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NULL CONTROLLABILITY OF NONLOCAL HILFER FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study exact null controllability of Hilfer fractional semilinear stochastic differential equations in Hilbert spaces. By using fractional calculus and fixed point approach, sufficient conditions of exact null controllability for such fractional systems are established. An example is given to show the application of our results.

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1. INTRODUCTION

The stochastic differential equations arise in many mathematical models [5, 14, 18, 20]. The problem of controllability of nonlinear stochastic or deterministic system has been discussed in [3, 4, 6, 8, 9, 16, 19].

Recently, basic theory of differential equations involving Caputo and Riemann-Liouville fractional derivatives can be found in [1, 2, 13, 21–24, 26–30] and the references cited therein. Beside Caputo and Riemann-Liouville fractional derivatives, there exists a new definition of fractional derivative introduced by Hilfer, which generalized the concept of Riemann-Liouville derivative and has many application in physics, for more details, see [10–12, 25].

In this paper, we investigate the exact null controllability of Hilfer fractional semilinear stochastic differential equation of the form

$$\begin{cases} D_{0+}^{\nu,\mu}x(t) = Ax(t) + Bu(t) \\ +F(t,x(t)) + G(t,x(t))\frac{d\omega(t)}{dt}, \quad t \in J = [0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}x(0) + h(x) = x_0, \end{cases} \tag{1.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \le \nu \le 1$, $\frac{1}{2} < \mu < 1$, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators

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$S(t),\ t \geq 0$, on a separable Hilbert space H with inner product $\langle .,. \rangle$ and norm $\| \,.\, \|$. There exists a $M \geq 1$ such that $\sup_{t\geq 0} \|S(t)\| \leq M$. The control function $u(\cdot)$ is given in $L_2(J,U)$, the Hilbert space of admissible control functions with U as a separable Hilbert space. The symbol B stands for a bounded linear operator from U into H. Here ω is an H-valued Wiener process associated with a positive, nuclear covariance operator Q, F is an H-valued map and G is a $L(K,H)$-valued map both defined on $J \times H$ (where K is a real separable Hilbert space with norn $\|\cdot\|_K$ and $L(K,H)$ is the space of all bounded, linear operators from K to H, we write simply $L(H)$ if $H = K$) and $h : C(J,H) \to H$.

2. Preliminaries

In this section, some definitions and results are given which will be used throughout this paper.

Definition 1 (see [15, 17])**.** The fractional integral operator of order $\mu > 0$ for a function f can be defined as

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)}{(t-s)^{1-\mu}} ds, \ \ t > 0$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 (see [11])**.** The Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ for a function f is defined by

$$D_{0+}^{\nu,\mu} f(t) = I_{0+}^{\nu(1-\mu)} \frac{d}{dt} I_{0+}^{(1-\nu)(1-\mu)} f(t).$$

Let (Ω, Υ, P) be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\Upsilon_t : t \in J\}$ satisfying $\Upsilon_t \subset \Upsilon$. An H-valued random variable is an Υ- measurable function $x(t) : \Omega \to H$ and a collection of random variables $\Psi = \{x(t,\omega) : \Omega \to H | t \in J\}$ is called a stochastic process. Usually we suppress the dependence on $\omega \in \Omega$ and write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \to H$ in the place of Ψ. Let $\beta_n(t)$ $(n = 1,2,...)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent over (Ω, Υ, P). Set

$$\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \ t \geq 0,$$

where λ_n, $(n = 1,2,...)$ are nonnegative real numbers and $\{e_n\}$ $(n = 1,2,...)$ is a complete orthonormal basis in K. Let $Q \in L(K,K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$, (Tr denotes the trace of the operator). Then the above K-valued stochastic process $\omega(t)$ is called Q-Wiener process.

We assume that $\Upsilon_t = \sigma\{\omega(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by ω.

For $\phi \in L(K,H)$ we define

$$\| \phi \|_Q^2 = Tr(\phi Q \phi^*) = \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \phi e_n \|^2 .$$

If $\| \phi \|_Q^2 < \infty$, then ϕ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\phi : K \to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $\| \, . \, \|_Q$ where $\| \phi \|_Q^2 = \|\phi,\phi\|$ is a Hilbert space with the above norm topology. The collection of all strongly-measurable, square-integrable, H-valued random variables, denoted by $L_2(\Omega,H)$, is a Banach space equipped with norm $\| x(\cdot) \|_{L_2(\Omega,H)} = (E \| x(.,\omega) \|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $E(x) = \int_\Omega x(\omega)dP$. An important subspace of $L_2(\Omega,H)$ is given by $L_2^0(\Omega,H) = \{x \in L_2(\Omega,H)$, x is Υ_0-measurable $\}$.

Let $C(J, L_2(\Omega,H))$ be the Banach space of all continuous maps from J into $L_2(\Omega,H)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$.

Define $Y = \{x : t^{(1-\nu)(1-\mu)}x(t) \in C(J, L_2(\Omega,H))\}$, with norm $\|\cdot\|_Y$ defined by $\|\cdot\|_Y = (\sup_{t\in J} E\|t^{(1-\nu)(1-\mu)}x(t)\|^2)^{\frac{1}{2}}$. Obviously, Y is a Banach space.

For $x \in H$, we define two families of operators $\{S_{\nu,\mu}(t) : t \geq 0\}$ and $\{P_\mu(t) : t \geq 0\}$ by

$$S_{\nu,\mu}(t) = I_{0+}^{\nu(1-\mu)} P_\mu(t),\ P_\mu(t) = t^{\mu-1}T_\mu(t),\ T_\mu(t) = \int_0^\infty \mu\theta\Psi_\mu(\theta)S(t^\mu\theta)d\theta,$$

where

$$\Psi_\mu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)},\ 0 < \mu < 1,\ \theta \in (0,\infty)$$

is a function of Wright-type which satisfies

$$\int_0^\infty \theta^\Psi \Psi_\mu(\theta)d\theta = \frac{\Gamma(1+\Psi)}{\Gamma(1+\mu\Psi)}$$

for $\theta \geq 0$.

Lemma 1 (see [10]). *The operator $S_{\nu,\mu}$ and P_μ have the following properties.*
(i) $\{P_\mu(t) : t > 0\}$ is continuous in the uniform operator topology.
(ii) For any fixed $t > 0$, $S_{\nu,\mu}(t)$ and $P_\mu(t)$ are linear and bounded operators, and

$$\|P_\mu(t)x\| \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)}\|x\|,\ \ \|S_{\nu,\mu}(t)x\| \leq \frac{Mt^{(\nu-1)(\mu-1)}}{\Gamma(\nu(1-\mu)+\mu)}\|x\|.$$

(iii) $\{P_\mu(t) : t > 0\}$ and $\{S_{\nu,\mu}(t) : t > 0\}$ are strongly continuous.

To study the exact null controllability of (1.1) we consider the fractional linear system

$$\begin{cases} D_{0+}^{\nu,\mu} y(t) = Ay(t) + Bu(t) + F(t) + G(t)\frac{d\omega(t)}{dt}, & t \in J = [0,b], \\ I_{0+}^{(1-\nu)(1-\mu)} y(0) = y_0, \end{cases} \tag{2.1}$$

associated with the system (1.1).

Define the operator

$$L_0^b u = \int_0^b P_\mu(b-s)Bu(s)ds \ : L_2(J,U) \to H,$$

where $L_0^b u$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L_2(J,U)/ker(L_0^b)$, and

$$\begin{aligned} N_0^b(y,F,G) &= S_{\nu,\mu}(b)y + \int_0^b P_\mu(b-s)F(s)ds \\ &\quad + \int_0^b P_\mu(b-s)G(s)d\omega(s) : H \times L_2(J,U) \to H. \end{aligned}$$

Definition 3. The system (2.1) is said to be exactly null controllable on J if

$$\operatorname{Im} L_0^b \supset \operatorname{Im} N_0^b.$$

By [7], the system (2.1) is exactly null controllable if there exists $\gamma > 0$ such that

$$\|(L_0^b)^* y\|^2 \geq \gamma \|(N_0^b)^* y\|^2$$

for all $y \in H$.

Lemma 2 (see [16]). *Suppose that the linear system (2.1) is exactly null controllable on J. Then the linear operator*

$$W = (L_0)^{-1} N_0^b : H \times L_2(J,H) \to L_2(J,U)$$

is bounded and the control

$$\begin{aligned} u(t) &= -(L_0)^{-1}\left[S_{\nu,\mu}(b)y_0 + \int_0^b P_\mu(b-s)F(s)ds + \int_0^b P_\mu(b-s)G(s)d\omega(s) \right] \\ &= -W(y_0,F,G) \end{aligned}$$

transfers the system (2.1) from y_0 to 0, where L_0 is the restriction of L_0^b to $[\ker L_0^b]^\perp$, $F \in L_2(J,H)$ and $G \in L_2(J,L(K,H))$.

3. EXACT NULL CONTROLLABILITY

In this section, we formulate sufficient conditions for exact null controllability for the system (1.1). First, we give the definitions of mild solution and exact null controllability for it.

Definition 4. We say $x \in C(J, L_2(\Omega, H))$ is a mild solution to (1.1) if it satisfies that

$$x(t) = S_{\nu,\mu}(t)[x_0 - h(0)] + \int_0^t P_\mu(t-s)[F(s,x(s)) + Bu(s)]ds$$
$$+ \int_0^t P_\mu(t-s)G(s,x(s))d\omega(s),\ t \in J.$$

Definition 5. The system (1.1) is said to be exact null controllable on the interval J if there exists a stochastic control $u \in L_2(J,U)$ such that the solution x of the system (1.1) satisfies $x(b) = 0$.

To prove the main result, we need the following hypotheses:

$(H1)$ The fractional linear system (2.1) is exactly null controllable on J.

$(H2)$ The function $F : J \times H \to H$ is locally Lipschitz continuous, for all $t \in J,\ x,\ x_1,\ x_2 \in H$, there exist constant $c_1 > 0$, such that

$$\|F(t,x_2) - F(t,x_1)\|^2 \le c_1\|x_2 - x_1\|^2, \quad \|F(t,x)\|^2 \le c_1(1 + \|x\|^2).$$

$(H3)$ The function $G : J \times H \to L(K,H)$ is locally Lipschitz continuous, for all $t \in J,\ x,\ x_1,\ x_2 \in H$, there exist constant $c_2 > 0$, such that

$$\|G(t,x_2) - G(t,x_1)\|_Q^2 \le c_2\|x_2 - x_1\|^2, \quad \|G(t,x)\|_Q^2 \le c_2(1 + \|x\|^2).$$

$(H4)$ The function $h : C(J,H) \to H$ is continuous, for any $x,\ x_1,\ x_2 \in C(J,H)$, there exist constant $c_3 > 0$, such that

$$\|h(x_2) - h(x_1)\|^2 \le c_3\|x_2 - x_1\|^2, \quad \|h(x)\|^2 \le c_3(1 + \|x\|^2).$$

Set $\varrho_1 := \frac{4M^2c_3}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{M^2b^{1+2\nu(\mu-1)}}{(2\mu-1)\Gamma^2(\mu)}(c_1 + c_2Tr(Q))$ and $\varrho_2 := 1 + \frac{4M^2b^{2\mu-1}\|W\|^2\|B\|^2}{(2\mu-1)\Gamma^2(\mu)}$.

Theorem 1. *If the hypotheses* $(H1)$-$(H4)$ *are satisfied, then the system* (1.1) *is exactly null controllable on* J *provided that*

$$\varrho := \varrho_1\varrho_2 < 1. \tag{3.1}$$

Proof. For an arbitrary x define the operator Φ on Y as follows

$$(\Phi x)(t) = S_{\nu,\mu}(t)[x_0 - h(x)] \tag{3.2}$$
$$+ \int_0^t P_\mu(t-s)[F(s,x(s)) - BW(x_0 - h(x), F, G)]ds$$

$$+\int_0^t P_\mu(t-s)G(s,x(s))d\omega(s), \quad t\in J,$$

where

$$\begin{aligned} u(t) &= W(x_0-h(x),F,G)(t)\\ &= -(L_0)^{-1}\{S_{\nu,\mu}(b)[x_0-h(x)]\\ &+\int_0^b P_\mu(b-s)F(s,x(s))ds+\int_0^b P_\mu(b-s)G(s,x(s))d\omega(s)\}. \end{aligned}$$

It will be shown that the operator Φ from Y into itself has a fixed point.
Step 1. The control $u(\cdot)=-W(x_0-h(x),F,G)$ is bounded on Y.
Indeed,

$$\begin{aligned} \|u\|_Y^2 &= \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|u\|^2 \\ &\le \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|W(x_0-h(x),F,G)(s)\|^2\\ &\le \|W\|^2\left\{\frac{M^2}{\Gamma^2(\nu(1-\mu)+\mu)}[E\|x_0\|^2+c_3(1+E\|x\|^2)]\right.\\ &\left.+\frac{M^2b^{1+2\nu(\mu-1)}}{(2\mu-1)\Gamma^2(\mu)}(1+E\|x\|^2)(c_1+c_2Tr(Q))\right\}. \end{aligned} \tag{3.3}$$

Step 2. We show that Φ maps Y into itself.
From (3.2) and (3.3) for $t\in J$, we have

$$\begin{aligned} \|(\Phi x)(t)\|_Y^2 &= \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|(\Phi x)(t)\|^2\\ &\le 4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\left\{E\|S_{\nu,\mu}(t)[x_0-h(x)]\|^2\right.\\ &\left.+E\left\|\int_0^t P_\mu(t-s)F(s,x(s))ds\right\|^2\right\}\\ &+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\\ &\times\left\{E\left\|\int_0^t P_\mu(t-s)[BW(x_0-h(x),F,G)(s)]ds\right\|^2\right.\\ &\left.+E\left\|\int_0^t P_\mu(t-s)G(s,x(s))d\omega(s)\right\|^2\right\}\\ &\le\left[\frac{4M^2}{\Gamma^2(\nu(1-\mu)+\mu)}[E\|x_0\|^2+c_3(1+E\|x\|^2)]\right. \end{aligned}$$

$$+\frac{4M^2b^{1+2\nu(\mu-1)}}{(2\mu-1)\Gamma^2(\mu)}(1+E\|x\|^2)(c_1+c_2Tr(Q))\Bigg]$$
$$\times\left[1+\frac{M^2\|B\|^2\|W\|^2\|b^{2\mu-1}}{(2\mu-1)\Gamma^2(\mu)}\right]<\infty.$$

Therefore Φ maps Y into itself.

Step 3. We prove $(\Phi x)(t)$ is continuous on J for any $x\in Y$.

Let $0<t\leq b$ and $\epsilon>0$ be sufficiently small, then,

$$\begin{aligned}
&\|(\Phi x)(t+\epsilon)-(\Phi x)(t)\|_Y^2 \qquad (3.4)\\
&=\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\|(\Phi x)(t+\epsilon)-(\Phi x)(t)\|^2\\
&\leq 4\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times E\|(S_{\nu,\mu}(t+\epsilon)-S_{\nu,\mu}(t))[x_0-h(x)]\|^2\\
&+4\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times E\left\|\int_0^{t+\epsilon}P_\mu(t+\epsilon-s)[BW(x_0-h(x),F,G)(s)]ds\right.\\
&\left.-\int_0^t P_\mu(t-s)[BW(x_0-h(x),F,G)(s)]ds\right\|^2\\
&+4\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times E\left\|\int_0^{t+\epsilon}P_\mu(t+\epsilon-s)F(s,x(s))ds-\int_0^t P_\mu(t-s)F(s,x(s))ds\right\|^2\\
&+4\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times E\left\|\int_0^{t+\epsilon}P_\mu(t+\epsilon-s)G(s,x(s))d\omega(s)-\int_0^t P_\mu(t-s)G(s,x(s))d\omega(s)\right\|^2.
\end{aligned}$$

Clearly, from Lemma 1, $(H2)$ and $(H3)$, the right hand side of (3.4) tends to zero as $\epsilon\to 0$. Hence, $(\Phi x)(t)$ is continuous on J.

Step 4. We show that $(\Phi x)(t)$ is a contraction on Y.

Let $x_1,\ x_2\in Y$, for any $t\in(0,b]$ be fixed, then

$$\begin{aligned}
&\|(\Phi x_2)(t)-(\Phi x_1)(t)\|_Y^2\\
&=\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\|(\Phi x_2)(t)-(\Phi x_1)(t)\|^2\\
&\leq 4\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\|(S_{\nu,\mu}(t)[h(x_2)-h(x_1)]\|^2
\end{aligned}$$

$$
\begin{aligned}
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\| \int_0^t P_\mu(t-s)[BW(x_0-h(x_2),F,G)(s)\right.\\
&\left. -BW(x_0-h(x_1),F,G)(s)]ds \right\|^2\\
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\| \int_0^t P_\mu(t-s)[F(s,x_2(s))-F(s,x_1(s))]ds \right\|^2\\
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\| \int_0^t P_\mu(t-s)[G(s,x_2(s))-G(s,x_1(s))]d\omega(s) \right\|^2\\
&\le \varrho E\|x_2-x_1\|^2.
\end{aligned}
$$

Hence, Φ is a contraction in Y via (3.1). From the Banach fixed point theorem, Φ has a unique fixed point. Therefore the system (1.1) is exact null controllable on J. $\square$

4. An example

Consider the following Hilfer fractional stochastic partial differential system

$$
\begin{cases}
D_{0+}^{\nu,\frac{2}{3}} x(t,z) = \frac{\partial^2}{\partial z^2} x(t,z) + u(t,z)\\
+f(t,x(t,z)) + g(t,x(t,z))\frac{d\omega(t)}{dt}, \quad t\in J,\ 0<z<1,\\
x(t,0)=x(t,1)=0,\ t\in J,\\
I_{0+}^{\frac{1}{3}(1-\nu)}(x(0,z)) + \sum_{i=1}^p k_i x(t_i,z) = x_0(z), \quad 0\le z\le 1,
\end{cases}
\tag{4.1}
$$

where p is a positive integer, $0<t_0<t_1<...<t_p<b$ and $\omega(t)$ is Wiener process, $u\in L_2(0,b)$, and $H=L_2([0,1])$. Let $f: R\times R\to R$ and $g: R\times R\to R$ are continuous and global Lipschitz continuous in the second variable. Also, let $A: H\to H$ be defined by $Ay=\frac{\partial^2}{\partial z^2}y$ with domain $D(A)=\{y\in H: y,\ \frac{\partial y}{\partial z}$ are absolutely continuous, and $\frac{\partial^2 y}{\partial z^2}\in H, y(0)=y(1)=0\}$.

It is known that A is self-adjoin and has the eigenvalues $\lambda_n=-n^2\pi^2$, $n\in N$, with the corresponding normalized eigenvectors $e_n(z)=\sqrt{2}\ \sin(n\pi z)$. Furthermore, A generates an analytic compact semigroup of bounded linear operator $S(t)$, $t\ge 0$, on a separable Hilbert space H which is given by

$$
S(t)y=\sum_{n=1}^{\infty}(y_n,e_n)e_n=\sum_{n=1}^{\infty}2e^{-n^2\pi^2 t}\ \sin(n\pi z)\int_0^1 \sin(n\pi\xi)y(\xi)d\xi,\ y\in H.
$$

If $u\in L_2(J,H)$, then $B=I,\ B^*=I$.

Now we consider

$$\begin{cases} D_{0+}^{\nu,\frac{2}{3}} y(t,z) = \frac{\partial^2}{\partial z^2} y(t,z) + u(t,z) \\ + f(t,z) + g(t,z) d\omega(t), \ \ t \in J, \ 0 < z < 1, \\ y(t,0) = y(t,1) = 0, \ t \in J, \\ I_{0+}^{\frac{1}{3}(1-\nu)}(y(0,z)) = y_0(z), \ \ 0 \le z \le 1, \end{cases} \tag{4.2}$$

The system (4.2) is exact null controllability if there is a $\gamma > 0$, such that

$$\int_0^b \|B^* P_\mu^*(b-s)y\|^2 ds \ge \gamma \left[\|S_{\mu,\nu}^*(b)y\|^2 + \int_0^b \|P_\mu^*(b-s)y\|^2 ds \right],$$

or equivalently

$$\int_0^b \|P_\mu(b-s)y\|^2 ds \ge \gamma \left[\|S_{\mu,\nu}(b)y\|^2 + \int_0^b \|P_\mu(b-s)y\|^2 ds \right].$$

If $f = 0$ and $g = 0$ in (4.2), then the fractional linear system is exactly null controllable if

$$\int_0^b \|P_\mu(b-s)y\|^2 ds \ge b\|S_{\mu,\nu}(b)y\|^2.$$

Therefore,

$$\int_0^b \|P_\mu(b-s)y\|^2 ds \ge \frac{b}{1+b} \left[\|S_{\mu,\nu}(b)y\|^2 + \int_0^b \|P_\mu(b-s)y\|^2 ds \right].$$

Hence, the linear fractional system (4.2) is exactly null controllable on $[0,b]$. So the hypothesis (H_1) is satisfied.

We define $F : J \times H \to H$, $G : J \times H \to L(K,H)$ and $h : C(J,H) \to H$ as follows: $F(t,x) = f(t,x(t,z))$, $G(t,x) = g(t,x(t,z))$ and $h(x) = \sum_{i=1}^{p} k_i x(t_i,z)$ Then $h(\cdot)$ satisfies $(H4)$.

By choosing the constants k_i, $i = 1,2,...,p,M,c_1$, c_2 and c_3 such that $\varrho < 1$. Hence, all the hypotheses of Theorem 1 are satisfied, so the Hilfer fractional stochastic partial differential system (4.1) is exact null controllable on $[0,b]$.

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