ON THE RECIPROCAL SUMS OF SQUARE OF GENERALIZED BI-PERIODIC FIBONACCI NUMBERS

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Abstract. Recently Basbük and Yazlik [1] proved identities related to the reciprocal sum of generalized bi-periodic Fibonacci numbers starting from 0 and 1, and raised an open question whether we can obtain similar results for the reciprocal sum of $m^{th}$ power ($m \geq 2$) of the same numbers. In this paper we derive identities for the reciprocal sum of square of generalized bi-periodic Fibonacci numbers with arbitrary initial conditions.

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1. INTRODUCTION

Throughout this paper we use the notation \( \{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b) \) to denote the generalized bi-periodic Fibonacci numbers \( \{G_n\}_{n=0}^{\infty} \) generated from the recurrence relation [4]

\[
G_n = \begin{cases} 
a G_{n-1} + G_{n-2}, & \text{if } n \in \mathbb{N}_e; 
b G_{n-1} + G_{n-2}, & \text{if } n \in \mathbb{N}_o, 
\end{cases} \quad (n \geq 2),
\]

with initial conditions \( G_0 \) and \( G_1 \), where \( G_0, G_1, a \) and \( b \) are real numbers, and \( \mathbb{N}_e \) (\( \mathbb{N}_o \), respectively) denotes the set of positive even (odd, respectively) integers.

Recently Ohtsuka and Nakamura [8] found interesting properties of the Fibonacci numbers \( \{F_n\}_{n=0}^{\infty} = S(0, 1, 1, 1) \) and proved (1.1) and (1.2) below, where \( \lfloor \cdot \rfloor \) indicates the floor function.

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right) \right]^{-1} = \begin{cases} 
F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; 
F_n - F_{n-1} - 1, & \text{if } n \geq 3 \text{ and } n \in \mathbb{N}_o.
\end{cases} \quad (1.1)
\]

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_k^2} \right) \right]^{-1} = \begin{cases} 
F_{n-1} F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; 
F_{n-1} F_n, & \text{if } n \geq 3 \text{ and } n \in \mathbb{N}_o.
\end{cases} \quad (1.2)
\]
The work of Ohtsuka and Nakamura was generalized by several authors \[1,2,5–7\]. In particular, Basbük and Yazlik \[1\] considered the reciprocal sum of generalized bi-periodic Fibonacci numbers \(G_n\) and proved the following theorem.

**Theorem 1.** Let \(a\) and \(b\) positive integers. Then, for \(G_n = S(0, 1, a, b)\), we have

\[
\left( \sum_{k=n}^{\infty} \frac{\psi(k)}{G_k} \right)^{-1} = \begin{cases} 
G_n - G_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\
G_n - G_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o.
\end{cases}
\]

(1.3)

where

\[
\psi(k) = \xi(k + 1) - \xi(n + 1) - (-1)^n \left\lfloor \frac{k - n}{2} \right\rfloor,
\]

and \(\xi(n)\) is the parity function such that

\[
\xi(n) = \begin{cases} 
0, & \text{if } n \in \{0\} \cup \mathbb{N}_e; \\
1, & \text{if } n \in \mathbb{N}_o.
\end{cases}
\]

In \[1\], Basbük and Yazlik raised an open question whether we can obtain similar results for the reciprocal sum of \(m^{th}\) power \((m \geq 2)\) of the same numbers.

In this paper we derive identities for the reciprocal sum of square of generalized bi-periodic Fibonacci numbers \(G_n = S(G_0, G_1, a, b)\), where \(G_0\) is a nonnegative integer and \(G_1\) is a positive integer.

2. **Main Results**

Lemma 1 will be used to prove our main results.

**Lemma 1.** For \(G_n = S(G_0, G_1, a, b)\), (a)-(c) below hold:

(a) \(G_n G_{n+1} - G_{n-1} G_{n+2} = (-1)^n (abG_0^2 + abG_0G_1 - aG_1^2)\).

(b) \(a^{\xi(n+1)} b^{\xi(n)} G_{n-1} G_{n+1} - a^{\xi(n)} b^{\xi(n+1)} G_n^2 = (-1)^n (aG_0^2 - abG_0G_1 - bG_1^2)\).

(c) \(G_{n+1} G_{n+2} - G_{n-1} G_n = a^{\xi(n)} b^{\xi(n+1)} G_n^2 + a^{\xi(n+1)} b^{\xi(n)} G_{n+1}^2\).

**Proof.** (a) and (b) are special cases of \[3, Theorem 2.2\]. Since

\[G_n = a^{\xi(n-1)} b^{\xi(n)} G_{n-1} + G_{n-2},\]

then (c) follows from the identity

\[G_n G_{n+1} = (G_{n+2} - a^{\xi(n+1)} b^{\xi(n)} G_{n+1}) G_{n+1} = G_n (a^{\xi(n)} b^{\xi(n)} G_n + G_{n-1}).\]

\(\Box\)
The main results of this paper are stated in Theorem 2. For the ease of presentation, we use the following notation for \( \{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b) \)
\[
\Phi(G) := b^2G_0^2 + ab^2G_0G_1 - abG_1^2.
\]

**Theorem 2.** Let \( G_0 \) be a nonnegative integer and let \( G_1, a \) and \( b \) positive integers. Then, for \( \{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b) \), (a) and (b) below hold:

(a) If
\[
\frac{\Phi(G)}{ab + 2} \notin \mathbb{Z},
\]
define
\[
g := \left\lfloor \frac{\Phi(G)}{ab + 2} \right\rfloor + \Delta,
\]
where
\[
\Delta = \begin{cases} 
1, & \text{if } \Phi(G) > 0; \\
0, & \text{if } \Phi(G) < 0.
\end{cases}
\]

(i) If \( \Phi(G) > 0 \), then there exist positive integers \( n_0 \) and \( n_1 \) such that
\[
\left\lfloor \left( \sum_{k=n}^{\infty} \left( \frac{a}{b} \right)^{1-k} \frac{G_k^2}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} 
bG_{n-1}G_n + g - 1, & \text{if } n \geq n_0 \text{ and } n \in \mathbb{N}_e; \\
bG_{n-1}G_n - g, & \text{if } n \geq n_1 \text{ and } n \in \mathbb{N}_o.
\end{cases}
\ ]
\]
(ii) If \( \Phi(G) < 0 \), then there exist positive integers \( n_2 \) and \( n_3 \) such that
\[
\left\lfloor \left( \sum_{k=n}^{\infty} \left( \frac{a}{b} \right)^{1-k} \frac{G_k^2}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} 
bG_{n-1}G_n + g, & \text{if } n \geq n_2 \text{ and } n \in \mathbb{N}_e; \\
bG_{n-1}G_n - g - 1, & \text{if } n \geq n_3 \text{ and } n \in \mathbb{N}_o.
\end{cases}
\]

(b) If
\[
\frac{\Phi(G)}{ab + 2} \in \mathbb{Z},
\]
then there exist positive integers \( n_4 \) and \( n_5 \) such that
\[
\left\lfloor \left( \sum_{k=n}^{\infty} \left( \frac{a}{b} \right)^{1-k} \frac{G_k^2}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} 
bG_{n-1}G_n + \hat{g}, & \text{if } n \geq n_4 \text{ and } n \in \mathbb{N}_e; \\
bG_{n-1}G_n - \hat{g}, & \text{if } n \geq n_5 \text{ and } n \in \mathbb{N}_o,
\end{cases}
\]
where
\[
\hat{g} := \frac{\Phi(G)}{ab + 2}.
\]

**Proof.** (a) To prove (2.1), assume that \( \Phi(G) > 0 \). Then
\[
\Phi(G) - g(ab + 2) < 0.
\]
Firstly, consider
Then with Lemma 2.1(a),(b), we have

\[
\frac{1}{bG_{n-1}G_n + (-1)^n g} - \frac{1}{bG_{n+1}G_{n+2} + (-1)^n g} - \left( \frac{a}{b} \right)^{1-\xi(n)} G_n - \left( \frac{a}{b} \right)^{1-\xi(n+1)} G_{n+1}
\]

\[
= \frac{(bG_{n-1}G_n + (-1)^n g)(bG_{n+1}G_{n+2} + (-1)^n g)G_n^2G_{n+1}^2}{Y_1}
\]

where, by Lemma 1(c)

\[
Y_1 = \left\{ \left( \frac{a}{b} \right)^{1-\xi(n+1)} G_n + \left( \frac{a}{b} \right)^{1-\xi(n)} G_{n+1} \right\} \hat{Y}_1.
\]

with

\[
\hat{Y}_1 = b^2(G_n^2G_{n+1}^2 - G_{n-1}G_nG_{n+1}G_{n+2})
\]

\[-(-1)^n gb(G_{n-1}G_n + G_{n+1}G_{n+2}) - g^2.
\]

By Lemma 2.1(a),(b), we have

\[
G_n^2G_{n+1}^2 - G_{n-1}G_nG_{n+1}G_{n+2}
\]

\[= (G_nG_{n+1} - G_{n-1}G_{n+2})G_nG_{n+1}
\]

\[= (-1)^n (bG_0^2 + abG_0G_1 - aG_1^2)G_nG_{n+1}.
\]

and

\[
G_{n-1}G_n + G_{n+1}G_{n+2}
\]

\[= (G_{n+1} - a\xi(n)b\xi(n+1)G_n)G_n + G_{n+1}(a\xi(n+1)b\xi(n)G_{n+1} + G_n)
\]

\[= (ab + 2)G_nG_{n+1} + a\xi(n+1)b\xi(n)G_{n-1}G_{n+1} - a\xi(n)b\xi(n+1)G_n^2
\]

\[= (ab + 2)G_nG_{n+1} + (-1)^n (aG_1^2 - abG_0G_1 - bG_0^2).
\]

Then

\[
\hat{Y}_1 = (-1)^n b^2(bG_0^2 + abG_0G_1 - aG_1^2)G_nG_{n+1}
\]

\[-(-1)^n gb \left\{ (ab + 2)G_nG_{n+1} + (-1)^n (aG_1^2 - abG_0G_1 - bG_0^2) \right\} - g^2
\]

\[= (-1)^n gbG_nG_{n+1} \left\{ \Phi(G) - g(ab + 2) \right\} + g^2 \Phi(G) - g^2.
\]

If \(n \in \mathbb{N}_e\), then there exists a positive integer \(m_0\) such that, for \(n \geq m_0\), \(X_1 < 0\), and

\[
\frac{1}{bG_{n-1}G_n + (-1)^n g} - \frac{1}{bG_{n+1}G_{n+2} + (-1)^n g} < \left( \frac{a}{b} \right)^{1-\xi(n)} G_n - \left( \frac{a}{b} \right)^{1-\xi(n+1)} G_{n+1}.
\]
Repeatedly applying the above inequality, we have
\[
\frac{1}{bG_{n-1}G_n + (-1)^n g} < \sum_{k=n}^{\infty} \left( \frac{a}{b} \right)^{1-\xi(k)} \frac{G_k^2}{G_k}, \quad \text{if } n \geq m_0 \text{ and } n \in \mathbb{N}. \tag{2.4}
\]
Similarly, we obtain, for some positive integer \( m_1 \),
\[
\sum_{k=n}^{\infty} \left( \frac{a}{b} \right)^{1-\xi(k)} \frac{G_k^2}{G_k} < \frac{1}{bG_{n-1}G_n + (-1)^n g}, \quad \text{if } n \geq m_1 \text{ and } n \in \mathbb{N}. \tag{2.5}
\]
Next, consider
\[
X_2 = \frac{1}{bG_{n-1}G_n + (-1)^n g - 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g - 1} - \frac{\left( \frac{a}{b} \right)^{1-\xi(n)}}{G_n^2}
\]
where
\[
Y_2 = bG_n^2G_{n+1} - ba^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_nG_{n+1} - bG_{n-1}G_n^3
\]
\[
- (-1)^n g (2G_n^2 - a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n + a^{1-\xi(n)}b^{\xi(n)}G_nG_{n+1})
\]
\[
+ a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1}) + a^{1-\xi(n)}b^{\xi(n)-1}(g^2 - 1).
\]
Using Lemma 1(a), we have
\[
bG_n^2G_{n+1} - ba^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_nG_{n+1} - bG_{n-1}G_n^3
\]
\[
= bG_n^2(G_n - a^{1-\xi(n)}b^{\xi(n)}G_{n-1}) - bG_{n-1}G_n^3
\]
\[
= bG_n^2(G_{n-2}G_{n+1} - G_{n-1}G_n)
\]
\[
= (-1)^n bG_n^2(bG_0^2 + abG_0G_1 - aG_1^2).
\]
and
\[
2G_n^2 - a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n + a^{1-\xi(n)}b^{\xi(n)}G_nG_{n+1}
\]
\[
= 2G_n^2 - a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n + a^{1-\xi(n)}b^{\xi(n)}G_n(a^{1-\xi(n+1)}b^{\xi(n+1)}G_n + G_{n-1})
\]
\[
= (ab + 2)G_n^2.
\]
Hence we obtain
\[
Y_2 = (-1)^n G_n^2 \Phi(G) - g(ab + 2)
\]
\[
+ a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1}) + a^{1-\xi(n)}b^{\xi(n)-1}(g^2 - 1).
\]
If \( n \in \mathbb{N}_e \), then there exists a positive integer \( m_2 \) such that, for \( n \geq m_2 \), \( X_2 > 0 \), and

\[
\frac{\left( \frac{a}{b} \right)^{1 - \xi(n)}}{G_n^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g - 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g - 1}.
\]

Repeatedly applying the above inequality, we have

\[
\sum_{k=n}^{\infty} \frac{\left( \frac{a}{b} \right)^{1 - \xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g - 1}, \quad \text{if } n \geq m_2 \text{ and } n \in \mathbb{N}_e. \tag{2.6}
\]

Similarly, consider

\[
X_3 = \frac{1}{bG_{n-1}G_n + (-1)^n g + 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g + 1} - \frac{\left( \frac{a}{b} \right)^{1 - \xi(n)}}{G_n^2} Y_3,
\]

where

\[
Y_3 = Y_2 - 2a^{1 - \xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1})
\]

\[
= (-1)^n G_n^2 \left\{ \Phi(G) - g(ab + 2) \right\}
\]

\[
- a^{1 - \xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1}) + a^{1 - \xi(n)}b^{\xi(n)-1}(g^2 - 1).
\]

If \( n \in \mathbb{N}_o \), then there exists a positive integer \( m_3 \) such that, for \( n \geq m_3 \), \( X_3 < 0 \), and

\[
\frac{1}{bG_{n-1}G_n + (-1)^n g + 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g + 1} < \frac{\left( \frac{a}{b} \right)^{1 - \xi(n)}}{G_n^2},
\]

from which we have

\[
\frac{1}{bG_{n-1}G_n + (-1)^n g + 1} < \sum_{k=n}^{\infty} \frac{\left( \frac{a}{b} \right)^{1 - \xi(k)}}{G_k^2}, \quad \text{if } n \geq m_3 \text{ and } n \in \mathbb{N}_o. \tag{2.7}
\]

Then (2.1) follows from (2.4), (2.5), (2.6) and (2.7).

Now suppose that \( \Phi(G) < 0 \). In this case, we have

\[
\Phi(G) - g(ab + 2) > 0,
\]

and (2.4), (2.5), (2.6) and (2.7) are respectively modified as

\[
\sum_{k=n}^{\infty} \frac{\left( \frac{a}{b} \right)^{1 - \xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g}, \quad \text{if } n \geq m_4 \text{ and } n \in \mathbb{N}_e. \tag{2.8}
\]
Reciprocal sums of generalized bi-periodic Fibonacci numbers

\[ \frac{1}{bG_{n-1}G_n + (-1)^n g} < \sum_{k=n}^{\infty} \frac{1}{G_k^2}, \text{ if } n \geq m_5 \text{ and } n \in \mathbb{N}_o. \quad (2.9) \]

\[ \sum_{k=n}^{\infty} \frac{(\frac{a}{b})^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g - 1}, \text{ if } n \geq m_6 \text{ and } n \in \mathbb{N}_o. \quad (2.10) \]

and

\[ \frac{1}{bG_{n-1}G_n + (-1)^n g + 1} < \sum_{k=n}^{\infty} \frac{1}{G_k^2}, \text{ if } n \geq m_7 \text{ and } n \in \mathbb{N}_e. \quad (2.11) \]

Then, (2.2) easily follows and the proof of (a) is completed.

(b) Suppose that

\[ \frac{\Phi(G)}{ab + 2} \in \mathbb{Z}. \]

We recall the proof of (a). Replacing \( g \) by \( \hat{g} \), we have

\[ \hat{Y}_1 = \hat{g} \Phi(G) - \hat{g}^2 = (ab + 1)\hat{g}^2 > 0. \]

Hence there exist positive integers \( m_8 \) and \( m_9 \) such that \( X_1 > 0 \) if \( n \geq m_8 \) and \( n \in \mathbb{N}_e \) or if \( n \geq m_9 \) and \( n \in \mathbb{N}_o \). Hence we obtain

\[ \sum_{k=n}^{\infty} \frac{(\frac{a}{b})^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n \hat{g}}, \text{ if } n \geq m_8 \quad (n \in \mathbb{N}_e) \text{ or if } n \geq m_9 \quad (n \in \mathbb{N}_o). \]

Similarly, there exist positive integers \( m_{10} \) and \( m_{11} \) such that \( X_3 < 0 \) if \( n \geq m_{10} \) and \( n \in \mathbb{N}_e \) or if \( n \geq m_{11} \) and \( n \in \mathbb{N}_o \), from which we have

\[ \frac{1}{bG_{n-1}G_n + (-1)^n \hat{g} + 1} < \sum_{k=n}^{\infty} \frac{1}{G_k^2}, \text{ if } n \geq m_{10} \quad (n \in \mathbb{N}_e) \text{ or if } n \geq m_{11} \quad (n \in \mathbb{N}_o). \]

Then, (2.3) follows from (2.12) and (2.13), and (b) is also proved.

□

Example 1. For \( \{G_n\}_{n=0}^\infty = S(2, 1, 2, 1) \), we have \( \Phi(G) = 6 \) and

\[ g = \left\lfloor \frac{6}{4} \right\rfloor + 1 = 2. \]

Then, from (2.1), we have

\[ \left\lfloor \left( \sum_{k=n}^{\infty} \frac{2^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} G_{n-1}G_n + 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ G_{n-1}G_n - 2, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases} \]
Example 2. Consider \( \{G_n\}_{n=0}^\infty = S(0,1,a,b) \) with \( a \) and \( b \) positive integers. In this case, we have \( \Phi(G) = -ab < 0 \) and 
\[
g = \left\lfloor \frac{-ab}{ab + 2} \right\rfloor = -1.
\]
Then, from (2.2), we obtain
\[
\left( \sum_{k=n}^{\infty} \left( \frac{a}{b} \right)^{1-\xi(k)} \frac{1}{G_k^2} \right)^{-1} = \begin{cases} 
bG_{n-1}G_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\
bG_{n-1}G_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}
\]

Example 3. For \( \{G_n\}_{n=0}^\infty = S(2,1,4,2) \), we have \( \Phi(G) = 40, \hat{g} = 4 \). Then, from (2.3), we have
\[
\left( \sum_{k=n}^{\infty} \frac{2^{1-\xi(k)}}{G_k^2} \right)^{-1} = \begin{cases} 
2G_{n-1}G_n + 4, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\
2G_{n-1}G_n - 4, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}
\]

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