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ON e-CONVEXITY

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Abstract. In this paper, we examine a generalized convexity type inequality, called e-convexity.

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1. Introduction

Throughout this paper denote by $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}$, and $\mathbb{N}$ the sets of real numbers, nonnegative real numbers, integers, and positive integers, respectively, and denote I by a nonempty subinterval of $\mathbb{R}$.

The stability theory of convexity started with the paper [2] of Hyers and Ulam who defined the ε-convex functions: If D is a convex subset of a real linear space X and ε is a nonnegative number, then a function $f : D \to R$ is called ε-convex, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)+\varepsilon$$

for all $x, y \in D$, $t \in [0,1]$. The basic result obtained by Hyers and Ulam states that if the underlying space X is finite dimension then f can be written as $f = g+h$, where g is a convex function and h is a bounded function whose supremum norm is not larger than $k_n\varepsilon$, where the positive constant k_n depends only on the dimension of the underlying space X.

In [7], Páles introduced a more general notion than ε-convexity. Let ε, δ be nonnegative constants. A function $f : D \to \mathbb{R}$ is called (ε, δ)-convex, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)+\delta+\varepsilon t(1-t)\|x-y\|$$

for every $x, y \in D$ and $t \in [0,1]$. The main results of the paper [7] obtain a complete characterization of (ε, δ)-convexity, if $D \subset \mathbb{R}$ is an open real interval by showing that these functions are of the form $f = g+h+l$, where g is convex, h is bounded with $|h| \leq \delta/2$ and l is Lipschitzian with Lipschitz modulus $\mathrm{Lip}(l) \leq \varepsilon$.

In [1], Alizadeh and Roohi introduced a general convexity notion, the so-called σ-convexity, namely let $\sigma : D \to \mathbb{R}$ be a nonnegative function. We say that a function

$f : D \to \mathbb{R}$ is σ-convex, if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+t(1-t)\min(\sigma(x),\sigma(y))\|x-y\|$$

for all $x,y \in D$ and $t \in [0,1]$.

In this paper the relations between the σ-monotonicity and σ-convexity were investigated. Moreover, some results on the sum and difference of two σ-monotone operator was considered. In this paper, we would like to generalize the notion of σ-convexity and we would like to consider the basic properties of this generalized convexity. Namely, we will characterize e-convexity in the real case, give a kind of strengthening of e-convexity, give Bernstein–Doetcsh type result, search relations between Hermite–Hadamard type inequalities and e-convexity.

2. Main results

Let X be a linear space and D be a nonempty convex subset of X, moreover let $e : D \times D \to [0,\infty[$ be a nonnegative, symmetric error function. We say that $f : D \to \mathbb{R}$ is *e-convex*, if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+t(1-t)e(x,y) \qquad (t \in [0,1],\, x,y \in D). \tag{2.1}$$

If the above inequality stands for a $t \in]0,1[$, we say that the function f is (t,e)*-convex*.

Remark 1. The e-convexity reduces to

1) convexity if $e(x,y) = 0$, for all $x,y \in D$;
2) ε-convexity if $e(x,y) = \varepsilon\|x-y\|$ for all $x,y \in D$ and for a fixed $\varepsilon \ge 0$;
3) paraconvexity if $e(x,y) = C\|x-y\|^2$ for all $x,y \in D$ and for a fixed $C \in \mathbb{R}$;
4) $\alpha(\cdot)$-paraconvexity if $e(x,y) = C\alpha(\|x-y\|)$, for all $x,y \in D$, where $C > 0$ and α is a nondecreasing function mapping the interval $[0,+\infty[$ into the interval $[0,\infty[$. (see [6])
5) σ-convexity, if $e(x,y) = \min(\sigma(x),\sigma(y))\|x-y\|$, if X is a normed space, and $\sigma : D \to \mathbb{R}$ be a nonnegative function.

In the following theorem, we would like to give a strengthening type result. This result is similar as in [3].

Theorem 1. *If the function f is e-convex on D, then the following e-convexity type inequality also holds,*

$$\begin{aligned} f(tx+(1-t)y) &\le tf(x)+(1-t)f(y)+t(1-t)e(x,y) \\ &\quad + \frac{t-r}{s-r}\cdot\frac{s-t}{s-r}\Big(e(sx+(1-s)y,rx+(1-r)y)-(s-r)^2e(x,y)\Big) \end{aligned} \tag{2.2}$$

for all $x,y \in D$, and $0 \le r \le t \le s \le 1$.

Proof. Write, in (2.1), x by $sx+(1-s)y$ and y by $rx+(1-r)y$, we can get that

$$f\big((ts+(1-t)r)x+(1-(ts+(1-t)r))y\big)$$

$$
\begin{aligned}
&= f\big(t(sx+(1-s)y)+(1-t)(rx+(1-r)y)\big)\\
&\le tf(sx+(1-s)y)+(1-t)f(rx+(1-r)y)\\
&\qquad +t(1-t)e(sx+(1-s)y,rx+(1-r)y)\\
&\le t\Big(sf(x)+(1-s)f(y)+s(1-s)e(x,y)\Big)\\
&\quad +(1-t)\Big(rf(x)+(1-r)f(y)+r(1-r)e(x,y)\Big)\\
&\qquad +t(1-t)e(sx+(1-s)y,rx+(1-r)y)\\
&\le (ts+(1-t)r)f(x)+(1-(ts+(1-t)r))f(y)\\
&\qquad +ts(1-s)e(x,y)+(1-t)r(1-r)e(x,y)\\
&\quad +t(1-t)e(sx+(1-s)y,rx+(1-r)y).
\end{aligned}
$$

Let $u = ts+(1-t)r$, then $t = \frac{u-r}{s-r}$ and from the previous inequality we have that,

$$
\begin{aligned}
f(ux+(1-u)y) \le\ & uf(x)+(1-u)f(y)+u(1-u)e(x,y)\\
&+\big(ts(1-s)+(1-t)r(1-r)-u(1-u)\big)e(x,y)\\
&+t(1-t)e(sx+(1-s)y,rx+(1-r)y).
\end{aligned}
$$

Applying the previous substitution, we have (2.2), which proves the statement. □

Remark 2. According to the previous theorem, we may assume that the plus error term is nonnegative, namely

$$
e(sx+(1-s)y,rx+(1-r)y)-(s-r)^2e(x,y)\ge 0 \quad \text{for all} \quad 0\le r\le s\le 1.
$$

If not, we can strengthen our error term with the e-type function in (2.2). This means that the error e has the property of superquadratic. For example, in the case of normed space, if $e(x,y)=\|x-y\|^p$, where $p>0$. We can get $p\le 2$.

Theorem 2. *The e-convexity of the function $f: D\to\mathbb{R}$ is equivalent with the following property: For all $x_1,\dots,x_n\in D$, $t_i\ge 0$ with $\sum_{i=1}^n t_i = 1$,*

$$
\begin{aligned}
&f\left(\sum_{i=1}^n t_ix_i\right)\\
&\le \sum_{i=1}^n t_if(x_i)+\sum_{j=1}^n\sum_{i=1}^{j-1}\frac{t_it_j}{\left(\sum_{k=1}^j t_k\right)^2}\cdot \\
&\quad \cdot e\left(\left(\sum_{k=1}^j t_k\right)x_i+\sum_{k=j+1}^n (t_kx_k),\left(\sum_{k=1}^j t_k\right)x_{i+1}+\sum_{k=j+1}^n (t_kx_k)\right).
\end{aligned}
\tag{2.3}
$$

Proof. Assume that f is e-convex on D. We will show (2.3) by induction. If $n=2$, we have the e-convexity of f. Let us assume that (2.3) satisfies for $n\in\mathbb{N}$. Let's consider the case $n+1$. Let $x_1,\dots,x_n,x_{n+1}\in I$ and $t_i\ge 0$ with $\sum_{i=1}^{n+1} t_i = 1$. If

$t_{n+1} = 1$, the statement is true. If it is not, then $1 - t_{n+1} = \sum_{i=1}^{n} t_i$. Then using the inductive assumption, and some simple computation, finally the e-convexity of f, we can get that,

$$
\begin{aligned}
&f\left(\sum_{j=1}^{n+1} t_j x_j\right) = f\left(\sum_{j=1}^{n} \frac{t_j}{1-t_{n+1}}\left((1-t_{n+1})x_j + t_{n+1}x_{n+1}\right)\right)\\
&\le \sum_{j=1}^{n} \frac{t_j}{1-t_{n+1}} f\left((1-t_{n+1})x_j + t_{n+1}x_{n+1}\right)\\
&+ \sum_{j=1}^{n}\sum_{i=1}^{j-1} \frac{\frac{t_i}{1-t_{n+1}}\frac{t_j}{1-t_{n+1}}}{\left(\sum_{k=1}^{j}\frac{t_k}{1-t_{n+1}}\right)^2} e\left(\left(\sum_{k=1}^{j}\tfrac{t_k}{1-t_{n+1}}\right)\left((1-t_{n+1})x_i + t_{n+1}x_{n+1}\right)\right.\\
&\qquad + \sum_{k=j+1}^{n} \tfrac{t_k}{1-t_{n+1}}\left((1-t_{n+1})x_k + t_{n+1}x_{n+1}\right),\\
&\left.\left(\sum_{k=1}^{j}\tfrac{t_k}{1-t_{n+1}}\right)\left((1-t_{n+1})x_{i+1} + t_{n+1}x_{n+1}\right) + \sum_{k=j+1}^{n}\tfrac{t_k}{1-t_{n+1}}\left((1-t_{n+1})x_k + t_{n+1}x_{n+1}\right)\right)\\
&= \sum_{j=1}^{n} \frac{t_j}{1-t_{n+1}} f\left((1-t_{n+1})x_j + t_{n+1}x_{n+1}\right)\\
&+ \sum_{j=1}^{n}\sum_{i=1}^{j-1} \frac{t_i t_j}{\left(\sum_{k=1}^{j} t_k\right)^2} e\left(\left(\sum_{k=1}^{j} t_k\right)x_i + \sum_{k=j+1}^{n}(t_k x_k) + t_{n+1}x_{n+1},\right.\\
&\qquad \left.\left(\sum_{k=1}^{j} t_k\right)x_{i+1} + \sum_{k=j+1}^{n}(t_k x_k) + t_{n+1}x_{n+1}\right)\\
&\le \sum_{j=1}^{n} \frac{t_j}{1-t_{n+1}}\left((1-t_{n+1})f(x_j) + t_{n+1}f(x_{n+1}) + t_{n+1}(1-t_{n+1})e(x_j, x_{n+1})\right)\\
&+ \sum_{j=1}^{n}\sum_{i=1}^{j-1} \frac{t_i t_j}{\left(\sum_{k=1}^{j} t_k\right)^2} e\left(\left(\sum_{k=1}^{j} t_k\right)x_i + \sum_{k=j+1}^{n+1}(t_k x_k), \left(\sum_{k=1}^{j} t_k\right)x_{i+1} + \sum_{k=j+1}^{n+1}(t_k x_k)\right)\\
&= \sum_{j=1}^{n+1} t_j f(x_j) + \sum_{j=1}^{n} t_j t_{n+1} e(x_j, x_{n+1})\\
&+ \sum_{j=1}^{n}\sum_{i=1}^{j-1} \frac{t_i t_j}{\left(\sum_{k=1}^{j} t_k\right)^2} e\left(\left(\sum_{k=1}^{j} t_k\right)x_i + \sum_{k=j+1}^{n+1}(t_k x_k), \left(\sum_{k=1}^{j} t_k\right)x_{i+1} + \sum_{k=j+1}^{n+1}(t_k x_k)\right)\\
&= \sum_{j=1}^{n+1} t_j f(x_j)
\end{aligned}
$$

$$+\sum_{j=1}^{n+1}\sum_{i=1}^{j-1}\frac{t_it_j}{\left(\sum_{k=1}^{j}t_k\right)^2}e\left(\left(\sum_{k=1}^{j}t_k\right)x_i+\sum_{k=j+1}^{n+1}(t_kx_k),\left(\sum_{k=1}^{j}t_k\right)x_{i+1}+\sum_{k=j+1}^{n+1}(t_kx_k)\right),$$

which proves the statement. The substitution $n=2$ give that the implication (iii)→(i) also holds. □

Theorem 3. *Let I be an open interval in $\mathbb{R}$, then $f: I\to\mathbb{R}$ is e-convex on I, if and only if for all $x<u<y$ from I,*

$$\frac{f(u)-f(x)}{u-x}\le\frac{f(y)-f(u)}{y-u}+\frac{e(x,y)}{y-x} \tag{2.4}$$

holds.

Proof. Assume that f is e-convex on I, then substituting $tx+(1-t)y$ by u, $x<u<y$ in (2.1), we can get that $t=\frac{y-u}{y-x}$ and

$$f(u)\le\frac{y-u}{y-x}f(x)+\frac{u-x}{y-x}f(y)+\frac{y-u}{y-x}\cdot\frac{u-y}{y-x}e(x,y).$$

Rearranging the above inequality we can get (2.4).

The implication (ii)→ (i) is also a simple calculation. Namely with the substitution $u=tx+(1-t)y$ we have the e-convexity of f. □

Corollary 1. *If $f: I\to\mathbb{R}$ is differentiable and e-convex on I, then*

$$f(x)-f(y)\ge f'(y)(x-y)-e(x,y)\qquad(x,y\in I). \tag{2.5}$$

Proof. Taking the limit $y\to u$ in (2.4), we have (2.5). □

Corollary 2. *If $f: I\to\mathbb{R}$ is differentiable and e-convex, then*

$$(f'(x)-f'(y))(x-y)\ge -2e(x,y)\qquad(x,y\in I). \tag{2.6}$$

Proof. Let $x,y\in I$, then using (2.5) and applying the substitution x by y and y by x, and adding the two inequalities, we have (2.6). □

Proposition 1. *Let $I=[a,b]$. If $e: I\times I\to[0,\infty[$ is upper semicontinuous and f is e-convex, then f is continuous.*

Proof. Assume that x_0 in I and (x_n) is a sequence in $]x_0,b[$, converging to x_0. Then,

$$x_n=\lambda_nb+(1-\lambda_n)x_0\qquad\text{with}\qquad\lambda_n\to 0.$$

On the other hand $x_n\in]a,x_0[$. Thus there exists $\lambda_n'\in[0,1]$, such that

$$x_0=\lambda_n'a+(1-\lambda_n')x_n\qquad\text{with}\qquad\lambda_n'\to 0.$$

Since f is e-convex, we have that

$$f(x_n)\le\lambda_nf(b)+(1-\lambda_n)f(x_0)+\lambda_n(1-\lambda_n)e(b,\lambda_nb+(1-\lambda_n)x_0).$$

Therefore, by taking the $\limsup$ in the above inequality we have that

$$\limsup_{n\to\infty} f(x_n) \le f(x_0).$$

However,

$$f(x_0) \le \lambda'_n f(a) + (1-\lambda'_n) f(x_n) + \lambda'_n(1-\lambda'_n) e(a, \lambda'_n a + (1-\lambda_n)x_n).$$

Taking the $\liminf$, we have that

$$f(x_0) \le \liminf f(x_n).$$

□

Remark 3. Assume that $0 \in I$ and $f : I \to \mathbb{R}$ is e-convex. If $f(0) \le 0$ and $e(x,0) = 0$ for all $x \in I$, then f is super-additive on $I \cap [0,\infty)$. Indeed, by the e-convexity of f, we have

$$f(tx) = f(tx + (1-t)0) \le tf(x) + (1-t)f(0) + t(1-t)e(x,0) \le tf(x).$$

On the other hand, for all $x, y \in I$

$$\begin{aligned} f(x) + f(y) &= f\left((x+y)\frac{x}{x+y}\right) + f\left((x+y)\frac{y}{x+y}\right) \\ &\le \frac{x}{x+y} f(x+y) + \frac{y}{x+y} f(x+y) = f(x+y) \end{aligned}$$

In what follows, we find connections between a lower Hermite–Hadamard type inequality and e-convexity. We will need the definition of hemi-property. The function $f : D \to \mathbb{R}$ has a *hemi-property*, if for all $x, y \in D$ the map

$$t \to f(tx + (1-t)y) \qquad t \in [0,1] \tag{2.7}$$

has got that property. For example $f : D \to \mathbb{R}$ is hemi-bounded, if for all $x, y \in D$ the function defined by (2.7) is bounded.

Now, we recall a theorem of [5].

Theorem 4. *Let D be a convex set of a linear space X. Let $\mathscr{A}$ be a sigma algebra containing the Borel subsets of $[0,1]$ and μ be a probability measure on the measure space $([0,1], \mathscr{A})$ such that the support of μ is not a singleton. Denote*

$$S(\mu) := \mu\big([0,\mu_1]\big) \int_{]\mu_1,1]} t\,d\mu(t) - \mu\big(]\mu_1,1]\big) \int_{[0,\mu_1]} t\,d\mu(t). \tag{2.8}$$

Assume that $f : D \to \mathbb{R}$ is an hemi-μ-integrable solution of the functional inequality

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y) + e_{x,y}(t) \quad ((x,y) \in D^2, t \in [0,1]), \tag{2.9}$$

where, for all $(x,y) \in D^{2*}$, $e_{x,y} : [0,1] \to \mathbb{R}$ is a function such that

$$I(x,y) := \int_{]\mu_1,1]} \int_{[0,\mu_1]} (t'' - t') e_{(1-t')x+t'y,(1-t'')x+t''y}\left(\frac{\mu_1 - t'}{t'' - t'}\right) d\mu(t') d\mu(t'') \tag{2.10}$$

exists in $[-\infty,\infty]$ *for all* $(x,y)\in D^{2*}$*. Then, for all* $(x,y)\in D^{2*}$*, the function* f *also satisfies the lower Hermite–Hadamard type inequality*

$$f((1-\mu_1)x+\mu_1 y)\leq \int\limits_{[0,1]} f\big((1-t)x+ty\big)d\mu(t)+E(x,y) \quad ((x,y)\in D^2), \tag{2.11}$$

where

$$E(x,y):=\frac{I(x,y)}{S(\mu)} \qquad ((x,y)\in D^{2*}). \tag{2.12}$$

The following result gives a lower Hermite–Hadamard type inequality for e-convex functions and it is a simple connection of the previous theorem.

Corollary 3. *Let* D *be a convex set of a linear space* X*. Let* $\mathscr{A}$ *be a sigma algebra containing the Borel subsets of* $[0,1]$ *and* μ *be a probability measure on the measure space* $([0,1],\mathscr{A})$ *such that the support of* μ *is not a singleton. Let* $S(\mu)$ *defined by* (2.8)*. Assume that* $f:D\to\mathbb{R}$ *is an hemi-*μ*-integrable solution of the e-convexity type inequality* (2.1)*, moreover let* $I(x,y)$ *- defined by* (2.10) *- exist in* $[-\infty,\infty]$*, for all* $(x,y)\in D^2$*. Then* f *satisfies the lower Hermite–Hadamard type inequality,* (2.11)*, where* E *is defined by* (2.12)*.*

Now, we apply this corollary for Lebesgue integral.

Corollary 4. *Let* D *be a convex set of a linear space* X*. Assume that* $f:D\to\mathbb{R}$ *is an hemi-Lebesgue-integrable solution of the e-convexity inequality* (2.1)*. Then* f *satisfies the following lower Hermite–Hadamard type inequality,*

$$f\Big(\frac{x+y}{2}\Big)\leq \int_0^1 f\big((1-t)x+ty\big)dt+4I(x,y) \quad ((x,y)\in D^2), \tag{2.13}$$

where

$$I(x,y):=\int\limits_{\frac12}^{1}\int\limits_{0}^{\frac12}\frac{(\frac12-t')(t''-\frac12)}{t''-t'}e((1-t')x+t'y,(1-t'')x+t''y)dt'dt''.$$

Proof. Denote by λ the Lebesgue measure on $[0,1]$. Then $\lambda_1=\int_0^1 t dt=\frac12$. On the other hand,

$$\begin{aligned}S(\lambda)&:=\lambda\big([0,\lambda_1]\big)\int\limits_{]\lambda_1,1]} tdt-\lambda\big(]\lambda_1,1]\big)\int\limits_{[0,\lambda_1]} tdt\\&=\lambda\big([0,\tfrac12]\big)\int\limits_{\frac12}^{1} tdt-\lambda\big(]\tfrac12,1]\big)\int\limits_{0}^{\frac12} tdt=\frac14.\end{aligned}$$

□

Now, we recall a result from [4].

Theorem 5. *Let μ be a Borel probability measure on $[0,1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Assume that, for all $(x,y) \in D^2$, $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality (2.13), where $E : D^2 \to \mathbb{R}$. Assume that, for all $(x,y) \in D^2$, $e_{x,y} : [0,1] \to \mathbb{R}$ is a lower semicontinuous function with $e_{x,y}(0) = e_{x,y}(1) = 0$ satisfying the following system of inequalities:*

$$e_{x,y}(s) \geq \begin{cases} \int\limits_{[0,1]} e_{x,y}\big(\frac{st}{\mu_1}\big)d\mu(t) + E\big(x, (1-\frac{s}{\mu_1})x + \frac{s}{\mu_1}y\big) & (s \in [0,\mu_1]), \\ \int\limits_{[0,1]} e_{x,y}\big(1 - \frac{(1-s)(1-t)}{1-\mu_1}\big)d\mu(t) + E\big(\frac{1-s}{1-\mu_1}x + (1-\frac{1-s}{1-\mu_1})y, y\big) & (s \in [\mu_1, 1]). \end{cases} \tag{2.14}$$

Then, for all $(x,y) \in D^2$ and $s \in [0,1]$, the function f also satisfies the approximate convexity inequality (2.9).

The following proposition states that from Hermite–Hadamard type inequality, we can get e-convexity.

Corollary 5. *Let μ be a Borel probability measure on $[0,1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$, $\mu_2 = \int_{[0,1]} t^2 d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Assume that, for all $(x,y) \in D^2$, $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality (2.13), where $E : D^2 \to \mathbb{R}$. Assume that, for all $(x,y) \in D^2$, $e : D \times D \to \mathbb{R}$ is a lower semicontinuous function with satisfying the following system of inequalities:*

$$\begin{cases} s^2\big(\frac{\mu_2}{\mu_1^2} - 1\big)e_{x,y}(s) \geq E\big(x, (1-\frac{s}{\mu_1})x + \frac{s}{\mu_1}y\big) & (s \in [0,\mu_1]), \\ (1-s)^2\big(\frac{1-2\mu_1+\mu_2}{(1-\mu_1)^2}\big)e(x,y) \geq E\big(\frac{1-s}{1-\mu_1}x + (1-\frac{1-s}{1-\mu_1})y, y\big) & (s \in [\mu_1, 1]). \end{cases} \tag{2.15}$$

Then, the function f is e-convex on D.

Proof. Define for $x, y \in D$ and $t \in [0,1]$, the function $e_{x,y}$ by the following formulae:

$$e_{x,y}(t) = t(1-t)e(x,y)$$

Simple calculations shows that (2.14) reduces (2.15). Using the previous theorem, we have the e-convexity of f. □

Corollary 6. *Let $d : D \times D \to [0,\infty[$ be a symmetric function. Assume that $f : D \to \mathbb{R}$ is a hemi-continuous solution of the functional inequality,*

$$f\Big(\frac{x+y}{2}\Big) \leq \int_0^1 f(tx + (1-t)y)dt + d(x,y), \qquad (x, y \in D)$$

with

$$\frac{1}{12}s^2 e(x,y) \geq d(x, (1-s)x + sy) \qquad (s \in [0,1], x, y \in D). \tag{2.16}$$

Then f is e-convex.

Proof. In this case, μ is the Lebesgue measure, which is denoted by λ. $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{3}$. Using the symmetry of the function d it is also easy to see that (2.15) reduces (2.16). Applying the previous corollary, we can get the e-convexity of f. □

The following proposition states a Bernstein–Doetsch type theorem for e-convexity.

Corollary 7. *Let X is normed space and D is a nonempty, open and convex subset of X. Let $d : D \times D \to [0,\infty[$ be a symmetric function. Let $f : D \to \mathbb{R}$ be a continuous solution of the following functional inequality,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + d(x,y)$$

Assume that $e : D \times D \to [0,\infty[$ is symmetric and it satisfies the following functional inequality,

$$\frac{s^2}{4} e(x,y) \geq d(x,(1-s)x+sy) \qquad s \in [0,1], s \in [0,1].$$

Then f is e-convex.

Proof. Let μ be the Dirac-measure which concentrated to $\frac{1}{2}$. Then, from Corollary 6, we can get the statement. □

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References

[1] M. H. Alizadeh and M. Roohi, "Some results on pre-monotone operators," *Bulletin of the Iranian Mathematical Society*, vol. 43, no. 6, pp. 2085–2097, 2017.

[2] D. Hyers and S. Ulam, "Approximately convex functions." *Proc. Am. Math. Soc.*, vol. 3, pp. 821–828, 1952, doi: 10.2307/2032186.

[3] J. Makó and Zs. Páles, "Strengthening of strong and approximate convexity." *Acta Math. Hung.*, vol. 132, no. 1-2, pp. 78–91, 2011, doi: 10.1007/s10474-010-0056-0.

[4] J. Makó and Zs. Páles, "Korovkin type theorems and approximate Hermite-Hadamard inequalities." *J. Approx. Theory*, vol. 164, no. 8, pp. 1111–1142, 2012, doi: 10.1016/j.jat.2012.05.010.

[5] J. Makó and Zs. Páles, "Approximate Hermite-Hadamard type inequalities for approximately convex functions." *Math. Inequal. Appl.*, vol. 16, no. 2, pp. 507–526, 2013, doi: 10.7153/mia-16-37.

[6] S. Rolewicz, "Paraconvex analysis," *Control and Cybernetics*, vol. 34, no. 3, pp. 951–965, 2005.

[7] Zs. Páles, "On approximately convex functions." *Proc. Am. Math. Soc.*, vol. 131, no. 1, pp. 243–252, 2003, doi: 10.1090/S0002-9939-02-06552-8.

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