Miskolc Mathematical Notes
Vol. 20 (2019), No. 1, pp. 489–509
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2019.2405

BASIC AND FRACTIONAL q-CALCULUS AND ASSOCIATED FEKETE-SZEGŐ PROBLEM FOR p-VALENTLY q-STARLIKE FUNCTIONS AND p-VALENTLY q-CONVEX FUNCTIONS OF COMPLEX ORDER

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Received 15 September, 2017

Abstract. In this paper, we introduce and study some subclasses of p-valently analytic functions in the open unit disk $\mathbb{U}$ by applying the q-derivative operator and the fractional q-derivative operator in conjunction with the principle of subordination between analytic functions. For the Taylor-Maclaurin coefficients $\{a_k\}_{k=p+1}^{\infty}$ of each of these subclasses of p-valently analytic functions, we derive sharp bounds for the Fekete-Szegő functional given by

$$\left|a_{p+2} - \mu a_{p+1}^2\right|.$$

Relevant connections of the results presented in this paper with those derived in earlier works are also considered.

2010 *Mathematics Subject Classification:* 26A33; 33C20; 30C45; 30C50

Keywords: analytic functions, univalent functions, p-valent functions, q-derivative operator, fractional q-derivative operator, q-starlike and fractional q-starlike functions, q-convex and fractional q-convex functions, Fekete-Szegő problem and Fekete-Szegő functional, principle of subordination between analytic functions

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The theory of the basic and the fractional quantum calculus (that is, the basic q-calculus and the fractional q-calculus) plays important roles in many diverse areas of the mathematical, physical and engineering sciences (see, for example, [5, 9, 16] and [21]). Our main objective in this paper is to introduce and study some subclasses of p-valently analytic functions in the open unit disk $\mathbb{U}$ by applying the q-derivative operator and the fractional q-derivative operator in conjunction with the principle of subordination between analytic functions (see, for details, [12]).

We begin by denoting by $\mathcal{A}(p)$ the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ (p \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

In particular, we write

$$\mathcal{A}(1) = \mathcal{A}.$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}_p^*(\alpha)$ of p-valently starlike of order α in $\mathbb{U}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (0 \leqq \alpha < p;\ z \in \mathbb{U}). \tag{1.2}$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{C}_p(\alpha)$ of p-valently convex of order α in $\mathbb{U}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (0 \leqq \alpha < p;\ z \in \mathbb{U}). \tag{1.3}$$

The p-valent function classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(\alpha)$ were studied by Owa [13]. From (1.2) and (1.3), it follows that

$$f(z) \in \mathcal{C}_p(\alpha) \iff \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha). \tag{1.4}$$

We now recall some basic definitions and concept details of the q-calculus which are used in this paper (see, for details, [7] and [8]; see also [5] and [21]).

Definition 1. Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ and the q-factorial $[n]_q!$ by

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\[2ex] \displaystyle\sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}) \end{cases} \tag{1.5}$$

and

$$[n]_q! = \begin{cases} 1 & (n = 0) \\[2ex] \displaystyle\prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}), \end{cases} \tag{1.6}$$

respectively.

Definition 2. The q-derivative (or the q-difference) $D_q f(z)$ of a function $f(z)$ is defined in a given subset of $\mathbb{C}$ by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{1.7}$$

provided that $f'(0)$ exists.

We note from Definition 2 that

$$\lim_{q \to 1-} D_q f(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z) \tag{1.8}$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1.1) and (1.7) that

$$D_q f(z) = [p]_q \, z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q \, a_k z^{k-1} \; (z \neq 0), \tag{1.9}$$

where $[\lambda]_q$ is given by (1.5) and the function $f(z) \in \mathcal{A}(p)$ is given by (1.1).

Making use of the q-derivative operator D_q given by (1.7), we introduce the subclass $\mathcal{S}_q^*(p,\alpha)$ of p-valently q-starlike functions of order α in $\mathbb{U}$ and the subclass $\mathcal{C}_q(p,\alpha)$ of p-valently q-convex functions of order α in $\mathbb{U}$ as follows:

$$f(z) \in \mathcal{S}_q^*(p,\alpha) \iff \Re\left(\frac{1}{[p]_q} \frac{z D_q f(z)}{f(z)}\right) > \alpha \tag{1.10}$$
$$(0 < q < 1;\ 0 \leqq \alpha < 1;\ z \in \mathbb{U})$$

and

$$f(z) \in \mathcal{C}_q(p,\alpha) \iff \Re\left(\frac{1}{[p]_q} \frac{D_q\big(z D_q f(z)\big)}{D_q f(z)}\right) > \alpha \tag{1.11}$$
$$(0 < q < 1;\ 0 \leqq \alpha < 1;\ z \in \mathbb{U}),$$

respectively. From (1.10) and (1.11), it follows that

$$f(z) \in \mathcal{C}_q(p,\alpha) \iff \frac{z D_q f(z)}{[p]_q} \in \mathcal{S}_q^*(p,\alpha). \tag{1.12}$$

We note also that

$$\lim_{q \to 1-} \mathcal{S}_q^*(p,\alpha) = \mathcal{S}_p^*(\alpha) \qquad \text{and} \qquad \lim_{q \to 1-} \mathcal{C}_q(p,\alpha) = \mathcal{C}_p(\alpha).$$

We next introduce the familiar principle of subordination between analytic functions. Indeed, for two given functions $f(z)$ and $g(z)$ which are analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to the function $g(z)$ and write

$$f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1$$

such that

$$f(z) = g\big(w(z)\big).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then it follows that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Throughout our present investigation, let the function $\varphi(z)$ be analytic with positive real part in $\mathbb{U}$ and satisfy the following conditions:

$$\varphi(0) = 1 \qquad \text{and} \qquad \varphi'(0) > 0,$$

which maps $\mathbb{U}$ onto a region which is starlike with respect to 1 and symmetric with respect to the real axis. Suppose now that $\mathcal{S}^*_{b,p}(\varphi)$ denotes the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 + \frac{1}{b}\left(\frac{1}{p}\,\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z) \qquad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\};\ z \in \mathbb{U}). \tag{1.13}$$

Also let $\mathcal{C}_{b,p}(\varphi)$ be the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \qquad (b \in \mathbb{C}^*;\ z \in \mathbb{U}). \tag{1.14}$$

The p-valent function classes $\mathcal{S}^*_{b,p}(\varphi)$ and $\mathcal{C}_{b,p}(\varphi)$ were introduced and studied by Ali *et al.* [2]. We note that each of the following function classes:

$$\mathcal{S}^*_{1,1}(\varphi) =: \mathcal{S}^*(\varphi) \qquad \text{and} \qquad \mathcal{C}_{1,1}(\varphi) =: \mathcal{C}(\varphi)$$

was introduced and studied by Ma and Minda [11]. In fact, the widely-investigated function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are the special cases of $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$, respectively, when

$$\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{U}).$$

Finally, for $b \in \mathbb{C}^*$, $0 < q < 1$, $0 \leqq \lambda \leqq 1$ and $p \in \mathbb{N}$, we define the subclass $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ of the p-valently analytic function class $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1) and satisfying the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\,\frac{zD_q f(z) + \lambda q z^2 D_q\big(D_q f(z)\big)}{(1-\lambda)f(z) + \lambda z D_q f(z)} - 1\right) \prec \varphi(z), \tag{1.15}$$

where D_q denotes the q-derivative operator given by Definition 2.

The following function classes are included in the class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ of p-valently q-starlike functions of *complex* order b in $\mathbb{U}$ (see also [1]):

(i) $\lim_{q\to 1-} \mathcal{S}_{0,q,b}(p,\varphi) =: \mathcal{S}^{*}_{b,p}(\varphi)$ and $\lim_{q\to 1-} \mathcal{S}_{1,q,b}(p,\varphi) =: \mathcal{C}_{b,p}(\varphi)$
(see Ali *et al.* [2]);

(ii) $\lim_{q\to 1-} \mathcal{S}_{\lambda,q,b}(p,\varphi) =: \mathcal{S}_{\lambda,b,p}(\varphi)$
(see Aouf *et al.* [3]) *with*

$$g(z) = \frac{z^p}{1-z};$$

(iii) $\mathcal{S}_{0,q,b}(1,\varphi) =: \mathcal{S}_{q,b}(\varphi)$ and $\mathcal{S}_{1,q,b}(1,\varphi) =: \mathcal{C}_{q,b}(\varphi)$
(see Seoudy and Aouf [19]);

(iv) $\lim_{q\to 1-} \mathcal{S}_{0,q,b}(1,\varphi) =: \mathcal{S}^{*}_{b}(\varphi)$ and $\lim_{q\to 1-} \mathcal{S}_{1,q,b}(1,\varphi) =: \mathcal{C}_{b}(\varphi)$
(see Ravichandran *et al.* [17]).

We also note here that

(i) $\mathcal{S}_{\lambda,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha}\cos\alpha}(p,\varphi) = \mathcal{S}^{\alpha,\gamma}_{q,p,\lambda}(\varphi)$

$$= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(\frac{zD_q f(z)+\lambda q z^2 D_q(D_q f(z))}{(1-\lambda)f(z)+\lambda z D_q f(z)}\right) - \gamma\cos\alpha - i\,[p]_q\,\sin\alpha}{([p]_q-\gamma)\cos\alpha} \prec \varphi(z) \right.$$

$$\left. \left(|\alpha| < \frac{\pi}{2};\ 0 \leqq \gamma < [p]_q\,;\ 0 \leqq \lambda \leqq 1\right) \right\};$$

(ii) $\mathcal{S}_{0,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha}\cos\alpha}(p,\varphi) = \mathcal{S}^{\alpha}_{q,p,\gamma}(\varphi)$

$$= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(\frac{zD_q f(z)}{f(z)}\right) - \gamma\cos\alpha - i\,[p]_q\sin\alpha}{([p]_q-\gamma)\cos\alpha} \prec \varphi(z) \right.$$

$$\left. \left(|\alpha| < \frac{\pi}{2};\ 0 \leqq \gamma < [p]_q\right) \right\};$$

(iii) $\mathcal{S}_{0,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha}\cos\alpha}(p,\varphi) = \mathcal{C}^{\alpha}_{q,p,\gamma}(\varphi)$

$$= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right) - \gamma\cos\alpha - i\,[p]_q\sin\alpha}{([p]_q-\gamma)\cos\alpha} \prec \varphi(z) \right.$$

$$\left. \left(|\alpha| < \frac{\pi}{2};\ 0 \leqq \gamma < [p]_q\right) \right\}.$$

In Geometric Function Theory, various subclasses of the normalized analytic function class $\mathcal{A}$ as well as the normalized p-valently analytic function class $\mathcal{A}(p)$ have been studied from different viewpoints. The above-defined q-calculus provides important tools that have been used in order to investigate various subclasses of $\mathcal{A}$ and $\mathcal{A}(p)$. Historically speaking, even though the q-derivative operator D_q was first applied by Ismail *et al.* [6] to study a q-extension of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [20, pp. 347 *et seq.*]).

2. Fekete-Szegő problem for the function class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$

Let Ω be the class of functions $w(z)$ of the form:

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \qquad (z \in \mathbb{U}), \tag{2.1}$$

which satisfy the following inequality:

$$|w(z)| < 1 \qquad (z \in \mathbb{U}).$$

Each of the following lemmas will be needed in our present investigation of the Fekete-Szegő problem for the function class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ which we have introduced by using the subordination condition (1.15) (see, for example, [4]; see also [22] and [24]).

Lemma 1 ([10])**.** *Let the function $w(z) \in \Omega$ be given by (2.1). Then*

$$\left|w_2 - \tau w_1^2\right| \leqq \max\{1, |\tau|\} \qquad (\tau \in \mathbb{C}).$$

The result is sharp for the function given by

$$w(z) = z \qquad or \qquad w(z) = z^2 \qquad (z \in \mathbb{U}).$$

Lemma 2 ([2], [11])**.** *Let the function $w(z) \in \Omega$ be given by (2.1). Then*

$$\left|w_2 - \kappa w_1^2\right| \leqq \begin{cases} -\kappa & (\kappa \leqq -1) \\ 1 & (-1 \leqq \kappa \leqq 1) \\ \kappa & (\kappa \geqq 1). \end{cases} \tag{2.2}$$

For $\kappa < -1$ or $\kappa > 1$, the equality holds true in (2.2) if and only if $w(z) = z$ or one of its rotations. If $-1 < \kappa < 1$, then the equality holds true in (2.2) if and only if $w(z) = z^2$ or one of its rotations. If $\kappa = -1$, then the equality holds true in (2.2) if and only if

$$w(z) = \frac{z(z+\eta)}{1+\eta z} \qquad (0 \leqq \eta \leqq 1)$$

or one of its rotations. If $\kappa = 1$, *then the equality holds true in* (2.2) *if and only if*

$$w(z) = -\frac{z(z+\eta)}{1+\eta z} \qquad (0 \leqq \eta \leqq 1)$$

or one of its rotations. The upper bound in (2.2) *is sharp and it can be improved as follows when* $-1 < \kappa < 1$:

$$\left|w_2 - \kappa w_1^2\right| + (\kappa+1)\left|w_1\right|^2 \leqq 1 \qquad (-1 < \kappa \leqq 0)$$

and

$$\left|w_2 - \kappa w_1^2\right| + (1-\kappa)\left|w_1\right|^2 \leqq 1 \qquad (0 < \kappa < 1).$$

Lemma 3 ([14])**.** *Let the function* $w(z) \in \Omega$ *be given by* (2.1)*. Then, for any real numbers* q_1 *and* q_2, *the following sharp estimates hold true*:

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \leqq H(q_1, q_2), \tag{2.3}$$

where

$$H(q_1,q_2) = \begin{cases} 1 & \left((q_1,q_2) \in D_1 \cup D_2\right) \\ |q_2| & \left((q_1,q_2) \in \bigcup_{k=3}^{7} D_k\right) \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \left((q_1,q_2) \in D_8 \cup D_9\right) \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \left((q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}\right) \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \left((q_1,q_2) \in D_{12}\right). \end{cases}$$

The extremal functions, up to rotations, are of the form given by

$$w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z) = \frac{z\left[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1\right]z - \varepsilon_1\varepsilon_2 z}{1 - \left[(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2\right]z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \quad w(z) = w_2(z) = \frac{z(t_2+z)}{1+t_2 z}, \quad |\varepsilon_1| = |\varepsilon_2| = 1,$$

$$\varepsilon_1 = t_0 - e^{-i\left(\frac{\theta_0}{2}\right)}(a \mp b), \quad \varepsilon_2 = -e^{-i\left(\frac{\theta_0}{2}\right)}(ia \pm b),$$

$$a = t_0 \cos\left(\frac{\theta_0}{2}\right), \quad b = \sqrt{1 - t_0^2 \sin^2\left(\frac{\theta_0}{2}\right)}, \quad \lambda = \frac{b \pm a}{2b},$$

$$t_0=\left(\frac{2q_2(q_1^2+2)-3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right)^{\frac{1}{2}},\quad t_1=\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2=\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}}\ \textit{and}\ \cos\left(\frac{\theta_0}{2}\right)=\frac{q_1}{2}\left(\frac{q_2(q_1^2+8)-2(q_1^2+2)}{2q_2(q_1^2+2)-3q_1^2}\right).$$

The sets D_k $(k=1,\cdots,12)$ *are defined as follows*:

$$D_1:=\left\{(q_1,q_2):|q_1|\leqq\frac{1}{2}\ \textit{and}\ |q_2|\leqq 1\right\},$$

$$D_2:=\left\{(q_1,q_2):\frac{1}{2}\leqq|q_1|\leqq 2\ \textit{and}\ \frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\leqq q_2\leqq 1\right\},$$

$$D_3:=\left\{(q_1,q_2):|q_1|\leqq\frac{1}{2}\ \textit{and}\ q_2\leqq -1\right\},$$

$$D_4:=\left\{(q_1,q_2):|q_1|\geqq\frac{1}{2}\ \textit{and}\ q_2\leqq -\frac{2}{3}(|q_1|+1)\right\},$$

$$D_5:=\{(q_1,q_2):|q_1|\leqq 2\ \ \textit{and}\ q_2\geqq 1\},$$

$$D_6:=\left\{(q_1,q_2):2\leqq|q_1|\leqq 4\ \textit{and}\ q_2\geqq\frac{1}{12}(q_1^2+8)\right\},$$

$$D_7:=\left\{(q_1,q_2):|q_1|\geqq 4\ \ \textit{and}\ q_2\geqq\frac{2}{3}(|q_1|-1)\right\},$$

$$D_8:=\left\{(q_1,q_2):\frac{1}{2}\leqq|q_1|\leqq 2\ \textit{and}\right.$$
$$\left.-\frac{2}{3}(|q_1|+1)\leqq q_2\leqq\frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\right\},$$

$$D_9:=\left\{(q_1,q_2):|q_1|\geqq 2\ \textit{and}\ \ -\frac{2}{3}(|q_1|+1)\leqq q_2\leqq\frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10}:=\left\{(q_1,q_2):2\leqq|q_1|\leqq 4\ \textit{and}\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\leqq q_2\leqq\frac{1}{12}(q_1^2+8)\right\},$$

$$D_{11}:=\left\{(q_1,q_2):|q_1|\geqq 4\ \ \textit{and}\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\leqq q_2\leqq\frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\}$$

and

$$D_{12} := \left\{ (q_1, q_2) : |q_1| \geqq 4 \ \text{ and } \ \frac{2|q_1|(|q_1|-1)}{q_1^2 - 2|q_1| + 4} \leqq q_2 \leqq \frac{2}{3}(|q_1|-1) \right\}.$$

Remark 1. Unless otherwise mentioned, we assume throughout this paper that

$$b \in \mathbb{C}^*, \quad 0 \leqq \lambda \leqq 1, \quad 0 < q < 1 \qquad \text{and} \qquad p \in \mathbb{N}.$$

Theorem 1. *Let the function $\varphi(z)$ be given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 > 0).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ and $\mu \in \mathbb{C}$, then

$$\begin{aligned} \left|a_{p+2} - \mu a_{p+1}^2\right| &\leqq \frac{[p]_q B_1 |b|}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda\big([p]_q - 1\big)}{1 + \lambda\big([p+2]_q - 1\big)} \right) \\ &\cdot \max\left\{ 1, \left| \frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right.\right. \\ &\cdot \left.\left. \left(1 - \mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{\big[1 + \lambda\big([p+2]_q - 1\big)\big]\big[1 + \lambda\big([p]_q - 1\big)\big]}{\big[1 + \lambda\big([p+1]_q - 1\big)\big]^2} \right) \right| \right\} \end{aligned} \tag{2.4}$$

and

$$\left|a_{p+3}\right| \leqq \frac{[p]_q \ B_1 |b|}{[p+3]_q - [p]_q} \left(\frac{1 + \lambda\big([p]_q - 1\big)}{1 + \lambda\big([p+3]_q - 1\big)} \right) H(q_1, q_2), \tag{2.5}$$

where

$$q_1 = \frac{2B_2}{B_1} - \frac{[p]_q \big(2[p]_q - [p+1]_q - [p+2]_q\big) B_1 b}{\big([p+1]_q - [p]_q\big)\big([p+2]_q - [p]_q\big)} \tag{2.6}$$

and

$$\begin{aligned} q_2 = \frac{B_3}{B_1} &- \left(\frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right)^2 - \frac{[p]_q \big(2[p]_q - [p+1]_q - [p+2]_q\big) B_1 b}{\big([p+1]_q - [p]_q\big)\big([p+2]_q - [p]_q\big)} \\ &\cdot \left(\frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right). \end{aligned} \tag{2.7}$$

The result is sharp.

Proof. If $f(z) \in \mathcal{S}_{\lambda,q,b}(p,\varphi)$, then there is a Schwarz function $w(z) \in \Omega$ given by

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$$

such that

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)+\lambda qz^2D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right)=\varphi\big(w(z)\big).$$

Since

$$\begin{aligned}
&\frac{1}{[p]_q}\frac{zD_qf(z)+\lambda qz^2D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}\\
&=1+\frac{1+\lambda\left([p+1]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}z\\
&+\left[\frac{1+\lambda\left([p+2]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\left(\frac{[p+2]_q}{[p]_q}-1\right)a_{p+2}-\left(\frac{1+\lambda\left([p+1]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\right)^2\right.\\
&\left.\cdot\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}^2\right]z^2+\left[\frac{1+\lambda\left([p+3]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\left(\frac{[p+3]_q}{[p]_q}-1\right)a_{p+3}\right.\\
&+\left(\frac{1+\lambda\left([p+1]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\right)^3\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}^3+\frac{1+\lambda\left([p+1]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\\
&\left.\cdot\frac{1+\lambda\left([p+2]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\left(2-\frac{[p+1]_q}{[p]_q}-\frac{[p+2]_q}{[p]_q}\right)a_{p+1}a_{p+2}\right]z^3+\cdots
\end{aligned}$$

and

$$\varphi\big(w(z)\big)=1+w_1B_1z+(w_1^2B_2+w_2B_1)z^2+(w_3B_1+w_1^3B_3+2w_1w_2B_2)z^3+\cdots,$$

we observe that

$$a_{p+1}=\frac{[p]_q\,B_1bw_1}{[p+1]_q-[p]_q}\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+1]_q-1\right)}\right), \tag{2.8}$$

$$\begin{aligned}
a_{p+2}=&\frac{[p]_q\,B_1b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\right)\\
&\cdot\left[w_2+w_1^2\left(\frac{B_2}{B_1}+\frac{[p]_q\,B_1b}{[p+1]_q-[p]_q}\right)\right]
\end{aligned} \tag{2.9}$$

and

$$a_{p+3}=\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+3]_q-1\right)}\frac{[p]_q\,B_1b}{[p+3]_q-[p]_q}\left\{w_3+\left[\frac{2B_2}{B_1}\right.\right.$$

$$-\frac{[p]_q\left(2[p]_q-[p+1]_q-[p+2]_q\right)B_1b}{\left([p+1]_q-[p]_q\right)\left([p+2]_q-[p]_q\right)}\Bigg]w_1w_2+\left[\frac{B_3}{B_1}-\left(\frac{[p]_q B_1 b}{[p+1]_q-[p]_q}\right)^2\right.$$
$$\left.-\frac{[p]_q\left(2[p]_q-[p+1]_q-[p+2]_q\right)B_1b}{\left([p+1]_q-[p]_q\right)\left([p+2]_q-[p]_q\right)}\left(\frac{B_2}{B_1}+\frac{[p]_q B_1 b}{[p+1]_q-[p]_q}\right)\right]w_1^3\Bigg\}. \tag{2.10}$$

Therefore, we have

$$a_{p+2}-\mu a_{p+1}^2=\frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\left[w_2-\upsilon w_1^2\right],$$

where

$$\upsilon=\frac{[p]_q B_1|b|}{[p+1]_q-[p]_q}\cdot\left(\mu\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}-1\right)-\frac{B_2}{B_1}. \tag{2.11}$$

The result (2.4) follows by an application of Lemma 1 and the result (2.5) follows by an application of Lemma 3. These results are sharp for the functions given by

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)+\lambda qz^2D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right)=\varphi(z^2)$$

and

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)+\lambda qz^2D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right)=\varphi(z),$$

respectively. This completes the proof of Theorem 1. □

By using Lemma 2, we can obtain the following theorem.

Theorem 2. *Let $b>0$ and let the function $\varphi(z)$ be given by*

$$\varphi(z)=1+B_1z+B_2z^2+\cdots(B_k>0;\ k\in\{1,2\}).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ and $\mu\in\mathbb{R}$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leqq$$

$$\begin{cases} \frac{[p]_q b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)}\right) \\ \cdot\left\{B_2+\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\left(1-\mu\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\frac{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]}{[1+\lambda([p+1]_q-1)]^2}\right)\right\} \\ (\mu \leqq \sigma_1) \\ \frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)}\right) \\ (\sigma_1 \leqq \mu \leqq \sigma_2) \\ \frac{[p]_q b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)}\right). \\ \cdot\left\{-B_2+\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\left(\mu\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\frac{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]}{[1+\lambda([p+1]_q-1)]^2}-1\right)\right\} \\ (\mu \geqq \sigma_2), \end{cases}$$

where

$$\sigma_1=\frac{\left([p+1]_q-[p]_q\right)\left[(B_2-B_1)\left([p+1]_q-[p]_q\right)+[p]_q B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b}$$

and

$$\sigma_2=\frac{\left([p+1]_q-[p]_q\right)\left[(B_2+B_1)\left([p+1]_q-[p]_q\right)+[p]_q B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b}.$$

The result is sharp.

Further, if we set

$$\sigma_3=\frac{\left([p+1]_q-[p]_q\right)\left[\left([p+1]_q-[p]_q\right)B_2+[p]_q B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b},$$

then each of the following assertions holds true:

(i) *If* $\sigma_1 \leqq \mu \leqq \sigma_3$, *then*

$$\begin{aligned} &\left|a_{p+2}-\mu a_{p+1}^2\right|+\frac{\left([p+1]_q-[p]_q\right)^2}{[p]_q\left([p+2]_q-[p]_q\right)B_1^2 b}\,\frac{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]} \\ &\quad\cdot\left\{(B_1-B_2)-\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\left[1-\mu\,\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right.\right. \\ &\quad\left.\left.\cdot\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\right]\right\}\left|a_{p+1}\right|^2 \leqq \frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda[p]_q-1}{1+\lambda\left([p+2]_q-1\right)}\right). \end{aligned}$$

(ii) *If* $\sigma_3 \leqq \mu \leqq \sigma_2$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{\left([p+1]_q-[p]_q\right)^2}{[p]_q\left([p+2]_q-[p]_q\right)B_1^2 b}\,\frac{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}$$

$$\cdot\left\{(B_1-B_2)-\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\left[\mu\,\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right.\right.$$

$$\left.\left.\cdot\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}-1\right]\right\}\left|a_{p+1}\right|^2 \leqq \frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda[p]_q-1}{1+\lambda\left([p+2]_q-1\right)}\right).$$

Proof. By applying Lemma 2 to (2.8), (2.9) and (2.11), we obtain the required results asserted by Theorem 2. In order to show that the bounds are sharp, we define the functions $K_{\varphi n}$ $(n \in \mathbb{N}\setminus\{1\})$ by

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\,\frac{zD_qK_{\varphi n}(z)+\lambda q z^2 D_q(D_qK_{\varphi n}(z))}{(1-\lambda)K_{\varphi n}(z)+\lambda z D_q K_{\varphi n}(z)}-1\right)=\varphi(z^{n-1})$$

$$\left(z^{1-p}\ K_{\varphi n}(z)\Big|_{z=0}=0=z^{1-p}\ K'_{\varphi n}(z)\Big|_{z=0}-p\right)$$

and the functions F_β and G_β $(0 \leqq \beta \leqq 1)$ by

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\,\frac{zD_qF_\beta(z)+\lambda q z^2 D_q(D_qF_\beta(z))}{(1-\lambda)F_\beta(z)+\lambda z D_q F_\beta(z)}-1\right)=\varphi\left(\frac{z(z+\beta)}{1+\beta z}\right)$$

$$\left(z^{1-p}\ F_\beta(z)\Big|_{z=0}=0=z^{1-p}\ F'_\beta(z)\Big|_{z=0}-p\right)$$

and

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\,\frac{zD_qG_\beta(z)+\lambda q z^2 D_q(D_qG_\beta(z))}{(1-\lambda)G_\beta(z)+\lambda z D_q G_\beta(z)}-1\right)=\varphi\left(-\frac{z(z+\beta)}{1+\beta z}\right).$$

$$\left(z^{1-p}\ G_\beta(z)\Big|_{z=0}=0=z^{1-p}\ G'_\beta(z)\Big|_{z=0}-p\right).$$

Clearly, the functions $K_{\varphi n}$, F_β and G_β are in the class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$. We also write

$$K_\varphi = K_{\varphi 2}.$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds true if and only if the function f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds true if the function f is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds true if and only if the function f is F_β or one of its rotations. If $\mu = \sigma_2$, then the equality holds true if and only if the function f is G_β or one of its rotations. This completes the proof of Theorem 2. □

Remark 2. For different choices of the parameters q, b and λ in Theorem 1 and Theorem 2, we can deduce the corresponding results derived earlier by Ali *et al.* [2], Aouf *et al.* [3] and Seoudy and Aouf [19].

3. Applications to functions defined by the fractional q-derivative operator

We first recall some definitions of the *fractional* q-calculus which we will be used in this section.

First of all, for $0 < q < 1$ and $\lambda, \mu \in \mathbb{C}$, the basic (or q-) shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example, [5, 21] and [23])

$$(\lambda;q)_\mu = \prod_{j=0}^{\infty}\left(\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}}\right) \qquad (0 < q < 1;\ \lambda, \mu \in \mathbb{C}),$$

so that

$$(\lambda;q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1}\left(1-\lambda q^j\right) & (n \in \mathbb{N}) \end{cases} \tag{3.1}$$

and

$$(\lambda;q)_\infty := \prod_{j=0}^{\infty}\left(1-\lambda q^j\right) \qquad (0 < q < 1;\ \lambda \in \mathbb{C}), \tag{3.2}$$

Furthermore, in terms of the basic (or q-) Gamma function $\Gamma_q(z)$ defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}\,(1-q)^{1-z} \qquad (0 < q < 1;\ z \in \mathbb{C}), \tag{3.3}$$

so that

$$\lim_{q \to 1-}\{\Gamma_q(z)\} = \Gamma(z)$$

for the familiar (Euler's) Gamma function $\Gamma(z)$, we find from (3.1) that

$$\left(q^\alpha;q\right)_n = \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}\,(1-q)^n \qquad (n \in \mathbb{N};\ \alpha \in \mathbb{C}).$$

For $0 < q < 1$, the (Jackson's) q-integral is defined (in general) by

$$\int_0^x f(t)\mathrm{d}_q t = (1-q)x \sum_{k=0}^{\infty} f(xq^k)q^k \qquad (x > 0), \tag{3.4}$$

provided that the series on the right-hand side converges absolutely. In the limit case when $q \to 1-$, the q-integral in (3.4) reduces to

$$\int_0^x f(t)\mathrm{d}t.$$

In general, for any closed interval $[a,b]$ $(0 \leqq a < b)$, we write

$$\int_a^b f(t)\mathrm{d}_q t = \int_0^b f(t)\mathrm{d}_q t - \int_0^a f(t)\mathrm{d}_q t. \tag{3.5}$$

In order to introduce the subclasses

$$\mathcal{S}^*_{q,b,\delta}(p,\varphi) \qquad \text{and} \qquad \mathcal{C}_{q,b,\delta}(p,\varphi),$$

we need the following definitions.

Definition 3 ([15], [16] and see also the references cited therein)**.** The *fractional* q-integral operator $\mathfrak{I}^\delta_{q,z}$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{I}^\delta_{q,z} f(z) = D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z-tq)_q^{\delta-1}\ f(t)\mathrm{d}_q t \qquad (\delta > 0), \tag{3.6}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the q-binomial $(\kappa + z)^n_q$ is defined by (see, for example, [21, p. 486])

$$(\kappa + z)^\mu_q = \sum_{k=0}^{\infty} \begin{bmatrix} \mu \\ k \end{bmatrix}_q q^{\binom{k}{2}}\ \kappa^{\mu-k}\ z^k = \kappa^\mu\ {}_1\Phi_0 \begin{bmatrix} q^{-\mu}; & \\ & q, -\dfrac{zq^\mu}{\kappa} \\ \text{——}; & \end{bmatrix} \tag{3.7}$$

in which the generalized basic (or q-) binomial coefficient $\left[\begin{smallmatrix}\lambda \\ n\end{smallmatrix}\right]_q$ is defined by

$$\begin{bmatrix} \mu \\ n \end{bmatrix}_q = \frac{(q^{-\mu};q)_n}{(q;q)_n} \left(-q^\mu\right)^n q^{-\binom{n}{2}} \qquad (0 < q < 1;\ \mu \in \mathbb{C};\ n \in \mathbb{N}_0) \tag{3.8}$$

and ${}_r\Phi_s$ denotes the generalized q-hypergeometric function with r numerator and s denominator parameters (see, for example, [23, p. 347, Eq. 9.4 (272)]).

Remark 3. It follows from Definition 3 that

$$(z-tq)_q^{\delta-1} = z^{\delta-1}\ {}_1\Phi_0 \begin{bmatrix} q^{1-\delta}; & \\ & q, \dfrac{tq^\delta}{z} \\ \text{——}; & \end{bmatrix}. \tag{3.9}$$

Since the series

$${}_1\Phi_0 \begin{bmatrix} \lambda; & \\ & q, z \\ \text{—}; & \end{bmatrix}$$

is single-valued when (see, for details, [5])

$$|\arg(-z)| < \pi \qquad \text{and} \qquad |z| < 1,$$

the q-binomial $(z-tq)_q^{\delta-1}$ in (3.9) is single-valued when

$$\left|\arg\left(-\frac{tq^\delta}{z}\right)\right| < \pi, \quad \left|\frac{tq^\delta}{z}\right| < 1 \qquad \text{and} \qquad |\arg(z)| < \pi.$$

Definition 4 ([15], [16] and see also the references cited therein)**.** The *fractional* q-derivative operator $\mathfrak{D}_{q,z}^{\delta}$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{D}_{q,z}^{\delta} f(z) = \mathfrak{D}_{q,z}^{\delta}\, \mathfrak{I}_{q,z}^{1-\delta} f(z) = \frac{1}{\Gamma_q(1-\delta)}\, \mathfrak{D}_{q,z} \int_0^z (z-tq)_q^{-\delta} f(t)\, \mathrm{d}_q t \qquad (3.10)$$

$$(0 \leqq \delta < 1),$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of the q-binomial $(z-tq)_q^{-\delta}$ is removed as in Definition 3.

Definition 5 ([15], [16] and see also the references cited therein)**.** Under the hypotheses of Definition 4, the *fractional* q-derivative operator $\mathfrak{D}_{q,z}^{\delta}$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{D}_{q,z}^{\delta} f(z) = \mathfrak{D}_{q,z}^{m}\, \mathfrak{I}_{q,z}^{m-\delta}\, f(z) \qquad (m-1 \leqq \delta < 1;\ m \in \mathbb{N}_0 = \mathbb{N} \setminus \{0\}). \qquad (3.11)$$

Definition 6 ([18])**.** In terms of the *fractional* q-derivative operator $\mathfrak{D}_{q,z}^{\delta}$ given by Definition 3.13, the *fractional* q-derivative operator

$$\Omega_{q,p}^{\delta} : \mathcal{A}(p) \to \mathcal{A}(p)$$

is defined as follows:

$$\begin{aligned} \Omega_{q,p}^{\delta} f(z) &= \frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)}\, z^{\delta}\, \mathfrak{D}_{q,z}^{\delta}\, f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\, a_k z^k, \end{aligned} \qquad (3.12)$$

where the function $f(z) \in \mathcal{A}(p)$ is given by (1.1).

Thus, for $b \in \mathbb{C}^*$, $0 < q < 1$, $0 \leqq \lambda \leqq 1$, $0 \leqq \delta < 1$ and $p \in \mathbb{N}$, we now let $\mathcal{S}_{\lambda,q,b,\delta}(p,\varphi)$ be the subclass of the normalized p -valently analytic function class $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1) and satisfying the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\, \frac{z D_q\left(\Omega_{q,p}^{\delta} f(z)\right) + \lambda q z^2 D_q^2\left(\Omega_{q,p}^{\delta} f(z)\right)}{(1-\lambda)\Omega_{q,p}^{\delta} f(z) + \lambda z D_q\left(\Omega_{q,p}^{\delta} f(z)\right)} - 1\right) \prec \varphi(z). \qquad (3.13)$$

Remark 4. By using arguments and analysis to those in the proofs of Theorem 1 and Theorem 2, we can analogously derive Theorem 3 and Theorem 4 below. The details involved are being left as an exercise for the interested reader.

Theorem 3. *Let the function* $\varphi(z)$ *be given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 > 0).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b,\delta}(p,\varphi)$ and $\mu \in \mathbb{C}$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqq \frac{[p]_q B_1 |b|}{[p+2]_q-[p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})} \left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\right)$$
$$\cdot \max\left\{1, \left|\frac{B_2}{B_1}+\frac{[p]_q B_1 b}{[p+1]_q-[p]_q}\left(1-\mu \frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\right.\right.\right.$$
$$\left.\left.\left.\cdot \frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})} \frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right)\right|\right\}$$

and

$$\left|a_{p+3}\right| \leqq \frac{[p]_q B_1 |b|}{[p+3]_q-[p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})(1-q^{p-\delta+3})}{(1-q^{p+1})(1-q^{p+2})(1-q^{p+3})}$$
$$\cdot \left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+3]_q-1\right)}\right) H(q_1,q_2),$$

where q_1 and q_2 are defined by (2.6) and (2.7). The result is sharp.

Theorem 4. *Let the function $\varphi(z)$ be given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_k > 0;\ k \in \{1,2\}).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b,\delta}(p,\varphi)$ and $\mu \in \mathbb{R}$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqq$$

$$\begin{cases} \dfrac{[p]_q\, b}{[p+2]_q-[p]_q}\,\dfrac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\,\dfrac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)} \\ \cdot\Bigg\{B_2+\dfrac{[p]_q\,B_1^2 b}{[p+1]_q-[p]_q}\Bigg(1-\mu\,\dfrac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\,\dfrac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})} \\ \cdot\dfrac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\Bigg)\Bigg\} \\ (\mu \leqq \sigma_1^*) \\ \\ \dfrac{[p]_q\, B_1 b}{[p+2]_q-[p]_q}\,\dfrac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\,\dfrac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)} \\ (\sigma_1^* \leqq \mu \leqq \sigma_2^*) \\ \\ \dfrac{[p]_q\, b}{[p+2]_q-[p]_q}\,\dfrac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\,\dfrac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)} \\ \cdot\Bigg\{B_2+\dfrac{[p]_q\,B_1^2 b}{[p+1]_q-[p]_q}\Bigg(\mu\,\dfrac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\,\dfrac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})} \\ \cdot\dfrac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}-1\Bigg)\Bigg\} \\ (\mu \leqq \sigma_2^*), \end{cases}$$

where

$$\sigma_1^* = \frac{\left([p+1]_q-[p]_q\right)\left[(B_2-B_1)\left([p+1]_q-[p]_q\right)+[p]_q\,B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b} \cdot\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}$$

and

$$\sigma_2^* = \frac{\left([p+1]_q-[p]_q\right)\left[(B_2+B_1)\left([p+1]_q-[p]_q\right)+[p]_q\,B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b} \cdot\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}.$$

The result is sharp.

Further, if we set

$$\sigma_3^* = \frac{\left([p+1]_q-[p]_q\right)\left[\left([p+1]_q-[p]_q\right)B_2+[p]_q\,B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b} \cdot\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}.$$

then each of the following assertions holds true:

(i) If $\sigma_1^* \leqq \mu \leqq \sigma_3^*$, *then*

$$\begin{aligned}&\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{\left([p+1]_q-[p]_q\right)^2}{[p]_q\left([p+2]_q-[p]_q\right)B_1^2 b}\,\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}\\&\quad\cdot\frac{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}\left\{(B_1-B_2)-\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\right.\\&\quad\cdot\left(1-\mu\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\,\frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})}\right.\\&\quad\left.\left.\cdot\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right)\right\}\left|a_{p+1}\right|^2\\&\leqq\frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\,\frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\,\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\end{aligned}$$

(ii) If $\sigma_3^* \leqq \mu \leqq \sigma_2^*$, *then*

$$\begin{aligned}&\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{\left([p+1]_q-[p]_q\right)^2}{[p]_q\left([p+2]_q-[p]_q\right)B_1^2 b}\,\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}\\&\quad\cdot\frac{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}\left\{(B_1-B_2)-\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\right.\\&\quad\cdot\left(\mu\,\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\,\frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})}\right.\\&\quad\left.\left.\cdot\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}-1\right)\right\}\left|a_{p+1}\right|^2\\&\leqq\frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\,\frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\,\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}.\end{aligned}$$

Remark 5. For different choices of the parameters p, q and λ in Theorem 3 and Theorem 4, we can obtain new results for each of the following p-valently analytic function classes:

$$\mathcal{S}_{b,p}^*(\varphi),\quad \mathcal{C}_{b,p}(\varphi),\quad \mathcal{S}_{\lambda,b,p}(\varphi),\quad \mathcal{S}_{q,b}(\varphi),\quad \mathcal{C}_{q,b}(\varphi),\quad \mathcal{S}_b^*(\varphi)\quad \text{and}\quad \mathcal{C}_b(\varphi),$$

which are defined in Section 1.

Remark 6. For different choices of the parameters b and λ in Theorem 3 and Theorem 4, we can deduce new results for each of the following p-valently analytic function classes:

$$\mathcal{S}^{\alpha,\gamma}_{q,p,\lambda}(\varphi), \quad \mathcal{S}^{\alpha}_{q,p,\lambda}(\varphi) \qquad \text{and} \qquad \mathcal{C}^{\alpha}_{q,p,\lambda}(\varphi),$$

which are defined in Section 1.

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