CONVERGENCE OF CESÁRO MEANS WITH VARYING PARAMETERS OF WALSH-FOURIER SERIES

ANAS AHMAD ABU JOUDEH AND GYÖRGY GÁT

Received 06 June, 2017

Abstract. In 2007 Akhobadze [1] (see also [2]) introduced the notion of Cesàro means of Fourier series with variable parameters. In the present paper we prove the almost everywhere convergence of the the Cesàro \( (C, a_n) \) means of integrable functions \( \sigma_n^a f \rightarrow f \), where \( N_{a,K} \ni n \rightarrow \infty \) for \( f \in L^1(I) \), where \( I \) is the Walsh group for every sequence \( a = (a_n) \), \( 0 < a_n < 1 \). This theorem for constant sequences \( a \) that is, \( a \equiv a_1 \) was proved by Fine [3].

2010 Mathematics Subject Classification: 42C10

1. INTRODUCTION AND MAIN RESULTS

We follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [9]) and others. Denote by \( \mathbb{N} := \{0,1,\ldots\} \), \( \mathbb{P} := \mathbb{N} \setminus \{0\} \), the set of natural numbers, the set of positive integers and \( I := [0,1) \) the unit interval. Denote by \( \lambda(B) = |B| \) the Lebesgue measure of the set \( B(B \subset I) \). Denote by \( L^p(I) \) the usual Lebesgue spaces and \( \| \cdot \|_p \) the corresponding norms \((1 \leq p \leq \infty)\). Set

\[
\mathcal{J} := \left\{ \left[ \frac{p}{2^n}, \frac{p+1}{2^n} \right) : p,n \in \mathbb{N} \right\}
\]

the set of dyadic intervals and for given \( x \in I \) and let \( I_n(x) \) denote the interval \( I_n(x) \in \mathcal{J} \) of length \( 2^{-n} \) which contains \( x (n \in \mathbb{N}) \). Also use the notion \( I_n := I_n(0) (n \in \mathbb{N}) \). Let

\[
x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}
\]

be the dyadic expansion of \( x \in I \), where \( x_n = 0 \) or \( 1 \) and if \( x \) is a dyadic rational number \((x \in \{ p/2^n : p,n \in \mathbb{N}\})\) we choose the expansion which terminates in 0’s. The

Research is supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. K111651 and by project EFOP-3.6.1-16-2016-00022 supported by the European Union, co-financed by the European Social Fund.

© 2018 Miskolc University Press
The notion of the Hardy space $H(I)$ is introduced in the following way [9]. A function $a \in L^\infty(I)$ is called an atom, if either $a = 1$ or $a$ has the following properties: $\text{supp} a \subseteq I_a$, $\|a\|_\infty \leq |I_a|^{-1}$, $\int_I a = 0$, for some $I_a \in \mathcal{J}$. We say that the function $f$ belongs to $H$, if $f$ can be represented as $f = \sum_{i=0}^{\infty} \lambda_i a_i$, where $a_i$'s are atoms and for the coefficients $(\lambda_i)$ the inequality $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ is true. It is known that $H$ is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H$.

Set the definition of the $n$th ($n \in \mathbb{N}$) Walsh-Paley function at point $x \in I$ as:

$$\omega_n(x) := \prod_{j=0}^{\infty} (-1)^{n_j},$$

where $\mathbb{N} \ni n = \sum_{n=0}^{\infty} n_j 2^j$ ($n_j \in \{0, 1\}$ ($j \in \mathbb{N}$)). It is known (see [8] or [10]) that the system $(\omega_n, n \in \mathbb{N})$ is the character system of $(I, +)$, where the group operation $+$ is the so-called dyadic or logical addition on $I$. That is, for any $x, y \in I$

$$x + y := \sum_{n=0}^{\infty} |x_n - y_n|2^{-(n+1)}.$$

Denote by

$$\hat{f}(n) := \int_I f \omega_n d\lambda, \quad D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n^1 := \frac{1}{n+1} \sum_{k=0}^{n} D_k$$

the Fourier coefficients, the Dirichlet and the Fejér or $(C,1)$ kernels, respectively. It is also known that the Fejér or $(C,1)$ means of $f$ is

$$\sigma_n^1 f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y) = \int_I f(x) K_n^1(y + x) d\lambda(x)$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \int_I f(x) D_k(y + x) d\lambda(x), \quad (n \in \mathbb{N}, y \in I).$$

It is known [9] that for $n \in \mathbb{N}, x \in I$ it holds

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases}$$

and also that

$$D_n(x) = \omega_n(x) \sum_{k=1}^{\infty} D_{2^k}(x) n_k (-1)^{x_k}.$$
where \( n = \sum_{i=1}^{\infty} n_i 2^i \), \( n_i = \{0, 1\} (i \in \mathbb{N}) \).

Denote by \( K_{n}^{\alpha} \) the kernel of the summability method \( (C, \alpha_n) \) and call it the \( (C, \alpha_n) \) kernel or the Cesàro kernel for \( \alpha_n \in \mathbb{R} \setminus \{-1, -2, \ldots\} \)

\[
K_{n}^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k,
\]

where

\[
A_k^{\alpha} = \frac{(\alpha_n + 1)(\alpha_n + 2) \cdots (\alpha_n + n)}{k!}.
\]

It is known [12] that \( A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1} \), \( A_k^{\alpha} - A_{k+1}^{\alpha} = -\frac{\alpha_n A_n^{\alpha}}{k+1} \). The \( (C, \alpha_n) \) Cesàro means of integrable function \( f \) is

\[
\sigma_n^{\alpha_n} f(y) := \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} A_{n-k}^{\alpha_n-1} S_k f(y) = \int_I f(x) K_n^{\alpha_n} (y + x) d\lambda(x).
\]

In [3] Fine proved the almost everywhere convergence \( \sigma_n^{\alpha_n} f \rightharpoonup f \) for all integrable function \( f \) with constant sequence \( \alpha_n = \alpha_1 > 0 \). On the rate of convergence of Cesàro means in this constant case see the paper of Fridli [4]. For the two-dimensional situation see the paper of Goginava [7].

Comment 1. With respect to other locally constant orthonormal systems for instance it was a question of Taibleson [8] open for a long time, that does the Fejér-Lebesgue theorem, that is the a.e. convergence \( \sigma_n^{1} f \rightharpoonup f \) hold for all integrable function \( f \) with respect to the character system of the group of 2-adic integers. In 1997 Gáat answered [1] this question in the affirmative. Zheng and Gát generalized this result [9,2] for more general orthonormal systems.

Set two variable function \( P(n, \alpha) := \sum_{i=0}^{\infty} n_i 2^{\alpha i} \) for \( n \in \mathbb{N}, \alpha \in \mathbb{R} \). For instance \( P(n, 1) = n \). Also set for sequences \( \alpha = (\alpha_n) \) and positive reals \( K \) the subset of natural numbers

\[
\mathbb{N}_{\alpha, K} := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq K \right\}.
\]

We can easily remark that for a sequence \( \alpha \) such that \( 1 > \alpha_n \geq \alpha_0 > 0 \) we have \( \mathbb{N}_{\alpha, K} = \mathbb{N} \) for some \( K \) depending only on \( \alpha_0 \). We also remark that \( 2^n \in \mathbb{N}_{\alpha, K} \) for every \( \alpha = (\alpha_n), 0 < \alpha_n < 1 \) and \( K \geq 1 \).

In this paper \( C \) denotes an absolute constant and \( C_K \) another one which may depend only on \( K \). The introduction of \( (C, \alpha_n) \) means due to Akhobadze investigated [1] the behavior of the \( L^1 \)-norm convergence of \( \sigma_n^{\alpha_n} f \rightharpoonup f \) for the trigonometric system. In this paper it is also supposed that \( 1 > \alpha_n > 0 \) for all \( n \).

The main aim of this paper is to prove :

**Theorem 1.** Suppose that \( 1 > \alpha_n > 0 \). Let \( f \in L^1(I) \). Then we have the almost everywhere convergence \( \sigma_n^{\alpha_n} f \rightharpoonup f \) provided that \( \mathbb{N}_{\alpha, K} \ni n \to \infty \).
The method we use to prove Theorem 1 is to investigate the maximal operator
\( \sigma_n^f := \sup_{n \in N, k} |\sigma_n^{\alpha_n} f| \). We also prove that this operator is a kind of type \((H, L)\) and of type \((L^p, L^p)\) for all \(1 < p \leq \infty\). That is,

**Theorem 2.** Suppose that \(1 > \alpha_n > 0\). Let \(|f| \in H(I)\). Then we have
\[ \|\sigma_n^f\|_1 \leq C_K \|f\|_H. \]
Moreover, the operator \(\sigma_n^f\) is of type \((L^p, L^p)\) for all \(1 < p \leq \infty\). That is,
\[ \|\sigma_n^f\|_p \leq C_{K,p} \|f\|_p \]
for all \(1 < p \leq \infty\).

For the sequence \(\alpha_n = 1\) Theorem 2 is due to Fujii [5]. For the more general but constant sequence \(\alpha_n = \alpha_1\) see Weisz [11].

Basically, in order to prove Theorem 1 we verify that the maximal operator
\( \sigma_n^f (\alpha = \alpha_n) \) is of weak type \((L^1, L^1)\). The way we get this, the investigation of kernel functions, and its maximal function on the unit interval \(I\) by making a hole around zero and some quasi locality issues (for the notion of quasi-locality see [9]). To have the proof of Theorem 2 is the standard way. We need several Lemmas in the next section.

2. PROOFS

**Lemma 1.** For \(j, n \in N, j < 2^n\) we have
\[ D_{2^n-j}(x) = D_{2^n}(x) - \omega_{2^n-1}(x)D_j(x). \]

**Proof.**
\[ D_{2^n}(x) = \sum_{k=0}^{2^n-1} \omega_k(x) = \sum_{k=0}^{2^n-j-1} \omega_k(x) + \sum_{k=2^n-j}^{2^n-1} \omega_k(x) = D_{2^n-j} + \sum_{k=2^n-j}^{2^n-1} \omega_k(x). \]

We have to prove:
\[ \sum_{k=2^n-j}^{2^n-1} \omega_k(x) = \omega_{2^n-1}(x)D_j(x). \]

For \(k < j, k = k_{n-1}2^{n-1} + \ldots + k_2 2^1 + k_0\) we have
\[ \omega_{2^n-1}(x)\omega_k = \omega_{2^n-1+\ldots+2^1+2^0}(x)\omega_{k_{n-1}2^{n-1}+\ldots+k_0}(x) \]
\[ = \omega(1+k_{n-1}(mod 2))2^{n-1}+\ldots+(1+k_0(mod 2))2^0(x) \]
\[ = \omega(1-k_{n-1}(mod 2))2^{n-1}+\ldots+(1-k_0(mod 2))2^0(x) \]
\[ = \omega_{2^n-1+2^n-2+\ldots+2^0}(x) = \omega_{2^n-1+k_0}(x) = \omega_{2^n-1-k}(x). \]
Thus,

\[ \omega_{2^n-1}(x)D_j(x) = \omega_{2^n-1}(x) \sum_{k=0}^{j-1} \omega_k(x) = \sum_{k=0}^{j-1} \omega_{2^n-1-k}(x) = \sum_{k=2^n-j}^{2^n-1} \omega_k(x). \]

This completes the proof of Lemma 1.

Introduce the following notations: for \( a, n, j \in \mathbb{N} \) let

\[ n.j/ = \sum_{j=0}^{n} n_i 2^i, \]

that is,

\[ n.0/ = n_0 \textrm{ and for } 2^B \leq n < 2^{B+1}, \quad n._{B+1}/ = n.0/ \]

Moreover, introduce the following functions and operators for \( n \in \mathbb{N} \) and \( 1 > \alpha_n > 0 \)

\[ T_n^{\alpha_a} := \frac{1}{A_n^{\alpha_a}} \sum_{j=0}^{2|n|-1} A_{n-j}^{\alpha_a-1} D_j, \]

\[ \tilde{T}_n^{\alpha_a} := \frac{1}{A_n^{\alpha_a}} D_2 B \sum_{j=0}^{2B-1} A_{n(B)+j}^{\alpha_a-1} \]

\[ + (1 - \alpha_a) \sum_{j=0}^{2B-1} A_{n(B)+j}^{\alpha_a-1} \frac{j+1}{n(2^B+1)} \left| K^j \right| + A_{n(B)+1}^{\alpha_a-1} 2^B \left| K^*_{2B-1} \right|, \]

\[ \tau_n^{\alpha_a} f(y) := \int_I f(x) T_n^{\alpha_a}(y + x) d\lambda(x), \]

\[ \tilde{\tau}_n^{\alpha_a} f(y) := \int_I f(x) \tilde{T}_n^{\alpha_a}(y + x) d\lambda(x). \]

Now, we need to prove the next Lemma which means that maximal operator \( \sup_{n,a} | \tilde{T}_n^{\alpha_a} | \) is quasi-local. This lemma together with the next one are the most important tools in the proof of the main results of this paper.

**Lemma 2.** Let \( 1 > \alpha_a > 0, \ f \in L^1(I) \) such that \( \text{supp } f \subset I_k(u), \int_{I_k(u)} f d\lambda = 0 \) for some dyadic interval \( I_k(u) \). Then we have

\[ \sup_{I \setminus I_k(u), n,a \in \mathbb{N}} | \tilde{T}_n^{\alpha_a} f | d\lambda \leq C \| f \|_1. \]

Moreover, \( | T_n^{\alpha_a} | \leq \tilde{T}_n^{\alpha_a} \).

**Proof.** It is easy to have that for \( n < 2^k \) and \( x \in I_k(u) \) we have \( \tilde{T}_n^{\alpha_a}(y + x) = \tilde{T}_n^{\alpha_a}(y + u) \) and

\[ \int_{I_k(u)} f(x) \tilde{T}_n^{\alpha_a}(y + x) d\lambda(x) = \tilde{T}_n^{\alpha_a}(y + u) \int_{I_k(u)} f(x) d\lambda(x) = 0. \]
Therefore,
\[
\int_{I \setminus I_k(n)} \sup_{n, a \in \mathbb{N}} \tilde{r}_n^a f d \lambda = \int_{I \setminus I_k(n) \geq 2^k} \sup_{n, a \in \mathbb{N}} \tilde{r}_n^a f d \lambda.
\]
Recall that \( B = |n| \). Then
\[
A_n^{a, T} T_n^{a, \alpha} = \sum_{j=0}^{2^B-1} A_{2^B+n(B)-j} D_j
\]
\[
= \sum_{j=0}^{2^B-1} A_{n(B)+j} D_{2^B-j}
\]
By Lemma 1 we have
\[
A_n^{a, T} T_n^{a, \alpha} = D_{2^B} \sum_{j=0}^{2^B-1} A_{n(B)+j} - \omega_{2^B-1} \sum_{j=0}^{2^B-1} A_{n(B)+j} D_j.
\]
It is easy to have that \( \frac{1}{A_n^a} D_{2^B}(z) \sum_{j=0}^{2^B-1} A_{n(B)+j} = 0 \), for any \( z \in I \setminus I_k \). This holds because \( D_{2^B}(z) = 0 \) for \( B = |n| \geq k \) and \( z \in I \setminus I_k \). By the help of the Abel transform we get:
These equalities above immediately proves inequality $|\mathcal{T}^{\alpha}_{n}| \leq \hat{\mathcal{T}}^{\alpha}_{n}$.

Since for any $j < 2^k$ we have that the Fejér kernel $K_j^1(y + x)$ depends (with respect to $x$) only on coordinates $x_0, \ldots, x_{k-1}$, then

$$f_k(u) f(x) |K_j^1(y + x)|d\lambda(x) = |K_j^1(y + u)|f_k(u) f(x)d\lambda(x) = 0$$

gives $f_k(u) f(x)I(y + x)d\lambda(x) = 0$. On the other hand,

$$II = (1 - \alpha_a) \sum_{j=2^k}^{2^{k+1} - 1} A_{n(B) + j}^{\alpha_a - 1} \frac{j + 1}{n(B) + j + 1} |K_j^1|$$

$$\leq \sup_{j \geq 2^k} |K_j^1|(1 - \alpha_a) \sum_{j=0}^{n} A_{n(B) + j}^{\alpha_a - 1} = A_{n(B)}^{\alpha_a} (1 - \alpha_a) \sup_{j \geq 2^k} |K_j^1|.$$
The situation with $III$ is similar. Namely,

$$\frac{A_n^{\alpha_a-1} n}{A_n^{\alpha_a}} = \frac{\alpha_a \cdot n}{(\alpha_a + n)} \leq \alpha_a < 1.$$  

So, just as in the case of $II$ we apply Lemma 3 in [6]

$$\int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} III d\lambda \leq \int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} |K_{2^n-1}^1| d\lambda \leq C.$$  

Therefore, substituting $z = x + y \in I \setminus I_k$ (since $x \in I_k(u)$ and $y \in I \setminus I_k(u)$)

$$\int_{I \setminus I_k(u)} \sup_{n \geq 2^k, a \in \mathbb{N}} \tilde{T}_n^{\alpha_a} f d\lambda$$

$$= \int_{I \setminus I_k(u)} \sup_{n \geq 2^k, a \in \mathbb{N}} \left| \int_{I_k(u)} f(x) \tilde{T}_n^{\alpha_a}(y + x)d\lambda(x) \right| d\lambda(y)$$

$$\leq \int_{I \setminus I_k(u)} \int_{I_k(u)} |f(x)| \sup_{n \geq 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} (III(y + x) + III(y + x)) d\lambda(x)$$

$$= \int_{I_k(u)} |f(x)| \int_{I \setminus I_k(n \geq 2^k, a \in \mathbb{N})} \frac{1}{A_n^{\alpha_a}} (III(z) + III(z)) d\lambda(z) d\lambda(x)$$

$$\leq C \int_{I_k(u)} |f(x)| d\lambda(x).$$

This completes the proof of Lemma 2. 

A straightforward corollary of this lemma is:

**Corollary 1.** Let $1 > \alpha_a > 0$. Then we have $\|T_n^{\alpha_a}\|_1 \leq \|\tilde{T}_n^{\alpha_a}\|_1 \leq C \cdot \|T_n^{\alpha_a} f\|_1$, $\|\tilde{T}_n^{\alpha_a} f\|_1 \leq C \|f\|_1$ and $\|T_n^{\alpha_a} g\|_{\infty}, \|\tilde{T}_n^{\alpha_a} g\|_{\infty} \leq C \|g\|_{\infty}$ for all natural numbers $a, n$, where $C$ is some absolute constant and $f \in L^1, g \in L^\infty$. That is, operators $\tilde{T}_n^{\alpha_a}, T_n^{\alpha_a}$ is of type $(L^1, L^1)$ and $(L^\infty, L^\infty)$ (uniformly in $n$).

**Proof.** The proof is a straightforward consequence of Lemma 2 and an easy estimation below. Let $B = |n|$. Then

$$\|A_n^{\alpha_a} \tilde{T}_n^{\alpha_a}\|_1 \leq \|D_{2^n} u\|_1 \sum_{j=0}^{2^n-1} A_n^{\alpha_a-1} j$$

$$+ (1 - \alpha_a) \sum_{j=0}^{2^n-1} A_n^{\alpha_a-1} j_{n(B)} + j + 1 \|K_j^1\|_1 + A_n^{\alpha_a-1} 2^B \|K_{2^n-1}^1\|_1.$$  

Then by $\|D_{2^n} u\|_1 = 1, \|K_j^1\|_1 \leq C$ we complete the proof of Corollary 1. 

□
Lemma 3. Let \( n, N \) be any natural numbers and \( 0 < \alpha < 1 \). Then we have

\[
\frac{A_n^\alpha}{A_N^\alpha} \leq 2 \left( \frac{n + 1}{N} \right)^\alpha.
\]

Proof. By definition we have

\[
\frac{A_n^\alpha}{A_N^\alpha} = \left( 1 - \frac{\alpha}{n + 1 + \alpha} \right) \cdots \left( 1 - \frac{\alpha}{N + \alpha} \right) \leq \left( 1 - \frac{\alpha}{n + 2} \right) \cdots \left( 1 - \frac{\alpha}{N + 1} \right).
\]

It is well-known that

\[
\left( 1 - \frac{\alpha}{i(n + 1) + 1} \right) \cdots \left( 1 - \frac{\alpha}{(i+1)(n+1)} \right) \leq \left( 1 - \frac{\alpha}{(i+1)(n+1)} \right)^{n+1} = (e^{-1})^{\frac{\alpha}{i+1}}
\]

for every \( n \in \mathbb{N} \). This gives

\[
\left( 1 - \frac{\alpha}{n + 2} \right) \cdots \left( 1 - \frac{\alpha}{N + 1} \right) \leq (e^{-1})^{\alpha \log_e \left( \frac{N}{n+1} \right) - 1 + c} \leq 2(e^{-1})^{\alpha \log_e \left( \frac{N}{n+1} \right)} = 2 \left( \frac{n + 1}{N} \right)^\alpha,
\]

where \( c \approx 0.5772 \) is the Euler-Mascheroni constant. This completes the proof of Lemma 3. \( \square \)

Recall that the two variable function \( P(n, \alpha) = \sum_{i=0}^{\infty} n_{i/2^i} \) for \( n \in \mathbb{N}, \alpha \in \mathbb{R} \) and \( K \in \mathbb{R} \) determines the set of natural numbers

\[
\mathbb{N}_{\alpha, K} = \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq K \right\}.
\]

Let \( n = 2^{h_s} + \cdots + 2^{h_0} \), where \( h_s > \cdots > h_0 \geq 0 \) are integers. That is, \( |n| = h_s \). Let \( n^{(i)} := 2^{h_i} + \cdots + 2^{h_0} \). This means \( n = n^{(s)} \). Define the following kernel function and operators

\[
\mathcal{K}_n^{\alpha_n} := \mathcal{T}^{\alpha_n}_{n^{(s)}} + \sum_{l=0}^{s} \left( \frac{A_{n^{\alpha_n}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{\alpha_n}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \mathcal{T}^{\alpha_n}_{n^{(s-l-1)}} \right)
\]

and

\[
\mathcal{S}_n^{\alpha_n} f := f * \mathcal{K}_n^{\alpha_n}, \quad \mathcal{S}_n^{\alpha} f := \sup_{n \in \mathbb{N}_{\alpha, K}} |f * \mathcal{K}_n^{\alpha_n}|.
\]

In the sequel we prove that maximal operator \( \mathcal{S}_n^{\alpha_n} f \) is quasi-local. This is the very base of the proof of the main results of this paper. That is, Theorem 1 and Theorem 2.
Lemma 4. Let $1 > \alpha > 0$, $f \in L^1(I)$ such that $\text{supp} f \subset I_k(u)$, $\int_{I_k(u)} f d\lambda = 0$ for some dyadic interval $I_k(u)$. Then we have

$$\int_{I \setminus I_k(u)} \tilde{\sigma}_*^\alpha f d\lambda \leq C K \|f\|_1,$$

where constant $C_K$ can depend only on $K$.

Proof. Recall that $n = 2^{h_s} + \cdots + 2^{h_0}$, where $h_s > \cdots > h_0 \geq 0$ are integers. That is, $|n| = h_s$. Let $n^{(j)} := 2^{h_j} + \cdots + 2^{h_0}$. This means $n = n^{(s)}$. Use also the notation

$$K_{n^{(s)}}^d = \tilde{T}_n^{\alpha_n} + \sum_{l=0}^s \left( \frac{A_{n^{(s)}}^{\alpha_n}}{A_{n^{(s)}} d} D_{2^{h_j}} + \frac{A_{n^{(s)}}^{\alpha_n}}{A_{n^{(s)}} d} \tilde{T}_{n^{(l-1)}}^{\alpha_n} \right)$$

\(= G_1 + G_2 + G_3.\)

Since $n^{(l-1)} < 2^{h_{j-1}+1}$, then by Lemma 3 we have

$$\frac{A_{n^{(s)}}^{\alpha_n}}{A_{n^{(s)}} d} \leq 2 \left( \frac{n^{(l-1)}+1}{n^{(s)}} \right)^\alpha \leq 2 \frac{2\alpha_n(h_{j-1}+1)}{2\alpha_n h_s} \leq C \frac{2^{h_{j-1}+1}}{n^{(s)}}.$$

That is, by the above written we also have

$$\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \left| \int_{I_k(u)} f(x) G_2(y + x)d\lambda(x) \right| d\lambda(y)$$

$$\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \sum_{l=0}^s \frac{2^{h_{j-1}+1}}{n^{(s)}} \left| \int_{I_k(u)} f(x) D_{2^{h_j}}(y + x)d\lambda(x) \right| d\lambda(y) = 0$$

since $f \ast D_{2^n} = 0$ for $h \leq k$ because of the $A_k$ measurability of $D_{2^n}$ and $f = 0$. Besides, for $h > k$ $D_{2^{h}}(y + x) = 0$ $(y + x \notin I_k)$.

As a result of these estimations above and by the help of Lemma 2, that is the quasi-locality of operator $\tilde{\tau}_n^\alpha = \sup_{n, \alpha \in \mathbb{N}} |\tilde{\tau}_n^{\alpha_n}|$ we conclude

$$\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \left| \int_{I_k(u)} f(x)(G_1(y + x) + G_3(y + x))d\lambda(x) \right| d\lambda(y)$$

$$\leq C K \int_{I \setminus I_k(u)} \sup_{n, \alpha \in \mathbb{N}} \left| \int_{I_k(u)} f(x) \tilde{T}_n^{\alpha_n}(y + x)d\lambda(x) \right| d\lambda(y)$$

$$\leq C K \|f\|_1.$$

This completes the proof of Lemma 4.

\(\square\)

Lemma 5. The operator $\tilde{\tau}_n^\alpha$ is of type $(L^\infty, L^\infty)$ $(\tilde{\tau}_n^\alpha f := \sup_{n \in \mathbb{N}, \alpha, K} |\tilde{\tau}_n^{\alpha_n} f|)$. 

\[ \| \hat{T}_{\alpha_n} \|_1 = \| \hat{T}_{\alpha_n}^{(r)} \|_1 \leq \| \hat{T}_{\alpha_n} \|_1 + \sum_{l=0}^{s} \left( \frac{A_{n(l)}^{\alpha_n}}{A_{n(l)}^{\alpha_n}} \| D_{2^{b_l}} \|_1 + \frac{A_{n(l-1)}^{\alpha_n}}{A_{n(l)}^{\alpha_n}} \| \hat{T}_{\alpha_n}^{(r)} \|_1 \right) \]

\[ \leq C + C \sum_{l=0}^{s} \frac{A_{n(l-1)}^{\alpha_n}}{A_{n(l)}^{\alpha_n}} \leq C_K \]

because \( n \in \mathbb{N}_{\sigma_K} \). Hence \( \hat{\sigma}_n^{\alpha} \) is of type \( (L^\infty, L^\infty) \) (with constant \( C_K \)). This completes the proof of Lemma 5.

Now, we can prove the main tool in order to have Theorem 1 for operator \( \hat{\sigma}_n^{\alpha} f := \sup_{n \in \mathbb{N}_{\sigma_K}} \| \sigma_n^{\alpha_n} f \| \).

**Lemma 6.** The operators \( \hat{\sigma}_n^{\alpha} \) and \( \sigma_n^{\alpha} \) are of weak type \( (L^1, L^1) \).

**Proof.** First, we prove Lemma 6 for operator \( \hat{\sigma}_n^{\alpha} \). We apply the Calderon-Zygmund decomposition lemma [9]. That is, let \( f \in L^1 \) and \( \| f \|_1 < \delta \). Then there is a decomposition:

\[ f = f_0 + \sum_{j=1}^{\infty} f_j \]

such that \( \| f_0 \|_\infty \leq C \delta \), \( \| f_0 \|_1 \leq C \| f \|_1 \) and \( I^j = I_{k_j}(u^j) \) are disjoint dyadic intervals for which

\[ \text{supp } f_j \subset I^j, \quad \int_{I^j} f_j d\lambda = 0, \quad |F| \leq \frac{C \| f \|_1}{\delta} \]

(\( u^j \in I, k_j \in \mathbb{N}, j \in \mathcal{P} \)), where \( F = \bigcup_{j=1}^{\infty} I^j \). By the \( \sigma \)-sublinearity of the maximal operator with an appropriate constant \( C_K \) we have

\[ \lambda(\hat{\sigma}_n^{\alpha} f > 2C_K \delta) \leq \lambda(\hat{\sigma}_n^{\alpha} f_0 > C_K \delta) + \lambda(\hat{\sigma}_n^{\alpha}(\sum_{i=1}^{\infty} f_i) > C_K \delta):= I + II. \]

Since by Lemma 5 \( \| \hat{\sigma}_n^{\alpha} f_0 \|_\infty \leq C_K \| f_0 \|_\infty \leq C_K \delta \) then we have \( I = 0 \). So,

\[ \lambda(\hat{\sigma}_n^{\alpha}(\sum_{i=1}^{\infty} f_i) > C_K \delta) \leq |F| + \lambda(F \cap \{\hat{\sigma}_n^{\alpha}(\sum_{i=1}^{\infty} f_i) > C_K \delta\}) \]

\[ \leq \frac{C_K \| f \|_1}{\delta} + \frac{C_K \| f \|_1}{\delta} \sum_{i=1}^{\infty} \hat{\sigma}_n^{\alpha} f_j d\lambda =: \frac{C_K \| f \|_1}{\delta} + \frac{C_K \| f \|_1}{\delta} \sum_{i=1}^{\infty} III_j, \]
where
\[
III_j := \int_{I \setminus I_j} \tilde{\sigma}_\alpha f_j d\lambda
\]
\[
\leq \int_{I \setminus I_{k_j}(u')} \sup_{n \in N_{a,K}} \left| \int_{I_{k_j}(u')} f_j(x) \tilde{\mathcal{K}}_{\alpha n}^a(y + x) d\lambda(x) \right| d\lambda(y).
\]
The forthcoming estimation of \(III_j\) is given by the help Lemma 4
\[
III_j \leq C_K \|f_j\|. 
\]
That is, operator \(\tilde{\sigma}_\alpha\) is of weak type \((L^1, L^1)\). Next, we prove the estimation
\[
|K_{\alpha n}^a| \leq K_{\alpha n}^a. \tag{1}
\]
To prove (1) recall again that \(n = 2^{h_s} + \cdots + 2^{h_0}, \) where \(h_s > \cdots > h_0 \geq 0\) are integers. Since \(n = 2^{h_s} + n^{(s-1)}\), then we have
\[
\sum_{j=2^{h_s}}^{2^{h_s} + n^{(s-1)}} A_{n^{(s-1)}+j}^a = \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)}-k}^a D_{2^{h_s}+k}
\]
\[
= D_{2^{h_s}} \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)}-k}^a + \omega_{2^{h_s}} \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)}-k}^a D_k
\]
\[
= D_{2^{h_s}} A_{n^{(s-1)}} + \omega_{2^{h_s}} A_{n^{(s-1)}} K_{\alpha n}^a.
\]
So, by the help of the equalities above we get
\[
K_{\alpha n}^a = T_{n^{(s)}}^a + A_{n^{(s)}}^a \left( D_{2^{h_s}} + \omega_{2^{h_s}} K_{\alpha n}^a \right).
\]
Apply this last formula recursively and Lemma 2. (Note that \(n^{(-1)} = 0, T_0^a = K_{\alpha 0}^a = 0, A_0^a = 1\).)
\[
|K_{\alpha n}^a| = |K_{\alpha n}^a| \leq |T_{n^{(s)}}^a| + \sum_{l=0}^{s} \left( \prod_{j=l}^{s} A_{n^{(l-1)}}^a D_{2^{h_j}} + \prod_{j=l}^{s} A_{n^{(l-1)}}^a \right) |T_{n^{(l-1)}}^a|
\]
\[
= |T_{n^{(s)}}^a| + \sum_{l=0}^{s} \left( \frac{A_{n^{(l-1)}}^a}{A_{n^{(s)}}^a} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^a}{A_{n^{(s)}}^a} |T_{n^{(l-1)}}^a| \right)
\]
\[
\leq \tilde{K}_{\alpha n}^a = \tilde{K}_{\alpha n}^a.
\]
This completes the proof of inequality (1). This inequality gives that the operator \(\sigma_\alpha^a\) is also of weak type \((L^1, L^1)\) since
\[ \lambda(\sigma_n^a f > 2C_K \delta) \leq \lambda(\tilde{\sigma}_n^a |f| > 2C_K \delta) \leq C_K \frac{\|f\|_1}{\delta} = C_K \frac{\|f\|_1}{\delta}. \]

This completes the proof of Lemma 6. \(\square\)

**Proof of Theorem 1.** Let \( P \in \mathbf{P} \) be a polynomial where \( P(x) = \sum_{i=0}^{2^k-1} c_i x^i \). Then for all natural number \( n \geq 2^k \), \( n \in \mathbb{N}_{\alpha,K} \) we have that \( S_n P \equiv P \). Consequently, the statement \( \sigma_n^a P \to P \) holds everywhere (of course not only for restricted \( n \in \mathbb{N}_{\alpha,K} \)). Now, let \( \epsilon, \delta > 0, f \in L^1 \). Let \( P \in \mathbf{P} \) be a polynomial such that \( \|f - P\|_1 < \delta \).

Then
\[
\frac{\lambda(\lim_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^a f - f| > \epsilon)}{\lambda(\lim_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^a (f - P)| > \epsilon/3) + \lambda(\lim_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^a P - P| > \epsilon/3)}
\]

\[
\leq \lambda(\sup_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^a (f - P)| > \epsilon/3) + \frac{3}{\epsilon} \|f - P\|_1 \leq C_K \frac{\|f\|_1}{\epsilon} \frac{3}{\epsilon} \frac{C_K}{\delta}
\]

because \( \sigma_n^a \) is of weak type \((L^1, L^1)\) (with any fixed \( K > 0 \)). This holds for all \( \delta > 0 \). That is, for an arbitrary \( \epsilon > 0 \) we have
\[
\lambda(\lim_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^a f - f| > \epsilon) = 0
\]

and consequently we also have
\[
\lambda(\lim_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^a f - f| > 0) = 0.
\]

This finally gives
\[
\lim_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^a f - f| = 0 \ a.e.,
\]
\[
\sigma_n^a f \to f \ a.e. \ (n \in \mathbb{N}_{\alpha,K}).
\]

This completes the proof of Theorem 1. \(\square\)

**Proof of Theorem 2.** Inequality (1), Lemma 5 and Lemma 6 by the interpolation theorem of Marcinkiewicz [9] give that the operator \( \sigma_n^a \) is of type \((L^p, L^q)\) for all \( 1 < p \leq \infty \). In the sequel we prove that operator \( \tilde{\sigma}_n^a f = \sup_{n \in \mathbb{N}_{\alpha,K}} |f \ast \tilde{K}_n^a| \) is of type \((H, L)\).

Let \( a \) be an atom \( (a \neq 1 \text{ can be supposed}) \), \( \text{supp} a \subset I_k(x), \|a\|_\infty \leq 2^k \) for some \( k \in \mathbb{N} \) and \( x \in I \). Then, \( n < 2^k, n \in \mathbb{N}_{\alpha,K} \) implies \( a \ast \tilde{K}_n^a = 0 \) because \( \tilde{K}_n^a \) is \( \mathcal{A}_k \) measurable for \( n < 2^k \) and \( \int_{I_k(x)} a(t) d\lambda(t) = 0 \). That is,
\[
\tilde{\sigma}_n^a a = \sup_{\mathbb{N}_{\alpha,K} \ni n \geq 2^k} |\tilde{\sigma}_n^a f|.
\]
By the help Lemma 4 we have

\[ \int_{I \setminus I_k(x)} \hat{\sigma}_n^a \ d \lambda = \int_{I \setminus I_k(x)} \sup_{\|a\|_1, K \geq 2^k} \left| \int_{I_k(x)} a(y) \hat{K}_n^a(z + y) d \lambda(y) \right| d \lambda(z) \]

\[ \leq C_K \int_{I_k(x)} |a(y)| d \lambda(y) \]

\[ \leq C_K \|a\|_1 \]

\[ \leq C_K. \]

Since the operator \( \hat{\sigma}_n^a \) is of type \((L^2, L^2)\) (i.e. \( \|\hat{\sigma}_n^a f\|_2 \leq C_K \|f\|_2 \) for all \( f \in L^2(I) \)), then we have

\[ \|\hat{\sigma}_n^a a\|_1 = \int_{I \setminus I_k(x)} \hat{\sigma}_n^a a + \int_{I_k(x)} \hat{\sigma}_n^a a \]

\[ \leq C_K + |I_k(x)|^{\frac{1}{2}} \|\hat{\sigma}_n^a a\|_2 \]

\[ \leq C_K + C_K 2^{\frac{-1}{2}} \|a\|_2 \]

\[ \leq C_K + C_K 2^{\frac{-1}{2}} 2^k \]

\[ \leq C_K. \]

That is \( \|\hat{\sigma}_n^a a\|_1 \leq C_K \) and consequently the \( \sigma \)-sublinearity of \( \hat{\sigma}_n^a \) gives

\[ \|\hat{\sigma}_n^a f\|_1 \leq \sum_{i=0}^{\infty} |\lambda_i| \|\hat{\sigma}_n^a a_i\|_1 \]

\[ \leq C_K \sum_{i=0}^{\infty} |\lambda_i| \]

\[ \leq C_K \|f\|_H \]

for all \( \sum_{i=0}^{\infty} \lambda_i a_i \in H \). That is, the operator \( \hat{\sigma}_n^a \) is of type \((H, L)\). This by inequality (1) and then by \( \|\sigma_n^a f\|_1 \leq \|\hat{\sigma}_n^a f\|_1 \leq C_K \|f\|_H \) completes the proof of Theorem 2. \( \square \)

REFERENCES


Authors’ addresses

**Anas Ahmad Abu Joudeh**
Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
E-mail address: anas.abujoudeh@mailbox.unideb.edu.hu, mr.anas_judeh@yahoo.com

**György Gát**
Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
E-mail address: gat.gyorgy@science.unideb.hu