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COMPONENTWISE PERTURBATION BOUNDS FOR THE LU, LDU AND LDL^T DECOMPOSITIONS

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Abstract. We improve a componentwise perturbation bound of Sun for the LU factorization and derive a new perturbation bound for the LDU factorization. The latter bound also improves a result of Sun given for the LDL^T factorization.

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1. Introduction

Perturbation bounds for the LU, LDL^T factorizations are given by many authors (e.g., see [1], [9], [7], [8], [2]). Here we improve the componentwise LU perturbation bound of Sun [9] and derive a new perturbation bound for the LDU decomposition. These bounds are used to investigate the stability of full rank factorizations produced by Egerváry's rank reduction procedure [4], [3]. The LDU perturbation bounds are then applied to positive definite symmetric matrices. The result is shown to be better than the LDL^T perturbation result of Sun [9].

We need the following notations. Let $A = [a_{ij}]_{i,j=1}^n$. Then $|A| = [|a_{ij}|]_{i,j=1}^n$,

$$diag\,(A) = diag\,(a_{11}, a_{22}, \ldots, a_{nn}),$$

$tril\,(A,l) = [\alpha_{ij}]_{i,j=1}^n$ and $triu\,(A,l) = \left[\beta_{ij}\right]_{i,j=1}^n$, where $0 \le |l| < n$ and

$$\alpha_{ij} = \begin{cases} a_{ij}, & i \ge j-l \\ 0, & i < j-l \end{cases}, \qquad \beta_{ij} = \begin{cases} a_{ij}, & i \le j-l \\ 0, & i > j-l \end{cases}.$$

We also use the special notations $tril\,(A) = tril\,(A,0)$, $tril^*\,(A) = tril\,(A,-1)$, $triu\,(A) = triu\,(A,0)$ and $triu^*\,(A) = triu\,(A,1)$. The spectral radius of A will be denoted by $\rho\,(A)$. For two matrices $A, B \in R^{n\times n}$ the relation $A \le B$ holds if and only if $a_{ij} \le b_{ij}$ for all $i,j = 1,\ldots,n$. Let $\widetilde{I}_k = \sum_{i=1}^k e_i e_i^T$ ($e_i \in R^n$ is the ith unit vector) for $1 \le k \le n$, $\widetilde{I}_k = 0$ for $k \le 0$ and $\min\,(A,B) = [\min\,(a_{ij}, b_{ij})]_{i,j=1}^n$.

In Sections 2 and 3 we derive the perturbation bound for the LU and LDU factorizations. A numerical example is shown in Section 4.

2. The LU factorization

We first prove the following

Lemma 1 *Assume that $A, B, C \in R^{n\times n}$ are such that $A, B, C \geq 0$ and $\rho(B) < 1$. The maximal solution of the inequality $A \leq C + Btriu(A,l)$ $(l \geq 0)$ is A^* $(A^* \geq C)$, where $A^*e_k = \left(I - B\widetilde{I}_{k-l}\right)^{-1} Ce_k$ $(k = 1, \dots, n)$. A^* is the unique solution of the fixed point problem $A = f(A) = C + Btriu(A,l)$. If $A_0 = (I-B)^{-1} C$, then $A_i = f(A_{i-1})$ converges to A^* monotonically decreasing as $i \to +\infty$ and $0 \leq A_i - A^* \leq (I-B)^{-1} B^i (A_0 - A_1)$ $(i \geq 1)$.*

Proof. It follows from $A \leq C + Btriu(A,l) \leq C + BA$ that $(I-B)A \leq C$. As $I-B$ is a nonsingular M-matrix by assumption we obtain the upper bound $A \leq A_0 = (I-B)^{-1} C$. As

$$\left|f(A) - f\left(\widetilde{A}\right)\right| = \left|B\left(triu(A,l) - triu\left(\widetilde{A},l\right)\right)\right| \leq B\left|A - \widetilde{A}\right|$$

for any two $n \times n$ matrices A and $\widetilde{A}$, the map $f(A)$ is a B-contraction [6] on $R^{n\times n}$ and there is a unique fixed point $A^* = f(A^*)$. Let $X_0 \in R^{n\times n}$ be arbitrary and $X_k = f(X_{k-1})$ $(k \geq 1)$. Then $|A^* - X_k| \leq (I-B)^{-1} B^k |X_1 - X_0|$ $(k \geq 1)$. As for any $0 \leq A \leq \widetilde{A}$, $f(A) \leq f\left(\widetilde{A}\right)$ holds and

$$A_1 = C + Btriu\left((I-B)^{-1} C, l\right) \leq C + B(I-B)^{-1} C = (I-B)^{-1} C = A_0,$$

the sequence $A_i = f(A_{i-1})$ tends to A^* and is monotonically decreasing. We prove that A^* is the maximal solution of the inequality. Assume that a solution $\widetilde{A}$ exists such that $\widetilde{A} \geq A^*$. Then $\widetilde{A} = A^* + L + U$, where $triu(U,l) = U$ and $tril(L, l-1) = L$. Then

$$\widetilde{A} = A^* + L + U \leq C + Btriu(A^* + L + U, l) \leq C + Btriu(A,l) + BU$$

must hold implying that $L + U \leq BU$ and $0 \leq U \leq -(I-B)^{-1} L \leq 0$. Hence $U = L = 0$. The kth column of A^* can be written as $A^*e_k = Ce_k + Btriu(A^*,l)e_k$, where $triu(A^*,l)e_k = \widetilde{I}_{k-l}A^*e_k$. Hence we obtain $A^*e_k = \left(I - B\widetilde{I}_{k-l}\right)^{-1} Ce_k$. ■

Remark 2 *The sequence $\{A_i\}_{i\geq 0}$ gives an improving sequence of upper estimates for the maximal solution A^* of the inequality.*

We will use the following notations: $A^* = \phi(B,C,l)$, $A_i = \phi_i(B,C,l)$, $\phi_0(B,C,l) = (I-B)^{-1} C$ and $\phi_i(B,C,l) = C + Btriu\left(\phi_{i-1}(B,C,l), l\right)$ $(i \geq 1)$. Notice that for any diagonal matrix $\widetilde{D}$, $\phi\left(B, C\widetilde{D}, l\right) = \phi(B,C,l)\widetilde{D}$ and $\phi_i\left(B, C\widetilde{D}, l\right) = \phi_i(B,C,l)\widetilde{D}$.

Remark 3 *Consider the inequality* $A \leq C + tril\,(A,-l)\,B$ $(l \geq 0)$ *with* $0 \leq A, B, C \in R^{n\times n}$ *and* $\rho\,(B) < 1$. *By transposition we obtain* $A^T \leq C^T + B^T tril\,(A,-l)^T = C^T + B^T triu\left(A^T, l\right)$ *the maximal solution of which is given by* $\phi\left(B^T, C^T, l\right)$. *The sequence* $\phi_i\left(B^T, C^T, l\right)$ *tends to* $\phi\left(B^T, C^T, l\right)$ *and is monotonically decreasing. Hence for the original inequality we have the maximal solution* $\phi\left(B^T, C^T, l\right)^T$ *and the monotone decreasing sequence* $\phi_i\left(B^T, C^T, l\right)^T$ *converging to* $\phi\left(B^T, C^T, l\right)^T$.

The next theorem improves the componentwise estimate of Sun [9].

Theorem 4 *Assume that the* $n \times n$ *matrix* A *has the* LU *decomposition* $A = L_1 U$, *where* L_1 *is unit lower triangular and* U *is upper triangular. Also assume that the perturbed matrix* $A + \delta_A$ *has the* LU *decomposition* $A + \delta_A = (L_1 + \delta_{L_1})(U + \delta_U)$, *where* $L_1 + \delta_{L_1}$ *is unit lower triangular and* $U + \delta_U$ *is upper triangular. Finally assume that* $\rho\left(\left|L_1 \delta_A U^{-1}\right|\right) < 1$. *Then we have*

$$|\delta_{L_1}| \leq |L_1|\, tril^*\left(\phi\left(\left|L_1^{-1}\delta_A U^{-1}\right|, \left|L_1^{-1}\delta_A U^{-1}\right|, 0\right)\right), \tag{2.1}$$

$$|\delta_U| \leq triu\left(\phi\left(\left|L_1^{-1}\delta_A U^{-1}\right|^T, \left|L_1^{-1}\delta_A U^{-1}\right|^T, 1\right)^T\right)|U|. \tag{2.2}$$

Proof. Using the relation

$$\delta_U\,(U+\delta_U)^{-1} + L_1^{-1}\delta_{L_1} = L_1^{-1}\delta_A\,(U+\delta_U)^{-1},$$

where $L_1^{-1}\delta_{L_1}$ is a strict lower triangular matrix, while $\delta_U\,(U+\delta_U)^{-1}$ is upper triangular, we can establish the relations

$$tril^*\left(L_1^{-1}\delta_A\,(U+\delta_U)^{-1}\right) = L_1^{-1}\delta_{L_1}, \tag{2.3}$$

$$triu\left(L_1^{-1}\delta_A\,(U+\delta_U)^{-1}\right) = \delta_U\,(U+\delta_U)^{-1}. \tag{2.4}$$

From relation

$$L_1^{-1}\delta_A\,(U+\delta_U)^{-1} = L_1^{-1}\delta_A U^{-1} - L_1^{-1}\delta_A U^{-1}\delta_U\,(U+\delta_U)^{-1} \tag{2.5}$$

we obtain the inequality

$$\left|L_1^{-1}\delta_A\,(U+\delta_U)^{-1}\right| \leq \left|L_1^{-1}\delta_A U^{-1}\right| + \left|L_1^{-1}\delta_A U^{-1}\right| triu\left(\left|L_1^{-1}\delta_A\,(U+\delta_U)^{-1}\right|\right).$$

Applying Lemma 1 we obtain the bound

$$\left|L_1^{-1}\delta_A\,(U+\delta_U)^{-1}\right| \leq A^* = \phi\left(\left|L_1^{-1}\delta_A U^{-1}\right|, \left|L_1^{-1}\delta_A U^{-1}\right|, 0\right).$$

Hence $\left|L_1^{-1}\delta_{L_1}\right| \leq tril^*(A^*)$ and $|\delta_{L_1}| \leq |L_1|\, tril^*(A^*)$.

Using the relation

$$\delta_U U^{-1} + (L_1+\delta_{L_1})^{-1}\delta_{L_1} = (L_1+\delta_{L_1})^{-1}\delta_A U^{-1},$$

where $(L_1+\delta_{L_1})^{-1}\delta_{L_1}$ is a strict lower triangular matrix, while $\delta_U U^{-1}$ is upper triangular, we can also establish the relations

$$tril^*\left((L_1+\delta_{L_1})^{-1}\delta_A U^{-1}\right) = (L_1+\delta_{L_1})^{-1}\delta_{L_1} \tag{2.6}$$

and

$$triu\left((L_1+\delta_{L_1})^{-1}\delta_A U^{-1}\right) = \delta_U U^{-1}. \tag{2.7}$$

>From relation

$$(L_1+\delta_{L_1})^{-1}\delta_A U^{-1} = L_1^{-1}\delta_A U^{-1} - (L_1+\delta_{L_1})^{-1}\delta_{L_1}L_1^{-1}\delta_A U^{-1} \tag{2.8}$$

we obtain the inequality

$$\left|(L_1+\delta_{L_1})^{-1}\delta_A U^{-1}\right| \leq \left|L_1^{-1}\delta_A U^{-1}\right| + tril^*\left(\left|(L_1+\delta_{L_1})^{-1}\delta_A U^{-1}\right|\right)\left|L_1^{-1}\delta_A U^{-1}\right|$$

the maximal solution of which is

$$\left|(L_1+\delta_{L_1})^{-1}\delta_A U^{-1}\right| \leq \widetilde{A}^* = \phi\left(\left|L_1^{-1}\delta_A U^{-1}\right|^T, \left|L_1^{-1}\delta_A U^{-1}\right|^T, 1\right)^T.$$

Hence $\left|\delta_U U^{-1}\right| \leq triu\left(\widetilde{A}^*\right)$ and $|\delta_U| \leq triu\left(\widetilde{A}^*\right)|U|$. This completes the proof. ■

Remark 5 *If function ϕ is replaced by ϕ_0 in (2.1)-(2.2) we obtain the theorem of Sun [9], Thm. 5.1). Hence, our result is sharper.*

Remark 6 *Assume that the LU factorizations $A = LU_1$ and*

$$A + \delta_A = (L+\delta_L)(U_1+\delta_{U_1})$$

are such that U_1 and $U_1+\delta_{U_1}$ are upper unit triangular. If $A^T = U_1^T L^T$ and $A^T+\delta_A^T = \left(U_1^T+\delta_{U_1}^T\right)\left(L^T+\delta_L\right)$ satisfy the conditions of the previous theorem we may write

$$|\delta_L| \leq |L|\, tril\left(\phi\left(\left|L^{-1}\delta_A U_1^{-1}\right|, \left|L^{-1}\delta_A U_1^{-1}\right|, 1\right)\right), \tag{2.9}$$

and

$$|\delta_{U_1}| \leq triu^*\left(\phi\left(\left|L^{-1}\delta_A U_1^{-1}\right|^T, \left|L^{-1}\delta_A U_1^{-1}\right|^T, 0\right)^T\right)|U_1|. \tag{2.10}$$

Hence, Theorem 4 is also true for the case $A = LU_1$ with unit upper triangular U_1. Notice, however, that we have here tril and triu instead of tril* and triu, respectively. This is due to the change of the unit triangular part in the LU factorization.*

3. The LDU factorization

Consider the LDU factorization $A = L_1 D U_1$ with unit lower triangular L_1, diagonal D and unit upper triangular U_1. Assume that $A + \delta_A$ can be factorized so that

$$A + \delta_A = (L_1 + \delta_{L_1})(D + \delta_D)(U_1 + \delta_{U_1})$$

where $L_1 + \delta_{L_1}$ is unit lower triangular and $U + \delta_{U_1}$ is unit upper triangular. For δ_{L_1} and δ_{U_1} we have the bounds (2.1) and (2.10), respectively. We now look for an estimate of δ_D. We use the relation

$$L_1^{-1}\delta_A (U_1 + \delta_{U_1})^{-1} = D\delta_{U_1}(U + \delta_{U_1})^{-1} + \delta_D + L_1^{-1}\delta_{L_1}(D + \delta_D),$$

where the matrix $D\delta_{U_1}(U + \delta_{U_1})^{-1}$ is strict upper triangular, δ_D is diagonal, and $L_1^{-1}\delta_{L_1}(D + \delta_D)$ is strict lower triangular. Hence

$$tril^*\left(L_1^{-1}\delta_A (U_1 + \delta_{U_1})^{-1}\right) = L_1^{-1}\delta_{L_1}(D + \delta_D), \tag{3.1}$$

$$diag\left(L_1^{-1}\delta_A (U_1 + \delta_{U_1})^{-1}\right) = \delta_D, \tag{3.2}$$

$$triu^*\left(L_1^{-1}\delta_A (U_1 + \delta_{U_1})^{-1}\right) = D\delta_{U_1}(U_1 + \delta_{U_1})^{-1}. \tag{3.3}$$

>From relation

$$L_1^{-1}\delta_A (U_1 + \delta_{U_1})^{-1} = L_1^{-1}\delta_A U_1^{-1} - L_1^{-1}\delta_A U_1^{-1}\delta_{U_1}(U_1 + \delta_{U_1})^{-1} \tag{3.4}$$

we obtain the inequality

$$\begin{aligned} \left|L_1^{-1}\delta_A (U_1 + \delta_{U_1})^{-1}\right| \leq\; & \left|L_1^{-1}\delta_A U_1^{-1} D^{-1}\right| |D| + \\ & + \left|L_1^{-1}\delta_A U_1^{-1} D^{-1}\right| triu^*\left(\left|L_1^{-1}\delta_A (U_1 + \delta_{U_1})^{-1}\right|\right) \end{aligned}$$

the maximal solution of which is given by the bound

$$\left|L_1^{-1}\delta_A (U_1 + \delta_{U_1})^{-1}\right| \leq \phi\left(\left|L_1^{-1}\delta_A U_1^{-1} D^{-1}\right|, \left|L_1^{-1}\delta_A U_1^{-1} D^{-1}\right| |D|, 1\right).$$

Hence $|\delta_D| \leq |D|\, diag\left(\phi\left(\left|L_1^{-1}\delta_A U_1^{-1} D^{-1}\right|, \left|L_1^{-1}\delta_A U_1^{-1} D^{-1}\right|, 1\right)\right)$.

We may get another estimate by using the expression

$$(L_1 + \delta_{L_1})^{-1}\delta_A U_1^{-1} = (D + \delta_D)\delta_{U_1} U_1^{-1} + \delta_D + (L_1 + \delta_{L_1})^{-1}\delta_{L_1} D,$$

where the matrix $(D+\delta_D)\,\delta_{U_1}U_1^{-1}$ is strict upper triangular, δ_D is diagonal, and $(L_1+\delta_{L_1})^{-1}\,\delta_{L_1}D$ is strict lower triangular. Hence

$$tril^*\left((L_1+\delta_{L_1})^{-1}\,\delta_A U_1^{-1}\right) = (L_1+\delta_{L_1})^{-1}\,\delta_{L_1}D, \tag{3.5}$$

$$diag\left((L_1+\delta_{L_1})^{-1}\,\delta_A U_1^{-1}\right) = \delta_D, \tag{3.6}$$

$$triu^*\left((L_1+\delta_{L_1})^{-1}\,\delta_A U_1^{-1}\right) = (D+\delta_D)\,\delta_{U_1}U_1^{-1}. \tag{3.7}$$

>From relation

$$(L_1+\delta_{L_1})^{-1}\,\delta_A U_1^{-1} = L_1^{-1}\delta_A U_1^{-1} - (L_1+\delta_{L_1})^{-1}\,\delta_{L_1}L_1^{-1}\delta_A U_1^{-1} \tag{3.8}$$

we obtain the inequality

$$\begin{aligned}\left|(L_1+\delta_{L_1})^{-1}\,\delta_A U_1^{-1}\right| \;&\le\; |D|\left|D^{-1}L_1^{-1}\delta_A U_1^{-1}\right| + \\ &+tril^*\left(\left|(L_1+\delta_{L_1})^{-1}\,\delta_A U_1^{-1}\right|\right)\left|D^{-1}L_1^{-1}\delta_A U_1^{-1}\right|.\end{aligned}$$

It has the maximal solution

$$\left|(L_1+\delta_{L_1})^{-1}\,\delta_A U_1^{-1}\right| \le \phi\left(\left|D^{-1}L_1^{-1}\delta_A U_1^{-1}\right|^T, \left|D^{-1}L_1^{-1}\delta_A U_1^{-1}\right|^T |D|, 1\right)^T.$$

Hence $|\delta_D| \le |D|\,diag\left(\phi\left(\left|D^{-1}L_1^{-1}\delta_A U_1^{-1}\right|^T, \left|D^{-1}L_1^{-1}\delta_A U_1^{-1}\right|^T, 1\right)^T\right)$. We now have two estimates for $|\delta_D|$. As in general $|AD| \ne |DA|$ these two estimates are different. We can establish

Theorem 7 *Assume that the $n\times n$ matrix A has the LDU decomposition $A = L_1 D U_1$, where L_1 is unit lower triangular, D is diagonal and U_1 is unit upper triangular. Also assume that the perturbed matrix $A+\delta_A$ has the LDU decomposition $A+\delta_A = (L_1+\delta_{L_1})(D+\delta_D)(U_1+\delta_{U_1})$, where $L_1+\delta_{L_1}$ is unit lower triangular and $U_1+\delta_{U_1}$ is unit upper triangular. Finally assume that $\max(\rho(\Gamma_{L_1}), \rho(\Gamma_{U_1})) < 1$ holds with $\Gamma_{L_1} = \left|L_1^{-1}\delta_A U_1^{-1}D^{-1}\right|$ and $\Gamma_{U_1} = \left|D^{-1}L_1^{-1}\delta_A U_1^{-1}\right|$. Then the following inequalities are satisfied:*

$$|\delta_{L_1}| \le |L_1|\,tril^*\left(\phi\left(\Gamma_{L_1}, \Gamma_{L_1}, 0\right)\right), \tag{3.9}$$

$$|\delta_D| \le |D|\min\left\{diag\left(\phi\left(\Gamma_{L_1}, \Gamma_{L_1}, 1\right)\right), diag\left(\phi\left(\Gamma_{U_1}^T, \Gamma_{U_1}^T, 1\right)^T\right)\right\}, \tag{3.10}$$

$$|\delta_{U_1}| \le triu^*\left(\phi\left(\Gamma_{U_1}^T, \Gamma_{U_1}^T, 0\right)^T\right)|U_1|. \tag{3.11}$$

Remark 8 *If ϕ is replaced by ϕ_0, we obtain the following weaker estimates:*

$$|\delta_{L_1}| \leq |L_1| \, tril^* \left((I - \Gamma_{L_1})^{-1} \Gamma_{L_1} \right), \tag{3.12}$$

$$|\delta_{U_1}| \leq triu^* \left(\Gamma_{U_1} (I - \Gamma_{U_1})^{-1} \right) |U_1|, \tag{3.13}$$

$$|\delta_D| \leq |D| \min \left(diag \left((I - \Gamma_{L_1})^{-1} \Gamma_{L_1} \right), diag \left(\Gamma_{U_1} (I - \Gamma_{U_1})^{-1} \right) \right). \tag{3.14}$$

Next we specialize the above result for symmetric and positive definite matrices. In such a case $\Gamma_{L_1} = \Gamma_{U_1}^T$ ($\Gamma_{L_1} = \left| L_1^{-1} \delta_A L_1^{-T} D^{-1} \right|$, $\Gamma_{U_1} = \left| D^{-1} L_1^{-1} \delta_A L_1^{-T} \right|$) and we have the following

Corollary 9 *Assume that A is symmetric and positive definite and its perturbation δ_A is such that $A + \delta_A$ remains symmetric and positive definite. If A and $A + \delta_A$ are written in the forms $A = L_1 D L_1^T$ $(D \geq 0)$ and*

$$A + \delta_A = (L_1 + \delta_{L_1}) (D + \delta_D) \left(L_1^T + \delta_{L_1}^T \right),$$

respectively, then

$$|\delta_{L_1}| \leq |L_1| \, tril^* \left(\phi \left(\Gamma_{L_1}, \Gamma_{L_1}, 0 \right) \right) \tag{3.15}$$

and

$$|\delta_D| \leq D diag \left(\phi \left(\Gamma_{L_1}, \Gamma_{L_1}, 1 \right) \right). \tag{3.16}$$

Replacing ϕ by the weaker estimate ϕ_0, we obtain the following bounds:

$$|\delta_{L_1}| \leq |L_1| \, tril^* \left((I - \Gamma_{L_1})^{-1} \Gamma_{L_1} \right) \tag{3.17}$$

and

$$|\delta_D| \leq D diag \left((I - \Gamma_{L_1})^{-1} \Gamma_{L_1} \right). \tag{3.18}$$

We recall that Sun ([9], Thm. 3.1) for symmetric positive definite matrices proved that

$$|\delta_{L_1}| \leq |L_1| \, tril^* \left(E_{ld} \left(I - diag \left(D^{-1} E_{ld} \right) \right)^{-1} D^{-1} \right), \tag{3.19}$$

$$|\delta_D| \leq diag \left(E_{ld} \right) \tag{3.20}$$

with

$$E_{ld} = \left(I - \left| L_1^{-1} \delta_A L_1^{-T} \right| D^{-1} \right)^{-1} \left| L_1^{-1} \delta_A L_1^{-T} \right|. \tag{3.21}$$

We compare now estimates (3.17)-(3.18) and (3.19)-(3.20), respectively. We exploit the fact that for any diagonal matrix D, $|AD| = |A|\,|D|$ and $\mathrm{diag}(AD) = diag\,(A)\,D$ hold. We can write

$$(I-\Gamma_{L_1})^{-1}\,\Gamma_{L_1} = \left(I - \left|L_1^{-1}\delta_A L_1^{-T}\right| D^{-1}\right)^{-1} \left|L_1^{-1}\delta_A L_1^{-T}\right| D^{-1} = E_{ld}D^{-1}$$

and then estimate (3.18) yield

$$|\delta_D| \leq diag\left(\left(I - \left|L_1^{-1}\delta_A L_1^{-T}\right| D^{-1}\right)^{-1} \left|L_1^{-1}\delta_A L_1^{-T}\right|\right) = diag\,(E_{ld})\,.$$

As $\left(I - diag\left(D^{-1}E_{ld}\right)\right)^{-1} \geq I$ and $E_{ld}\left(I - diag\left(D^{-1}E_{ld}\right)\right)^{-1} D^{-1} \geq E_{ld}D^{-1}$, the bound (3.19) satisfies

$$|L_1|\,tril^*\left(E_{ld}\left(I - diag\left(D^{-1}E_{ld}\right)\right)^{-1} D^{-1}\right) \geq |L_1|\,tril^*\left((I-\Gamma_{L_1})^{-1}\,\Gamma_{L_1}\right).$$

Thus it follows that Theorem 7 improves the special LDL^T perturbation result of Sun ([9], Thm. 3.1).

4. Final remarks

Computer experiments on symmetric positive definite MATLAB test matrices indicate that estimate ϕ_1 is often so good as ϕ itself. We could observe significant difference between the estimates if Γ_{L_1} was relatively large. A typical result is shown in Figure 4.1.

Here we display the maximum difference between the components of the bound and the true error matrix for Example 6.1 of [9] to which we added 20 random symmetric matrices with elements of the magnitude 5×10^{-3}. Hence, the line marked with $+$ denotes estimate (3.19) of Sun, the line with triangles denotes the estimate (3.17), the solid line denotes estimate ϕ_1, while the line with circles denotes the best estimate.

The estimates of Theorems 4 and 7 are optimal, if one accepts inequalities of the form $A \leq C + Btriu\,(A,l)\quad (A,B,C \geq 0)$ in the estimation process. We can solve, however, the equation $A = C + Btriu\,(A,l)$ without any nonnegativity condition. Hence we can give exact expressions for the perturbation errors. For example, in case of Theorem 4 we can prove the following result.

Theorem 10 *For $k = 1,\dots,n$ we have*

$$\delta_{L_1}e_k = \begin{bmatrix} 0 \\ -\left(L_2^{(k)}\left(L_1^{(k)}\right)^{-1}\delta_1^{(k)} - \delta_2^{(k)}\right)\left(L_1^{(k)}U_1^{(k)} + \delta_1^{(k)}\right)^{-1}\widetilde{e}_k \end{bmatrix}$$

and

$$e_k^T\delta_{U_1} = \left[0, -\widetilde{e}_k^T\left(L_1^{(k)}U_1^{(k)} + \delta_1^{(k)}\right)^{-1}\left(\delta_1^{(k)}\left(U_1^{(k)}\right)^{-1}U_2^{(k)} - \delta_4^{(k)}\right)\right],$$

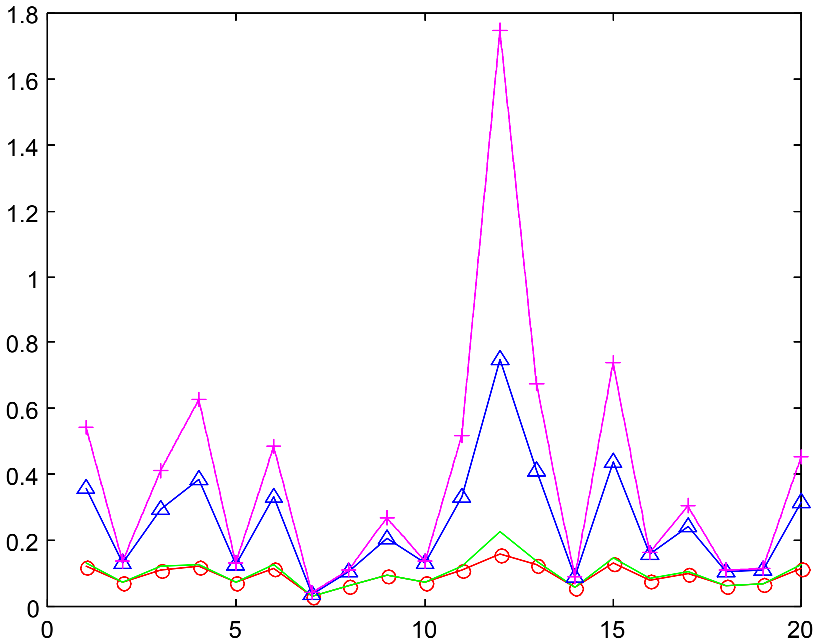

Figure 1. Perturbation bounds for the LDL^T factorization

where $\widetilde{e}_k \in R^k$ *is the kth unit vector,*

$$L_1 = \begin{bmatrix} L_1^{(k)} & 0 \\ L_2^{(k)} & L_3^{(k)} \end{bmatrix}, \qquad U = \begin{bmatrix} U_1^{(k)} & U_2^{(k)} \\ 0 & U_3^{(k)} \end{bmatrix}, \qquad \delta_A = \begin{bmatrix} \delta_1^{(k)} & \delta_4^{(k)} \\ \delta_2^{(k)} & \delta_3^{(k)} \end{bmatrix}$$

and $L_1^{(k)}, U_1^{(k)}, \delta_1^{(k)} \in R^{k\times k}$.

It does not seem easy to find componentwise estimates better than those of Theorem 4. We can obtain, however, better result than those of Chang and Paige [2].

Finally we remark that either from Theorem 4 or Theorem 7 we can easily obtain normwise perturbation estimates slightly weaker than those of Barrlund [1] by simply using the relation $\||A|\| = \|A\|_F$ and ϕ_0 instead of ϕ.

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