



IMPROVEMENT OF FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITY FOR CONVEX FUNCTIONS

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Abstract. In this paper, it is proved that fractional Hermite-Hadamard inequality and fractional Hermite-Hadamard-Fejér inequality are just results of Hermite-Hadamard-Fejér inequality. After this, a new fractional Hermite-Hadamard inequality which is not a result of Hermite-Hadamard-Fejér inequality and better than given in [9] by Sarikaya et al. is obtained. Also, a new equality is proved and some new fractional midpoint type inequalities are given. Our results generalizes the results given in [5] by Kirmacı.

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1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1.1)$$

is well known in the literature as Hermite-Hadamard's inequality [2,3].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequality or its weighted versions, the so-called Hermite-Hadamard-Fejér inequality.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be convex function. Then, the inequality*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \int_a^b f(x) g(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx \quad (1.2)$$

holds, where $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$ (i.e. $g(x) = g(a+b-x)$ for all $x \in [a, b]$).

In [5], Kirmacı used the following equality to obtain midpoint type inequalities and some applications:

Lemma 1. *Let $a, b \in I$ with $a < b$ and $f : I^\circ \rightarrow \mathbb{R}$ is a differentiable mapping (I° the interior of I). If $f' \in L[a, b]$, then we have*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \\ &= (b-a) \int_0^{1/2} t f'(ta + (1-t)b) dt + \int_{1/2}^1 (t-1) f'(ta + (1-t)b) dt. \end{aligned} \quad (1.3)$$

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 1. Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right Riemann-Liouville fractional integrals $R_{a+}^\theta f$ and $R_{b-}^\theta f$ of order $\theta > 0$ are defined by

$$R_{a+}^\theta f(x) = \frac{1}{\Gamma(\theta)} \int_a^x (x-t)^{\theta-1} f(t) dt, \quad x > a$$

and

$$R_{b-}^\theta f(x) = \frac{1}{\Gamma(\theta)} \int_x^b (t-x)^{\theta-1} f(t) dt, \quad x < b$$

respectively, where $\Gamma(\theta)$ is the Gamma function defined by $\Gamma(\theta) = \int_0^\infty e^{-t} t^{\theta-1} dt$ (see [6, page 69] and [10, page 4]).

In [9], Sarıkaya et al. proved the following fractional Hermite-Hadamard type inequality:

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\theta+1)}{2(b-a)^\theta} \left[R_{a+}^\theta f(b) + R_{b-}^\theta f(a) \right] \leq \frac{f(a) + f(b)}{2} \quad (1.4)$$

with $\theta > 0$.

Remark 1. In Theorem 2, it is not necessary supposing that f be a positive function and a, b are positive real numbers. From the Definition 1, it is clear that a, b are any real numbers such as $a < b$.

In [4], İşcan proved the following fractional Hermite-Hadamard-Fejér type inequality:

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \left[R_{a+}^{\theta} g(b) + R_{b-}^{\theta} g(a) \right] &\leq \left[R_{a+}^{\theta} (fg)(b) + R_{b-}^{\theta} (fg)(a) \right] \\ &\leq \frac{f(a) + f(b)}{2} \left[R_{a+}^{\theta} g(b) + R_{b-}^{\theta} g(a) \right] \end{aligned} \quad (1.5)$$

with $\theta > 0$.

In [7], Kunt et al. proved the following left Riemann-Liouville fractional Hermite-Hadamard type inequality and next equality:

Theorem 4. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. If $f \in L[a, b]$, then the following inequality for the left Riemann-Liouville fractional integral holds:

$$f\left(\frac{\theta a + b}{\theta + 1}\right) \leq \frac{\Gamma(\theta + 1)}{(b-a)^{\theta}} R_{a+}^{\theta} f(b) \leq \frac{\theta f(a) + f(b)}{\theta + 1} \quad (1.6)$$

with $\theta > 0$.

Lemma 2. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) . If $f' \in L[a, b]$, then the following equality for the left Riemann-Liouville fractional integrals holds:

$$\begin{aligned} \frac{\Gamma(\theta + 1)}{(b-a)^{\theta}} R_{a+}^{\theta} f(b) - f\left(\frac{\theta a + b}{\theta + 1}\right) \\ = (b-a) \left[\int_0^{\frac{\theta}{\theta+1}} t^{\theta} f'(ta + (1-t)b) dt \right. \\ \left. + \int_{\frac{\theta}{\theta+1}}^1 (t^{\theta} - 1) f'(ta + (1-t)b) dt \right] \end{aligned} \quad (1.7)$$

with $\theta > 0$.

In [8], Kunt et al. proved the following right Riemann-Liouville fractional Hermite-Hadamard type inequality and next equality:

Theorem 5. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. If $f \in L[a, b]$, then the following inequality for the right Riemann-Liouville fractional integral holds:

$$f\left(\frac{a + \theta b}{\theta + 1}\right) \leq \frac{\Gamma(\theta + 1)}{(b-a)^{\theta}} R_{b-}^{\theta} f(a) \leq \frac{f(a) + \theta f(b)}{\theta + 1} \quad (1.8)$$

with $\theta > 0$.

Lemma 3. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) . If $f' \in L[a, b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:

$$\begin{aligned} & \frac{\Gamma(\theta + 1)}{(b-a)^\theta} R_{b-}^\theta f(a) - f\left(\frac{a + \theta b}{\theta + 1}\right) \\ &= (b-a) \left[\int_0^{\frac{\theta}{\theta+1}} -t^\theta f'(tb + (1-t)a) dt \right. \\ & \quad \left. + \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta f'(tb + (1-t)a) dt \right] \end{aligned} \quad (1.9)$$

with $\theta > 0$.

In our studies we noticed that fractional Hermite-Hadamard type inequality given in Theorem 2 and fractional Hermite-Hadamard-Fejér type inequality given in Theorem 3 are just result of Hermite-Hadamard-Fejér inequality (given in Theorem 1), with a special selection of the weighted function. This show how strong the Hermite-Hadamard-Fejér inequality is. However, we will prove new fractional Hermite-Hadamard type inequality which is not a result of Theorem 1. Also, we will have new fractional midpoint type inequalities.

2. RESULTS OF HERMITE-HADAMARD-FEJÉR INEQUALITY

Proposition 1. Theorem 2 is a result of Theorem 1.

Proof. In Theorem 1, let we choose $g(x) = (x-a)^{\theta-1} + (b-x)^{\theta-1}$ for $\theta > 0$, $a, b \in \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ (It is clear $g(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$). Computing the following integrals, we have

$$\int_a^b g(x) dx = \int_a^b (x-a)^{\theta-1} + (b-x)^{\theta-1} dx = \frac{2(b-a)^\theta}{\theta}, \quad (2.1)$$

$$\begin{aligned} \int_a^b f(x)g(x) dx &= \int_a^b \left[(x-a)^{\theta-1} + (b-x)^{\theta-1} \right] f(x) dx \\ &= \int_a^b (x-a)^{\theta-1} f(x) dx + \int_a^b (b-x)^{\theta-1} f(x) dx \\ &= \Gamma(\theta) \left[R_{a+}^\theta f(b) + R_{b-}^\theta f(a) \right]. \end{aligned} \quad (2.2)$$

Combining (1.2), (2.1) and (2.2) we have (1.4). This completes the proof. \square

Proposition 2. Theorem 3 is a result of Theorem 1.

Proof. In Theorem 1, let we choose $w(x) = \left[(x-a)^{\theta-1} + (b-x)^{\theta-1} \right] g(x)$ for $\theta > 0$, $a, b \in \mathbb{R}$, $g : [a, b] \rightarrow \mathbb{R}$ and $g(x)$ nonnegative, integrable and symmetric to

$\frac{a+b}{2}$ (It is clear $w(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$). Computing the following integrals, we have

$$\begin{aligned} \int_a^b w(x) dx &= \int_a^b [(x-a)^{\theta-1} + (b-x)^{\theta-1}] g(x) dx & (2.3) \\ &= \int_a^b (x-a)^{\theta-1} g(x) dx + \int_a^b (b-x)^{\theta-1} g(x) dx \\ &= \Gamma(\theta) [R_{a+}^\theta g(b) + R_{b-}^\theta g(a)], \end{aligned}$$

$$\begin{aligned} \int_a^b f(x)w(x)dx &= \int_a^b [(x-a)^{\theta-1} + (b-x)^{\theta-1}] f(x)g(x) dx & (2.4) \\ &= \int_a^b (x-a)^{\theta-1} f(x)g(x) dx + \int_a^b (b-x)^{\theta-1} f(x)g(x) dx \\ &= \Gamma(\theta) [R_{a+}^\theta (fg)(b) + R_{b-}^\theta (fg)(a)]. \end{aligned}$$

Combining (1.2), (2.3) and (2.4) we have (1.5). This completes the proof. \square

Remark 2. Theorem 4 and Theorem 5 are not results of Theorem 1.

3. IMPROVEMENT OF FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITY

We will use Theorem 4 and Theorem 5 to have new fractional Hermite-Hadamard type inequality better than (1.4).

Theorem 6. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. If $f \in L[a, b]$, then the following inequality for fractional integral holds:

$$\frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \leq \frac{\Gamma(\theta+1)}{2(b-a)^\theta} [R_{a+}^\theta f(b) + R_{b-}^\theta f(a)] \leq \frac{f(a) + f(b)}{2} \tag{3.1}$$

with $\theta > 0$.

Proof. If (1.6) and (1.8) gather side by side and dividing into 2, it is hold the desired result. \square

Remark 3. Since, f is a convex function on $[a, b]$, it is clear $f\left(\frac{a+b}{2}\right) \leq \frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2}$ for $\theta > 0$. It means that

- (1) Theorem 6 is better than Theorem 2,
- (2) In Theorem 6 if one takes $\theta = 1$, one has (1.1),
- (3) Theorem 6 is not a result of Theorem 1.

4. NEW FRACTIONAL MIDPOINT TYPE INEQUALITIES

We will now prove an equality to have new fractional midpoint type inequalities.

Lemma 4. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) . If $f' \in L[a, b]$, then the following equality for the fractional integrals holds:*

$$\begin{aligned} & \frac{\Gamma(\theta+1)}{2(b-a)^\theta} \left[R_{a+}^\theta f(b) + R_{b-}^\theta f(a) \right] - \frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \\ &= \frac{b-a}{2} \left[\begin{aligned} & \int_0^{\frac{\theta}{\theta+1}} t^\theta f'(ta + (1-t)b) dt \\ & + \int_{\frac{\theta}{\theta+1}}^1 (t^\theta - 1) f'(ta + (1-t)b) dt \\ & + \int_0^{\frac{\theta}{\theta+1}} -t^\theta f'(tb + (1-t)a) dt \\ & + \int_{\frac{\theta}{\theta+1}}^1 (1-t^\theta) f'(tb + (1-t)a) dt \end{aligned} \right] \end{aligned} \quad (4.1)$$

Proof. If (1.7) and (1.9) gather side by side and dividing into 2, it is hold the desired result. \square

Corollary 1. *In Lemma 4, if one takes $\theta = 1$, one has Lemma 1.*

Theorem 7. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) . If $|f'|$ is convex on $[a, b]$, then the following fractional midpoint type inequality holds:*

$$\begin{aligned} & \left| \frac{\Gamma(\theta+1)}{2(b-a)^\theta} \left[R_{a+}^\theta f(b) + R_{b-}^\theta f(a) \right] - \frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \right| \\ & \leq (b-a) \frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}} [|f'(a)| + |f'(b)|] \end{aligned} \quad (4.2)$$

with $\theta > 0$.

Proof. Using Lemma 4 and the convexity of $|f'|$, we have

$$\begin{aligned} & \left| \frac{\Gamma(\theta+1)}{2(b-a)^\theta} \left[R_{a+}^\theta f(b) + R_{b-}^\theta f(a) \right] - \frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \right| \\ & \frac{b-a}{2} \left[\begin{aligned} & \int_0^{\frac{\theta}{\theta+1}} t^\theta |f'(ta + (1-t)b)| dt \\ & + \int_{\frac{\theta}{\theta+1}}^1 (1-t^\theta) |f'(ta + (1-t)b)| dt \\ & + \int_0^{\frac{\theta}{\theta+1}} t^\theta |f'(tb + (1-t)a)| dt \\ & + \int_{\frac{\theta}{\theta+1}}^1 (1-t^\theta) |f'(tb + (1-t)a)| dt \end{aligned} \right] \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{b-a}{2} \left[\int_0^{\frac{\theta}{\theta+1}} t^\theta [t|f'(a)| + (1-t)|f'(b)|] dt \right. \\
 &\quad + \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta [t|f'(a)| + (1-t)|f'(b)|] dt \\
 &\quad + \int_0^{\frac{\theta}{\theta+1}} t^\theta [t|f'(b)| + (1-t)|f'(a)|] dt \\
 &\quad \left. + \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta [t|f'(b)| + (1-t)|f'(a)|] dt \right] \\
 &= \frac{b-a}{2} \left[\int_0^{\frac{\theta}{\theta+1}} t^\theta [|f'(a)| + |f'(b)|] dt \right. \\
 &\quad \left. + \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta [|f'(a)| + |f'(b)|] dt \right] \\
 &= \frac{b-a}{2} \left[\int_0^{\frac{\theta}{\theta+1}} t^\theta dt + \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right] [|f'(a)| + |f'(b)|] \\
 &= (b-a) \frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}} [|f'(a)| + |f'(b)|].
 \end{aligned}$$

This completes the proof. □

Corollary 2. *In Theorem 7, if one takes $\theta = 1$, one has [5, Theorem 2.2].*

Theorem 8. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) . If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following fractional midpoint type inequality holds:*

$$\begin{aligned}
 &\left| \frac{\Gamma(\theta+1)}{2(b-a)^\theta} [R_{a+}^\theta f(b) + R_{b-}^\theta f(a)] - \frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \right| \tag{4.3} \\
 &\leq \frac{b-a}{2} \frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}} \left[\begin{aligned}
 &\left(\frac{\theta}{\theta+2} |f'(a)|^q + \frac{2}{\theta+2} |f'(b)|^q \right)^{\frac{1}{q}} \\
 &+ \left(\frac{(\theta+1)^\theta + 2\theta^{\theta+1}}{2\theta^\theta(\theta+2)} |f'(a)|^q + \frac{4\theta^\theta - (\theta+1)^\theta}{2\theta^\theta(\theta+2)} |f'(b)|^q \right)^{\frac{1}{q}} \\
 &\quad + \left(\frac{\theta}{\theta+2} |f'(b)|^q + \frac{2}{\theta+2} |f'(a)|^q \right)^{\frac{1}{q}} \\
 &+ \left(\frac{(\theta+1)^\theta + 2\theta^{\theta+1}}{2\theta^\theta(\theta+2)} |f'(b)|^q + \frac{4\theta^\theta - (\theta+1)^\theta}{2\theta^\theta(\theta+2)} |f'(a)|^q \right)^{\frac{1}{q}}
 \end{aligned} \right]
 \end{aligned}$$

with $\theta > 0$.

Proof. Using Lemma 4, power mean inequality and the convexity of $|f'|^q$, we have

$$\left| \frac{\Gamma(\theta+1)}{2(b-a)^\theta} [R_{a+}^\theta f(b) + R_{b-}^\theta f(a)] - \frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \right|$$

$$\begin{aligned}
 &\leq \frac{b-a}{2} \left[\begin{aligned} &\int_0^{\frac{\theta}{\theta+1}} t^\theta |f'(ta + (1-t)b)| dt \\ &+ \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta |f'(ta + (1-t)b)| dt \\ &+ \int_0^{\frac{\theta}{\theta+1}} t^\theta |f'(tb + (1-t)a)| dt \\ &+ \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta |f'(tb + (1-t)a)| dt \end{aligned} \right] \\
 &\leq \frac{b-a}{2} \left[\begin{aligned} &\left(\int_0^{\frac{\theta}{\theta+1}} t^\theta dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{\theta}{\theta+1}} t^\theta |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\ &+ \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\ &+ \left(\int_0^{\frac{\theta}{\theta+1}} t^\theta dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{\theta}{\theta+1}} t^\theta |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ &+ \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \end{aligned} \right] \\
 &\leq \frac{b-a}{2} \left(\frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}} \right)^{1-\frac{1}{q}} \left[\begin{aligned} &\left(\int_0^{\frac{\theta}{\theta+1}} t^\theta [t|f'(a)|^q + (1-t)|f'(b)|^q] dt \right)^{\frac{1}{q}} \\ &+ \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta [t|f'(a)|^q + (1-t)|f'(b)|^q] dt \right)^{\frac{1}{q}} \\ &+ \left(\int_0^{\frac{\theta}{\theta+1}} t^\theta [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \right)^{\frac{1}{q}} \\ &+ \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \right)^{\frac{1}{q}} \end{aligned} \right] \\
 &\leq \frac{b-a}{2} \frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}} \left[\begin{aligned} &\left(\frac{\theta}{\theta+2} |f'(a)|^q + \frac{2}{\theta+2} |f'(b)|^q \right)^{\frac{1}{q}} \\ &+ \left(\frac{(\theta+1)^\theta + 2\theta^{\theta+1}}{2\theta^\theta(\theta+2)} |f'(a)|^q + \frac{4\theta^\theta - (\theta+1)^\theta}{2\theta^\theta(\theta+2)} |f'(b)|^q \right)^{\frac{1}{q}} \\ &+ \left(\frac{\theta}{\theta+2} |f'(b)|^q + \frac{2}{\theta+2} |f'(a)|^q \right)^{\frac{1}{q}} \\ &+ \left(\frac{(\theta+1)^\theta + 2\theta^{\theta+1}}{2\theta^\theta(\theta+2)} |f'(b)|^q + \frac{4\theta^\theta - (\theta+1)^\theta}{2\theta^\theta(\theta+2)} |f'(a)|^q \right)^{\frac{1}{q}} \end{aligned} \right].
 \end{aligned}$$

This completes the proof. □

Corollary 3. *In Theorem 8, if one takes $\theta = 1$, one has the following midpoint type inequality,*

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left[\begin{aligned} &\left(\frac{1}{3} |f'(a)|^q + \frac{2}{3} |f'(b)|^q \right)^{\frac{1}{q}} \\ &+ \left(\frac{2}{3} |f'(a)|^q + \frac{1}{3} |f'(b)|^q \right)^{\frac{1}{q}} \end{aligned} \right]. \tag{4.4}$$

Theorem 9. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) . If $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following fractional midpoint*

type inequality holds:

$$\left| \frac{\Gamma(\theta + 1)}{2(b-a)^\theta} \left[R_{a+}^\theta f(b) + R_{b-}^\theta f(a) \right] - \frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \right| \tag{4.5}$$

$$\leq \frac{b-a}{2} \left[\begin{aligned} & \left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p} dt \right)^{\frac{1}{p}} \left(\frac{\theta^2}{2(\theta+1)^2} |f'(a)|^q + \frac{\theta^2+2\theta}{2(\theta+1)^2} |f'(b)|^q \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{\frac{1}{p}} \left(\frac{2\theta+1}{2(\theta+1)^2} |f'(a)|^q + \frac{1}{2(\theta+1)^2} |f'(b)|^q \right)^{\frac{1}{q}} \\ & + \left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p} dt \right)^{\frac{1}{p}} \left(\frac{\theta^2}{2(\theta+1)^2} |f'(b)|^q + \frac{\theta^2+2\theta}{2(\theta+1)^2} |f'(a)|^q \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{\frac{1}{p}} \left(\frac{2\theta+1}{2(\theta+1)^2} |f'(b)|^q + \frac{1}{2(\theta+1)^2} |f'(a)|^q \right)^{\frac{1}{q}} \end{aligned} \right]$$

with $\frac{1}{p} + \frac{1}{q} = 1$ and $\theta > 0$.

Proof. Using Lemma 4, Holder inequality and the convexity of $|f'|^q$, we have

$$\left| \frac{\Gamma(\theta + 1)}{2(b-a)^\theta} \left[R_{a+}^\theta f(b) + R_{b-}^\theta f(a) \right] - \frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \right|$$

$$\leq \frac{b-a}{2} \left[\begin{aligned} & \int_0^{\frac{\theta}{\theta+1}} t^\theta |f'(ta + (1-t)b)| dt \\ & + \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta |f'(ta + (1-t)b)| dt \\ & + \int_0^{\frac{\theta}{\theta+1}} t^\theta |f'(tb + (1-t)a)| dt \\ & + \int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta |f'(tb + (1-t)a)| dt \end{aligned} \right]$$

$$\leq \frac{b-a}{2} \left[\begin{aligned} & \left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{\theta}{\theta+1}} |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{\frac{1}{p}} \left(\int_{\frac{\theta}{\theta+1}}^1 |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\ & + \left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{\theta}{\theta+1}} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{\frac{1}{p}} \left(\int_{\frac{\theta}{\theta+1}}^1 |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \end{aligned} \right]$$

$$\leq \frac{b-a}{2} \left[\begin{aligned} & \left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{\theta}{\theta+1}} [t |f'(a)|^q + (1-t) |f'(b)|^q] dt \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{\frac{1}{p}} \left(\int_{\frac{\theta}{\theta+1}}^1 [t |f'(a)|^q + (1-t) |f'(b)|^q] dt \right)^{\frac{1}{q}} \\ & + \left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{\theta}{\theta+1}} [t |f'(b)|^q + (1-t) |f'(a)|^q] dt \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{\frac{1}{p}} \left(\int_{\frac{\theta}{\theta+1}}^1 [t |f'(b)|^q + (1-t) |f'(a)|^q] dt \right)^{\frac{1}{q}} \end{aligned} \right]$$

$$\leq \frac{b-a}{2} \left[\begin{aligned} & \left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p} dt \right)^{\frac{1}{p}} \left(\frac{\theta^2}{2(\theta+1)^2} |f'(a)|^q + \frac{\theta^2+2\theta}{2(\theta+1)^2} |f'(b)|^q \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{\frac{1}{p}} \left(\frac{2\theta+1}{2(\theta+1)^2} |f'(a)|^q + \frac{1}{2(\theta+1)^2} |f'(b)|^q \right)^{\frac{1}{q}} \\ & + \left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p} dt \right)^{\frac{1}{p}} \left(\frac{\theta^2}{2(\theta+1)^2} |f'(b)|^q + \frac{\theta^2+2\theta}{2(\theta+1)^2} |f'(a)|^q \right)^{\frac{1}{q}} \\ & + \left(\int_{\frac{\theta}{\theta+1}}^1 (1-t)^\theta dt \right)^{\frac{1}{p}} \left(\frac{2\theta+1}{2(\theta+1)^2} |f'(b)|^q + \frac{1}{2(\theta+1)^2} |f'(a)|^q \right)^{\frac{1}{q}} \end{aligned} \right].$$

This completes the proof. \square

Corollary 4. *In Theorem 9, if one takes $\theta = 1$, one has [5, Theorem 2.3].*

5. COMPETING INTERESTS

The authors declare that they have no competing interests.

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