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SOME INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR s-GEOMETRICALLY CONVEX FUNCTIONS

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Abstract. In the paper, the authors present some integral inequalities of the Hermite–Hadamard type for s-geometrically convex functions and for the product of two s-geometrically convex functions.

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1. Introduction

We recall the definitions of classical convex functions and geometrically convex functions.

Definition 1. Let $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$. If the inequality

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

is valid for all $x, y \in I$ and $\lambda \in [0,1]$, then f is called the convex function on I; if the inequality (1.1) reverses, then f is called the concave function on I.

Definition 2. Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}_+$. If the inequality

$$f\left(x^{\lambda} y^{1-\lambda}\right) \le f^{\lambda}(x) f^{1-\lambda}(y) \tag{1.2}$$

is sound for any $x, y \in I$ and $\lambda \in [0,1]$, then f is called the geometrically convex function; if the inequality of (1.2) reverses, then f is called the geometrically concave function.

The concept of classical convex functions has been generalized or extended widely in recent decades. Some of them can be reformulated as follows.

Definition 3 ([3,5])**.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_0 = [0, \infty)$ and $s \in (0,1]$. If the inequality

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$, then f is called the s-convex function on I.

Definition 4 ([14, 15]). Let $s \in (0,1]$ and $f : I \subseteq R_+ \to R_+$. If the inequality

$$f\left(x^{\lambda} y^{1-\lambda}\right) \le f^{\lambda^s}(x) f^{(1-\lambda)^s}(y)$$

validates for any $x, y \in I$ and $\lambda \in [0,1]$, then f is called the s-geometrically convex function on I.

For classical convex functions, we have the famous Hermite–Hadamard integral inequality below.

Theorem 1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

If f is a concave function on I, then the inequality (1.3) reverses.

In the literature, there have existed some integral inequalities of the Hermite–Hadamard type on classical convex functions and s-convex functions. Some of them can be reformulated as follows.

Theorem 2 ([4, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)}{8}\big(|f'(a)| + |f'(b)|\big).$$

Theorem 3 ([7, Theorems 2.3 and 2.4]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^p$ is s-convex on $[a,b]$ for some $s \in (0,1]$ and $p > 1$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\big(|f'(a)| + |f'(b)|\big)$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\Big\{\big[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}\big]^{1-1/p} + \big[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\big]^{1-1/p}\Big\}.$$

Theorem 4 ([6, Theorem 3]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for some $s \in (0,1]$ and $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2}\left[\frac{q-1}{2(2q-1)}\right]^{1/p}\left(\frac{1}{s+1}\right)^{1/q} \times \left\{\left[|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q} + \left[|f'(b)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\}.$$

In recent decades, many new integral inequalities of the Hermite–Hadamrd type for diverse new kinds of convex functions have been created and established. For detailed information, please refer to [1–4, 6–15] and closely related references therein.

In this paper, we will establish some new integral inequalities of the Hermite–Hadamard type for s-geometrically convex functions.

2. Some new integral inequalities

We now start out to establish some integral inequalities of the Hermite–Hadamard type for s-geometrically convex functions.

Theorem 5. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable function, $a, b \in I$ with $a < b$, and $s \in (0,1]$. If f is an s-geometrically convex function, then*

$$f^{(1/2)^{1-s}}\big(\sqrt{ab}\big) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x \le [f(a)f(b)]^{1-s} L\big(f^s(a), f^s(b)\big),$$

where the logarithmic mean $L(u,v)$ is defined by

$$L(u,v) = \begin{cases} \dfrac{v-u}{\ln v - \ln u}, & u \ne v; \\ u, & u = v. \end{cases} \tag{2.1}$$

Proof. By changing the variable $x = a^t b^{1-t}$ for $t \in [0,1]$ and by the s-geometric convexity, we have

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x = \int_0^1 f\big(a^t b^{1-t}\big)\,\mathrm{d}t \le \int_0^1 f^{t^s}(a) f^{(1-t)^s}(b)\,\mathrm{d}t.$$

In [2], it was obtained that the inequality

$$\eta^{t^s} \le \eta^{st+1-s} \tag{2.2}$$

is valid for $\eta \ge 1$, $0 \le t \le 1$, and $0 < s \le 1$. When $s \in (0,1)$, the s-geometrically convex function satisfies $f(x) \ge 1$. Consequently, since $f(a), f(b) \ge 1$, we arrive at

$$\begin{aligned}\int_0^1 f^{t^s}(a) f^{(1-t)^s}(b)\,\mathrm{d}t &\le \int_0^1 f^{st+1-s}(a) f^{s(1-t)+1-s}(b)\,\mathrm{d}t \\ &= [f(a)f(b)]^{1-s} L\big(f^s(a), f^s(b)\big).\end{aligned}$$

Due to $\sqrt{ab} = \sqrt{a^t b^{1-t} b^t a^{1-t}}$ for all $t \in [0,1]$, by the s-geometric convexity, we find

$$f^{(1/2)^{1-s}}\big(\sqrt{ab}\big) \le \big[f\big(a^t b^{1-t}\big) f\big(b^t a^{1-t}\big)\big]^{1/2} \le \frac{f\big(a^t b^{1-t}\big) + f\big(b^t a^{1-t}\big)}{2}.$$

Integrating with respect to $t \in [0,1]$ on both sides of the above inequality gains

$$f^{(1/2)^{1-s}}\big(\sqrt{ab}\big) \le \frac{1}{2}\int_0^1 \big[f\big(a^t b^{1-t}\big) + f\big(b^t a^{1-t}\big)\big]\,\mathrm{d}t = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x.$$

The proof of Theorem 5 is complete. □

Theorem 6. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable function, $a,b \in I$ with $a < b$, and $s \in (0,1]$. If f is an s-geometrically convex function on $[a,b]$, then*

$$\sqrt{ab}\, f^{(1/2)^{1-s}}\big(\sqrt{ab}\big) \le \frac{1}{\ln b - \ln a}\int_a^b f(x)\,\mathrm{d}x \le [f(a)f(b)]^{1-s} L\big(af^s(a), bf^s(b)\big).$$

where $L(u,v)$ is the logarithmic mean defined by (2.1).

Proof. Taking $x = a^t b^{1-t}$ for $t \in [0,1]$, using Definition 4, and employing the inequality (2.2) lead to

$$\begin{aligned}\frac{1}{\ln b - \ln a}\int_a^b f(x)\,\mathrm{d}x &= \int_0^1 a^t b^{1-t} f\big(a^t b^{1-t}\big)\,\mathrm{d}t \le \int_0^1 a^t b^{1-t} f^{t^s}(a) f^{(1-t)^s}(b)\,\mathrm{d}t\\ &\le \int_0^1 a^t b^{1-t} f^{st+1-s}(a) f^{s(1-t)+1-s}(b)\,\mathrm{d}t\\ &= [f(a)f(b)]^{1-s} L\big(af^s(a), bf^s(b)\big)\end{aligned}$$

and

$$\begin{aligned}\sqrt{ab}\, f^{(1/2)^{1-s}}\big(\sqrt{ab}\big) &= \int_0^1 \sqrt{a^t b^{1-t} b^t a^{1-t}}\, f^{(1/2)^{1-s}}\Big(\sqrt{a^t b^{1-t} b^t a^{1-t}}\Big)\,\mathrm{d}t\\ &\le \int_0^1 \sqrt{a^t b^{1-t} b^t a^{1-t}}\, \big[f\big(a^t b^{1-t}\big) f\big(b^t a^{1-t}\big)\big]^{1/2}\,\mathrm{d}t\\ &\le \frac{1}{2}\int_0^1 \big[a^t b^t f\big(a^t b^{1-t}\big) + b^t a^{1-t} f\big(b^t a^{1-t}\big)\big]\,\mathrm{d}t\\ &= \frac{1}{\ln b - \ln a}\int_a^b f(x)\,\mathrm{d}x.\end{aligned}$$

The proof of Theorem 6 is complete. □

Theorem 7. *Let $f,g : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be integrable functions, $a,b \in I$ with $a < b$, and $s_1, s_2 \in (0,1]$. If f is an s_1-geometrically convex function and g is an s_2-geometrically convex function on $[a,b]$, then*

$$\begin{aligned}f^{(1/2)^{1-s_1}}\big(\sqrt{ab}\big) g^{(1/2)^{1-s_2}}\big(\sqrt{ab}\big) &\le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)g(x)}{x}\,\mathrm{d}x\\ &\le [f(a)f(b)]^{1-s_1}[g(a)g(b)]^{1-s_2} L\big(f^{s_1}(a)g^{s_2}(a), f^{s_1}(b)g^{s_2}(b)\big),\end{aligned}$$

where $L(u,v)$ is the logarithmic mean defined by (2.1).

Proof. Letting $x = a^t b^{1-t}$ for $t \in [0,1]$, using the s-geometric convexity, and utilizing the inequality (2.2) result in

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)g(x)}{x}\,\mathrm{d}x = \int_0^1 f\big(a^t b^{1-t}\big) g\big(a^t b^{1-t}\big)\,\mathrm{d}t$$

$$\begin{aligned}
&\le \int_0^1 f^{t^{s_1}}(a) f^{(1-t)^{s_1}}(b) g^{t^{s_1}}(a) g^{(1-t)^{s_2}}(b)\,\mathrm{d}t \\
&\le \int_0^1 f^{s_1 t+1-s_1}(a) f^{s_2(1-t)+1-s_2}(b) g^{s_1 t+1-s_1}(a) g^{s_2(1-t)+1-s_2}(b)\,\mathrm{d}t \\
&= [f(a)f(b)]^{1-s_1}[g(a)g(b)]^{1-s_2} L\big(f^{s_1}(a)g^{s_2}(a), f^{s_1}(b)g^{s_2}(b)\big).
\end{aligned}$$

For $t \in [0,1]$, we have

$$\begin{aligned}
& f^{(1/2)^{1-s_1}}\big(\sqrt{ab}\big) g^{(1/2)^{1-s_2}}\big(\sqrt{ab}\big) \\
&= f^{(1/2)^{1-s_1}}\big(\sqrt{a^t b^{1-t} b^t a^{1-t}}\big) g^{(1/2)^{1-s_2}}\big(\sqrt{a^t b^{1-t} b^t a^{1-t}}\big) \\
&\le \big[f\big(a^t b^{1-t}\big) f\big(b^t a^{1-t}\big) g\big(a^t b^{1-t}\big) g\big(b^t a^{1-t}\big)\big]^{1/2} \\
&\le \frac{f\big(a^t b^{1-t}\big) g\big(a^t b^{1-t}\big) + f\big(b^t a^{1-t}\big) g\big(b^t a^{1-t}\big)}{2}.
\end{aligned}$$

Integrating on both sides with respect to $t \in [0,1]$ leads to

$$\begin{aligned}
& f^{(1/2)^{1-s_1}}\big(\sqrt{ab}\big) g^{(1/2)^{1-s_2}}\big(\sqrt{ab}\big) \\
&\le \frac{1}{2}\int_0^1 \big[f\big(a^t b^{1-t}\big) g\big(a^t b^{1-t}\big) + f\big(b^t a^{1-t}\big) g\big(b^t a^{1-t}\big)\big]\,\mathrm{d}t \\
&= \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)g(x)}{x}\,\mathrm{d}x.
\end{aligned}$$

The proof of Theorem 7 is complete. □

Theorem 8. *Let $f, g : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be integrable functions, $a, b \in I$ with $a < b$, and $s_1, s_2 \in (0,1]$. If f is an s_1-geometrically convex function and g is an s_2-geometrically convex function on $[a,b]$, then*

$$\begin{aligned}
\sqrt{ab}\, f^{(1/2)^{1-s_1}}\big(\sqrt{ab}\big) g^{(1/2)^{1-s_2}}\big(\sqrt{ab}\big) &\le \frac{1}{\ln b - \ln a}\int_a^b f(x)g(x)\,\mathrm{d}x \\
&\le [f(a)f(b)]^{1-s_1}[g(a)g(b)]^{1-s_2} L\big(a f^{s_1}(a) g^{s_2}(a), b f^{s_1}(b) g^{s_2}(b)\big),
\end{aligned}$$

where $L(u,v)$ is the logarithmic mean defined by (2.1).

Proof. Setting $x = a^t b^{1-t}$ for $t \in [0,1]$ and making use of Definition 4 arrive at

$$\begin{aligned}
\frac{1}{\ln b - \ln a}\int_a^b f(x)g(x)\,\mathrm{d}x &= \int_0^1 a^t b^{1-t} f\big(a^t b^{1-t}\big) g\big(a^t b^{1-t}\big)\,\mathrm{d}t \\
&\le \int_0^1 a^t b^{1-t} f^{t^{s_1}}(a) f^{(1-t)^{s_1}}(b) g^{t^{s_1}}(a) g^{(1-t)^{s_2}}(b)\,\mathrm{d}t \\
&\le [f(a)f(b)]^{1-s_1}[g(a)g(b)]^{1-s_2} L\big(a f^{s_1}(a) g^{s_2}(a), b f^{s_1}(b) g^{s_2}(b)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{ab}\, f^{(1/2)^{1-s_1}}\big(\sqrt{ab}\big) g^{(1/2)^{1-s_2}}\big(\sqrt{ab}\big) \\
&\le \int_0^1 \big(a^t b^{1-t} b^t a^{1-t}\big)^{1/2} \big[f\big(a^t b^{1-t}\big) f\big(b^t a^{1-t}\big) g\big(a^t b^{1-t}\big) g\big(b^t a^{1-t}\big)\big]^{1/2} \,\mathrm{d}t \\
&\le \frac{1}{2}\int_0^1 \big[a^t b^{1-t} f\big(a^t b^{1-t}\big) g\big(a^t b^{1-t}\big) + b^t a^{1-t} f\big(b^t a^{1-t}\big) g\big(b^t a^{1-t}\big)\big] \,\mathrm{d}t \\
&= \frac{1}{\ln b - \ln a} \int_a^b f(x) g(x) \,\mathrm{d}x.
\end{aligned}$$

The proof of Theorem 8 is thus complete. □

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