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ON THE RECIPROCAL SUMS OF SQUARE OF GENERALIZED BI-PERIODIC FIBONACCI NUMBERS

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Abstract. Recently Basbük and Yazlik [1] proved identities related to the reciprocal sum of generalized bi-periodic Fibonacci numbers starting from 0 and 1, and raised an open question whether we can obtain similar results for the reciprocal sum of m^{th} power ($m \geq 2$) of the same numbers. In this paper we derive identities for the reciprocal sum of square of generalized bi-periodic Fibonacci numbers with arbitrary initial conditions.

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1. INTRODUCTION

Throughout this paper we use the notation $\{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b)$ to denote the generalized bi-periodic Fibonacci numbers $\{G_n\}_{n=0}^{\infty}$ generated from the recurrence relation [4]

$$G_n = \begin{cases} aG_{n-1} + G_{n-2}, & \text{if } n \in \mathbb{N}_e; \\ bG_{n-1} + G_{n-2}, & \text{if } n \in \mathbb{N}_o, \end{cases} \quad (n \geq 2),$$

with initial conditions G_0 and G_1, where G_0, G_1, a and b are real numbers, and $\mathbb{N}_e$ ($\mathbb{N}_o$, respectively) denotes the set of positive even (odd, respectively) integers.

Recently Ohtsuka and Nakamura [8] found interesting properties of the Fibonacci numbers $\{F_n\}_{n=0}^{\infty} = S(0,1,1,1)$ and proved (1.1) and (1.2) below, where $\lfloor \cdot \rfloor$ indicates the floor function.

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, \text{ if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, \text{ if } n \geq 3 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{1.1}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, \text{ if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1}F_n, \text{ if } n \geq 3 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{1.2}$$

The work of Ohtsuka and Nakamura was generalized by several authors [1,2,5–7]. In particular, Basbük and Yazlik [1] considered the reciprocal sum of generalized bi-periodic Fibonacci numbers $\{G_n\}_{n=0}^{\infty} = S(0,1,a,b)$ and proved the following theorem.

Theorem 1. *Let a and b positive integers. Then, for* $\{G_n\}_{n=0}^{\infty} = S(0,1,a,b)$*, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\psi(k)}}{G_k} \right)^{-1} \right\rfloor = \begin{cases} G_n - G_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ G_n - G_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{1.3}$$

where

$$\psi(k) = \xi(k+1) - \xi(n+1) - (-1)^n \left\lfloor \frac{k-n}{2} \right\rfloor,$$

and $\xi(n)$ *is the parity function such that*

$$\xi(n) = \begin{cases} 0, & \text{if } n \in \{0\} \cup \mathbb{N}_e; \\ 1, & \text{if } n \in \mathbb{N}_o. \end{cases}$$

In [1], Basbük and Yazlik raised an open question whether we can obtain similar results for the reciprocal sum of m^{th} power ($m \geq 2$) of the same numbers.

In this paper we derive identities for the reciprocal sum of square of generalized bi-periodic Fibonacci numbers $\{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b)$, where G_0 is a nonnegative integer and G_1 is a positive integer.

2. Main results

Lemma 1 below will be used to prove our main results.

Lemma 1. *For* $\{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b)$*, (a)-(c) below hold:*
(a) $G_n G_{n+1} - G_{n-1} G_{n+2} = (-1)^n (bG_0^2 + abG_0G_1 - aG_1^2)$.
(b) $a^{\xi(n+1)} b^{\xi(n)} G_{n-1} G_{n+1} - a^{\xi(n)} b^{\xi(n+1)} G_n^2 = (-1)^n (aG_1^2 - abG_0G_1 - bG_0^2)$.
(c) $G_{n+1} G_{n+2} - G_{n-1} G_n = a^{\xi(n)} b^{\xi(n+1)} G_n^2 + a^{\xi(n+1)} b^{\xi(n)} G_{n+1}^2$.

Proof. (a) and (b) are special cases of [3, Theorem 2.2]. Since

$$G_n = a^{\xi(n-1)} b^{\xi(n)} G_{n-1} + G_{n-2},$$

then (c) follows from the identity

$$\begin{aligned} G_n G_{n+1} &= (G_{n+2} - a^{\xi(n+1)} b^{\xi(n)} G_{n+1}) G_{n+1} \\ &= G_n (a^{\xi(n)} b^{\xi(n+1)} G_n + G_{n-1}). \end{aligned}$$

□

The main results of this paper are stated in Theorem 2. For the ease of presentation, we use the following notation for $\{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b)$

$$\Phi(G) := b^2 G_0^2 + ab^2 G_0 G_1 - ab G_1^2.$$

Theorem 2. *Let G_0 be a nonnegative integer and let G_1, a and b positive integers. Then, for $\{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b)$, (a) and (b) below hold:*
(a) If

$$\frac{\Phi(G)}{ab+2} \notin \mathbb{Z},$$

define

$$g := \left\lfloor \frac{\Phi(G)}{ab+2} \right\rfloor + \Delta,$$

where

$$\Delta = \begin{cases} 1, & \text{if } \Phi(G) > 0; \\ 0, & \text{if } \Phi(G) < 0. \end{cases}$$

(i) If $\Phi(G) > 0$, then there exist positive integers n_0 and n_1 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} bG_{n-1}G_n + g - 1, & \text{if } n \geq n_0 \text{ and } n \in \mathbb{N}_e; \\ bG_{n-1}G_n - g, & \text{if } n \geq n_1 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{2.1}$$

(ii) If $\Phi(G) < 0$, then there exist positive integers n_2 and n_3 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} bG_{n-1}G_n + g, & \text{if } n \geq n_2 \text{ and } n \in \mathbb{N}_e; \\ bG_{n-1}G_n - g - 1, & \text{if } n \geq n_3 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{2.2}$$

(b) If

$$\frac{\Phi(G)}{ab+2} \in \mathbb{Z},$$

then there exist positive integers n_4 and n_5 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} bG_{n-1}G_n + \hat{g}, & \text{if } n \geq n_4 \text{ and } n \in \mathbb{N}_e; \\ bG_{n-1}G_n - \hat{g}, & \text{if } n \geq n_5 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{2.3}$$

where

$$\hat{g} := \frac{\Phi(G)}{ab+2}.$$

Proof. (a) To prove (2.1), assume that $\Phi(G) > 0$. Then

$$\Phi(G) - g(ab+2) < 0.$$

Firstly, consider

$$X_1 = \frac{1}{bG_{n-1}G_n+(-1)^n g} - \frac{1}{bG_{n+1}G_{n+2}+(-1)^n g} - \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2} - \frac{\left(\frac{a}{b}\right)^{1-\xi(n+1)}}{G_{n+1}^2} = \frac{Y_1}{(bG_{n-1}G_n+(-1)^n g)(bG_{n+1}G_{n+2}+(-1)^n g)G_n^2G_{n+1}^2},$$

where, by Lemma 1(c)

$$Y_1 = \left\{\left(\frac{a}{b}\right)^{1-\xi(n+1)} G_n^2 + \left(\frac{a}{b}\right)^{1-\xi(n)} G_{n+1}^2\right\}\hat{Y}_1,$$

with

$$\hat{Y}_1 = b^2(G_n^2G_{n+1}^2 - G_{n-1}G_nG_{n+1}G_{n+2}) - (-1)^n gb(G_{n-1}G_n + G_{n+1}G_{n+2}) - g^2.$$

By Lemma 2.1(a),(b), we have

$$G_n^2G_{n+1}^2 - G_{n-1}G_nG_{n+1}G_{n+2} = (G_nG_{n+1} - G_{n-1}G_{n+2})G_nG_{n+1} = (-1)^n(bG_0^2 + abG_0G_1 - aG_1^2)G_nG_{n+1},$$

and

$$\begin{aligned} G_{n-1}G_n + G_{n+1}G_{n+2} &= (G_{n+1} - a^{\xi(n)}b^{\xi(n+1)}G_n)G_n + G_{n+1}(a^{\xi(n+1)}b^{\xi(n)}G_{n+1} + G_n) \\ &= (ab+2)G_nG_{n+1} + a^{\xi(n+1)}b^{\xi(n)}G_{n-1}G_{n+1} - a^{\xi(n)}b^{\xi(n+1)}G_n^2 \\ &= (ab+2)G_nG_{n+1} + (-1)^n(aG_1^2 - abG_0G_1 - bG_0^2). \end{aligned}$$

Then

$$\begin{aligned} \hat{Y}_1 &= (-1)^n b^2(bG_0^2 + abG_0G_1 - aG_1^2)G_nG_{n+1} \\ &\quad - (-1)^n gb\left\{(ab+2)G_nG_{n+1} + (-1)^n(aG_1^2 - abG_0G_1 - bG_0^2)\right\} - g^2 \\ &= (-1)^n bG_nG_{n+1}\left\{\Phi(G) - g(ab+2)\right\} + g\Phi(G) - g^2. \end{aligned}$$

If $n \in \mathbb{N}_e$, then there exists a positive integer m_0 such that, for $n \geq m_0$, $X_1 < 0$, and

$$\frac{1}{bG_{n-1}G_n+(-1)^n g} - \frac{1}{bG_{n+1}G_{n+2}+(-1)^n g} < \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2} + \frac{\left(\frac{a}{b}\right)^{1-\xi(n+1)}}{G_{n+1}^2}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{bG_{n-1}G_n+(-1)^n g}<\sum_{k=n}^{\infty}\frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n\geq m_0 \text{ and } n\in\mathbb{N}_e. \tag{2.4}$$

Similarly, we obtain, for some positive integer m_1,

$$\sum_{k=n}^{\infty}\frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}<\frac{1}{bG_{n-1}G_n+(-1)^n g}, \text{ if } n\geq m_1 \text{ and } n\in\mathbb{N}_o. \tag{2.5}$$

Next, consider

$$\begin{aligned}X_2&=\frac{1}{bG_{n-1}G_n+(-1)^n g-1}-\frac{1}{bG_nG_{n+1}+(-1)^{n+1}g-1}-\frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2}\\&=\frac{Y_2}{(bG_{n-1}G_n+(-1)^n g-1)(bG_nG_{n+1}+(-1)^{n+1}g-1)G_n^2},\end{aligned}$$

where

$$\begin{aligned}Y_2&=bG_n^3G_{n+1}-ba^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n^2G_{n+1}-bG_{n-1}G_n^3\\&\quad-(-1)^n g(2G_n^2-a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n+a^{1-\xi(n)}b^{\xi(n)}G_nG_{n+1})\\&\quad+a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n+G_nG_{n+1})+a^{1-\xi(n)}b^{\xi(n)-1}(g^2-1).\end{aligned}$$

Using Lemma 1(a), we have

$$\begin{aligned}&bG_n^3G_{n+1}-ba^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n^2G_{n+1}-bG_{n-1}G_n^3\\&=bG_n^2G_{n+1}(G_n-a^{1-\xi(n)}b^{\xi(n)}G_{n-1})-bG_{n-1}G_n^3\\&=bG_n^2(G_{n-2}G_{n+1}-G_{n-1}G_n)\\&=(-1)^n bG_n^2(bG_0^2+abG_0G_1-aG_1^2),\end{aligned}$$

and

$$\begin{aligned}&2G_n^2-a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n+a^{1-\xi(n)}b^{\xi(n)}G_nG_{n+1}\\&=2G_n^2-a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n+a^{1-\xi(n)}b^{\xi(n)}G_n(a^{1-\xi(n+1)}b^{\xi(n+1)}G_n+G_{n-1})\\&=(ab+2)G_n^2.\end{aligned}$$

Hence we obtain

$$\begin{aligned}Y_2&=(-1)^n G_n^2\Big\{\Phi(G)-g(ab+2)\Big\}\\&\quad+a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n+G_nG_{n+1})+a^{1-\xi(n)}b^{\xi(n)-1}(g^2-1).\end{aligned}$$

If $n \in \mathbb{N}_e$, then there exists a positive integer m_2 such that, for $n \geq m_2$, $X_2 > 0$, and

$$\frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g - 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g - 1}.$$

Repeatedly applying the above inequality, we have

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g - 1}, \text{ if } n \geq m_2 \text{ and } n \in \mathbb{N}_e. \tag{2.6}$$

Similarly, consider

$$X_3 = \frac{1}{bG_{n-1}G_n + (-1)^n g + 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g + 1} - \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2}$$
$$= \frac{Y_3}{(bG_{n-1}G_n + (-1)^n g + 1)(bG_nG_{n+1} + (-1)^{n+1} g + 1)G_n^2},$$

where

$$Y_3 = Y_2 - 2a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1})$$
$$= (-1)^n G_n^2 \Big\{\Phi(G) - g(ab+2)\Big\}$$
$$-a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1}) + a^{1-\xi(n)}b^{\xi(n)-1}(g^2 - 1).$$

If $n \in \mathbb{N}_o$, then there exists a positive integer m_3 such that, for $n \geq m_3$, $X_3 < 0$, and

$$\frac{1}{bG_{n-1}G_n + (-1)^n g + 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g + 1} < \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2},$$

from which we have

$$\frac{1}{bG_{n-1}G_n + (-1)^n g + 1} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_3 \text{ and } n \in \mathbb{N}_o. \tag{2.7}$$

Then (2.1) follows from (2.4), (2.5), (2.6) and (2.7).

Now suppose that $\Phi(G) < 0$. In this case, we have

$$\Phi(G) - g(ab+2) > 0,$$

and (2.4), (2.5), (2.6) and (2.7) are respectively modified as

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g}, \text{ if } n \geq m_4 \text{ and } n \in \mathbb{N}_e, \tag{2.8}$$

$$\frac{1}{bG_{n-1}G_n+(-1)^n g} < \sum_{k=n}^{\infty}\frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_5 \text{ and } n \in \mathbb{N}_o, \tag{2.9}$$

$$\sum_{k=n}^{\infty}\frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n+(-1)^n g-1}, \text{ if } n \geq m_6 \text{ and } n \in \mathbb{N}_o, \tag{2.10}$$

and

$$\frac{1}{bG_{n-1}G_n+(-1)^n g+1} < \sum_{k=n}^{\infty}\frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_7 \text{ and } n \in \mathbb{N}_e. \tag{2.11}$$

Then, (2.2) easily follows and the proof of (a) is completed.

(b) Suppose that

$$\frac{\Phi(G)}{ab+2} \in \mathbb{Z}.$$

We recall the proof of (a). Replacing g by $\hat{g}$, we have

$$\hat{Y}_1 = \hat{g}\Phi(G) - \hat{g}^2 = (ab+1)\hat{g}^2 > 0.$$

Hence there exist positive integers m_8 and m_9 such that $X_1 > 0$ if $n \geq m_8$ and $n \in \mathbb{N}_e$ or if $n \geq m_9$ and $n \in \mathbb{N}_o$. Hence we obtain

$$\sum_{k=n}^{\infty}\frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n+(-1)^n \hat{g}}, \text{ if } n \geq m_8\ (n \in \mathbb{N}_e) \text{ or if } n \geq m_9\ (n \in \mathbb{N}_o). \tag{2.12}$$

Similarly, there exist positive integers m_{10} and m_{11} such that $X_3 < 0$ if $n \geq m_{10}$ and $n \in \mathbb{N}_e$ or if $n \geq m_{11}$ and $n \in \mathbb{N}_o$, from which we have

$$\frac{1}{bG_{n-1}G_n+(-1)^n \hat{g}+1} < \sum_{k=n}^{\infty}\frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_{10}\ (n \in \mathbb{N}_e) \text{ or if } n \geq m_{11}\ (n \in \mathbb{N}_o). \tag{2.13}$$

Then, (2.3) follows from (2.12) and (2.13), and (b) is also proved.

□

Example 1. For $\{G_n\}_{n=0}^{\infty} = S(2,1,2,1)$, we have $\Phi(G) = 6$ and

$$g = \left\lfloor \frac{6}{4} \right\rfloor + 1 = 2.$$

Then, from (2.1), we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{2^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} G_{n-1}G_n + 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ G_{n-1}G_n - 2, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}$$

Example 2. Consider $\{G_n\}_{n=0}^{\infty} = S(0,1,a,b)$ with a and b positive integers. In this case, we have $\Phi(G) = -ab < 0$ and

$$g = \left\lfloor \frac{-ab}{ab+2} \right\rfloor = -1.$$

Then, from (2.2), we obtain

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} bG_{n-1}G_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ bG_{n-1}G_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}$$

Example 3. For $\{G_n\}_{n=0}^{\infty} = S(2,1,4,2)$, we have $\Phi(G) = 40$, $\hat{g} = 4$. Then, from (2.3), we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{2^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} 2G_{n-1}G_n + 4, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ 2G_{n-1}G_n - 4, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}$$

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REFERENCES

[1] M. Basbük and Y. Yazlik, "On the sum of reciprocal of generalized bi-periodic Fibonacci numbers." *Miskolc Math. Notes.*, vol. 17, no. 1, pp. 35–41, 2016, doi: 10.18514/MMN.2016.1667.
[2] Y. Choo, "On the reciprocal sums of generalized Fibonacci numbers." *Int. J. Math. Anal.*, vol. 10, no. 28, pp. 1365–1373, 2016, doi: https://doi.org/ 10.12988/ijma.2016.610118.
[3] Y. Choo, "On the generalizations of Fibonacci identities." *Results. Math.*, vol. 71, no. 1-2, pp. 347–356, 2017, doi: 10.1007/s00025-015-0515-6.
[4] E. Edson and O. Yayenie, "A new generalization of Fibonacci sequences and extended Binet's formula." *Integers*, vol. 9, pp. 639–654, 2009, doi: 10.1515/INTEG.2009.051.
[5] S. Holliday and T. Komatsu, "On the sum of reciprocal generalized Fibonacci numbers." *Integers*, vol. 11A, p. Article 11, 2011, doi: 10.1515/INTEG.2009.051.
[6] E. Kilic and T. Arican, "More on the infinite sum of reciprocal Fibonacci, Pell and higher order recurrences." *Appl. Math. Comput.*, vol. 219, pp. 7783–7788, 2013, doi: http://doi.org/10.1016/j.amc.2013.02.003.
[7] T. Komatsu and V. Laohakosol, "On the sum of reciprocals of numbers satisfying a recurrence relation of order *s*." *J. Integer Seq.*, vol. 13, no. 5, p. Article 10.5.8, 2010, doi: http://doi.org/10.1016/j.amc.2013.02.003.
[8] H. Ohtsuka and S. Nakamura, "On the sum of reciprocal Fibonacci numbers." *Fibonacci Quart.*, vol. 46/47, no. 2, pp. 153–159, 2008/2009, doi: http://doi.org/10.1016/j.amc.2013.02.003.

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