	x	$f_{11}(x)$
43	$8065-8064(y+z)+109yz$	47	$8833-8832(y+z)+137yz$
53	$9985-9984(y+z)+179yz$	59	$11137-11136(y+z)+221yz$
67	$12673-12672(y+z)+277yz$	73	$13825-13824(y+z)+319yz$

TABLE 9. Diophantine equations for Lemma 9 Case 3-f

x	$f_{12}(x)=0$	x	$f_{12}(x)=0$
47	$10305=10304(y+z)-199yz$	53	$11649=11648(y+z)-253yz$
59	$12993=12992(y+z)-307yz$	67	$14785=14784(y+z)-379yz$
x	$f_{13}(x)=0$	x	$f_{13}(x)=0$
53	$38275=38272(y+z)-907yz$	59	$42691=42688(y+z)-1093yz$
x	$f_{14}(x)=0$		
59	$48259=48256(y+z)-1351yz$		

TABLE 10. Diophantine equations for Lemma 9 Case 3-g

x	$f_{16}(x)$	x	$f_{16}(x)$
31	$2401-2400(y+z)+13yz$	37	$2881-2880(y+z)+31yz$
41	$3201-3200(y+z)+43yz$	47	$3681-3680(y+z)+61yz$
53	$4161-4160(y+z)+79yz$	59	$4641-4640(y+z)+97yz$
61	$4801-4800(y+z)+103yz$	73	$5761-5760(y+z)+139yz$

TABLE 11. Diophantine equations for Lemma 9 Case 3-h

x	$f_{17}(x)=0$	x	$f_{17}(x)=0$
23	$2113=2112(y+z)-19yz$	31	$2881=2880(y+z)-59yz$
37	$3457=3456(y+z)-89yz$	41	$3841=3840(y+z)-109yz$
47	$4417=4416(y+z)-139yz$		

TABLE 12. Diophantine equations for Lemma 9 Case 3-i

x	$f_{18}(x)=0$	x	$f_{18}(x)=0$
19	$2305=2304(y+z)-43yz$	23	$2817=2816(y+z)-79yz$
31	$3841=3840(y+z)-151yz$	37	$4609=4608(y+z)-205yz$
x	$f_{19}(x)=0$	x	$f_{19}(x)=0$
23	$3169=3168(y+z)-109yz$	31	$4321=4320(y+z)-197yz$

TABLE 13. Diophantine equations for Lemma 9 Case 3-k

x	$f_{21}(x)=0$	x	$f_{22}(x)=0$
13	$1921=1920(y+z)-61yz$	19	$11523=11520(y+z)-861yz$
17	$2561=2560(y+z)-129yz$		

TABLE 14. Diophantine equations for Lemma 9 Case 4-a

x	$f_{24}(x)$	x	$f_{24}(x)$
257	$65537-65536(y+z)+yz$	263	$67073-67072(y+z)+7yz$
269	$68609-68608(y+z)+13yz$	293	$74753-74752(y+z)+37yz$
317	$80897-80896(y+z)+61yz$	347	$88577-88576(y+z)+91yz$
353	$90113-90112(y+z)+97yz$	359	$91649-91648(y+z)+103yz$
383	$97793-97792(y+z)+127yz$	389	$99329-99328(y+z)+133yz$
419	$107009-107008(y+z)+163yz$	449	$114689-114688(y+z)+193yz$
467	$119297-119296(y+z)+211yz$	479	$122369-122368(y+z)+223yz$
503	$128513-128512(y+z)+247yz$	509	$130049-130048(y+z)+253yz$
557	$142337-142336(y+z)+301yz$	563	$143873-143872(y+z)+307yz$
569	$145409-145408(y+z)+313yz$	587	$150017-150016(y+z)+331yz$
593	$151553-151552(y+z)+337yz$	599	$153089-153088(y+z)+343yz$
617	$157697-157696(y+z)+361yz$	653	$166913-166912(y+z)+397yz$
659	$168449-168448(y+z)+403yz$	677	$173057-173056(y+z)+421yz$
683	$174593-174592(y+z)+427yz$	719	$183809-183808(y+z)+463yz$
743	$189953-189952(y+z)+487yz$		

TABLE 15. Diophantine equations for Lemma 9 Case 4-b

x	$f_{25}(x)$	x	$f_{25}(x)$
53	$18305-18304(y+z)+19yz$	59	$20417-20416(y+z)+61yz$
83	$28865-28864(y+z)+229yz$	89	$30977-30976(y+z)+271yz$
107	$37313-37312(y+z)+397yz$	113	$39425-39424(y+z)+439yz$
137	$47873-47872(y+z)+607yz$		

TABLE 16. Diophantine equations for Lemma 9 Case 4-c

x	$f_{26}(x)$	x	$f_{26}(x)$
47	$20609-20608(y+z)+163yz$	53	$23297-23296(y+z)+241yz$
83	$36737-36736(y+z)+631yz$	89	$39425-39424(y+z)+709yz$

TABLE 17. Diophantine equations for Lemma 9 Case 4-d

x	$f_{28}(x)=0$	x	$f_{29}(x)=0$
53	$38273=38272(y+z)-907yz$	59	$48257=48256(y+z)-1351yz$
59	$42689=42688(y+z)-1093yz$		

TABLE 18. Values of $U(n)=U(n+1)$ for $\left(1\le n\le 10^5\right)$. Except $1,\ 12\,|U(n)=U(n+1)$

n	$n+1$	*Factor of n*	*Factor of n+1*	$U(n)=U(n+1)$
168	169	$2^3\times3\times7$	13^2	12
194	195	2×97	$3\times5\times13$	96
350	351	$2\times5^2\times7$	$3^3\times13$	24
1368	1369	$2^3\times3^2\times19$	37^2	36
1628	1629	$2^2\times11\times37$	$3^2\times181$	360
3705	3706	$3\times5\times13\times19$	$2\times17\times109$	1728
5186	5187	2×2593	$3\times7\times13\times19$	2592
6929	6930	$13^2\times41$	$2\times3^2\times5\times7\times11$	480
7475	7476	$5^2\times13\times23$	$2^2\times3\times7\times89$	1056
25545	25546	$3\times5\times13\times131$	$2\times53\times241$	12480
26047	26048	7×61^2	$2^6\times11\times37$	360
26864	26865	$2^4\times23\times73$	$3^3\times5\times199$	1584
28251	28252	$3^2\times43\times73$	$2^2\times7\times1009$	6048
34936	34937	$2^3\times11\times397$	$7^2\times23\times31$	3960
37248	37249	$2^7\times3\times97$	193^2	192
56574	56575	$2^2\times3\times29\times163$	$5^2\times2269$	9072
65575	65576	$5^2\times37\times71$	$2^2\times3\times13\times421$	10080
81732	81733	$2^2\times3\times7^2\times139$	37×47^2	1656
82368	82969	$2^6\times3^2\times11\times13$	$7^2\times41^2$	240
87308	87309	$2^2\times13\times23\times73$	$3^2\times89\times109$	19008
87367	87368	$7^2\times1783$	$2^3\times67\times163$	10692
88450	88451	$2\times5^2\times29\times61$	$11^2\times17\times43$	6720
91539	91539	$3^2\times7\times1453$	$2^2\times5\times23\times199$	17424

TABLE 19. Values of $\varphi(n)=\varphi(n+1)=U(n)=U(n+1)$ for $\left(1\le n\le 10^6\right)$

n	$n+1$	*Factor of n*	*Factor of n+1*	$U(n)$	*Factor of U(n)*
1	2	•	•	1	•
194	195	2×97	$3\times5\times13$	96	$2^5\times3$
3705	3706	$3\times5\times13\times19$	$2\times17\times109$	1728	$2^6\times3^3$
5186	5187	2×2593	$3\times7\times13\times19$	2592	$2^5\times3^4$
25545	25546	$3\times5\times13\times131$	$2\times53\times141$	12480	$2^6\times3\times5\times13$
388245	388246	$3\times5\times11\times13\times181$	$2\times17\times19\times601$	172800	$2^8\times3^3\times5^2$

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