



A CLASS OF FOURTH-ORDER KIRCHHOFF TYPE ELLIPTIC PROBLEMS WITH FOUR SOLUTIONS

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Abstract. In this paper, we establish the existence of at least four solutions for a perturbed fourth-order Kirchhoff type elliptic problem by using a recent variational principle due to Ricceri.

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1. INTRODUCTION

This paper is concerned with the existence of weak solutions to the following fourth-order Kirchhoff type elliptic problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \left[M\left(\int_{\Omega} |\nabla u|^p dx\right) \right]^{p-1} \Delta_p u + \eta|u|^{p-2}u \\ = \lambda a(x)f(u) + b(x)g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded smooth domain in \mathbb{R}^N ($N \geq 1$), $p > \max\{1, \frac{N}{2}\}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian operator, $\eta > 0$, $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous non-constant functions, $\lambda \in \mathbb{R}$, $a, b \in L^1(\Omega)$ are non-negative non-constant functions and the function $M : [0, +\infty[\rightarrow \mathbb{R}$ is a continuous function.

The fourth-order equation furnishes a model to study travelling waves in suspension bridges, so it is important to physics. Due to this, many researchers have discussed the existence of at least one solution, or multiple solutions, or even infinitely many solutions for fourth-order boundary value problems by using lower and upper solution methods, Morse theory, the mountain-pass theorem, constrained minimization and concentration-compactness principle, fixed point theorems and degree theory, and variational methods and critical point theory. We refer the reader to [1, 8, 10, 13, 22, 23].

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This problem models several physical and biological systems where u describes a process which depends on the average of itself, such as the population density, see [2].

The problem (1.1) is a generalization of the stationary problem of a model introduced by Kirchhoff [9]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

which L, E, h, ρ, P_0 are constants, which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations.

However, to be accurate, the problem (1.1) is related to the models of extensible beam and plates by Woinowsky-Krieger [20] and Berger [4]. In addition, the first stationary study of such fourth order nonlocal boundary value problem was given by Ma [11]. In [13], it was named first time *fourth order problem of Kirchhoff type*. The problem is also related to the so-called p -Kirchhoff problems.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer interested readers to [2, 3, 7, 8, 12, 16, 19, 21, 24]. For instance, B. Ricceri in an interesting paper [16] established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problems using Theorem A of [14].

The starting point of our approach to problem (1.1) has been [15], in which the author considers the following ordinary Neumann problem

$$\begin{cases} -u'' + u = \lambda a(x)f(u) + b(x)g(u) & \text{in } [0, 1], \\ u'(0) = u'(1) = 0, \end{cases} \quad (1.3)$$

where $\lambda \in \mathbb{R}$ and $f, g : \mathbb{R} \rightarrow \mathbb{R}, a, b : [0, 1] \rightarrow [0, +\infty[$ are four continuous non-constant functions. Ricceri has proved that the above problem admits at least three non-zero solutions.

Our results here are motivated by the recent papers [5, 15, 17, 18]. We establish some sufficient conditions under which the problem (1.1) possesses four weak solutions.

The paper is arranged as follows. In Section 2, we give preliminaries and our main tool, that is, Theorem 1, while in Section 3, we present our main results.

2. FUNCTIONAL SETTING

As already said in the introduction, our approach in facing problem (1.1) is based on the multiplicity result established in [15], that we recall below for the reader's convenience (see also Theorem 1 of [17] for a similar result).

Theorem 1 ([15, Theorem 1]). *Let X be a reflexive real Banach space; $\Phi : X \rightarrow \mathbb{R}$ be a coercive and sequentially weakly lower semicontinuous C^1 functional whose*

derivative admits a continuous inverse on X^* , $\Psi_1, \Psi_2 : X \rightarrow \mathbb{R}$ two C^1 functionals with compact derivative. Assume that there exist two points $u_0, v_0 \in X$ with the following properties:

- (i) u_0 is a strict local minimum of Φ and $\Phi(u_0) = \Psi_1(u_0) = \Psi_2(u_0) = 0$;
- (ii) $\Phi(v_0) \leq \Psi_1(v_0)$ and $\Psi_2(v_0) > 0$.

Moreover, suppose that, for some $\rho \in \mathbb{R}$, one has either

$$\sup_{\lambda > 0} \inf_{u \in X} (\lambda(\Phi(u) - \Psi_1(u) - \rho) - \Psi_2(u)) < \inf_{u \in X} \sup_{\lambda > 0} (\lambda(\Phi(u) - \Psi_1(u) - \rho) - \Psi_2(u)) \tag{2.1}$$

or

$$\sup_{\lambda > 0} \inf_{u \in X} (\Phi(u) - \Psi_1(u) - \lambda(\rho + \Psi_2(u))) < \inf_{u \in X} \sup_{\lambda > 0} (\Phi(u) - \Psi_1(u) - \lambda(\rho + \Psi_2(u))). \tag{2.2}$$

Finally, assume that

$$\max \left\{ \limsup_{\|u\| \rightarrow +\infty} \frac{\Psi_1(u)}{\Phi(u)}, \limsup_{u \rightarrow u_0} \frac{\Psi_1(u)}{\Phi(u)} \right\} < 1 \tag{2.3}$$

and

$$\max \left\{ \limsup_{\|u\| \rightarrow +\infty} \frac{\Psi_2(u)}{\Phi(u)}, \limsup_{u \rightarrow u_0} \frac{\Psi_2(u)}{\Phi(u)} \right\} \leq 0. \tag{2.4}$$

Under such hypotheses, there exists $\lambda^* > 0$ such that the equation $\Phi'(u) = \Psi_1'(u) + \lambda^* \Psi_2'(u)$ has at least four solutions in X . More precisely, among them, one is u_0 as a strict local, not global minimum and two are global minima of the functional $\Phi - \Psi_1 - \lambda^* \Psi_2$.

Remark 1. It is important to remark that, in view of Theorem 1 of [6], condition (2.1) is equivalent to the existence of $u_1, v_1 \in X$ satisfying

$$\Phi(u_1) - \Psi_1(u_1) < \rho < \Phi(v_1) - \Psi_1(v_1)$$

and

$$\frac{\sup_{(\Phi - \Psi_1)^{-1}([-\infty, \rho])} \Psi_2 - \Psi_2(u_1)}{\rho - \Phi(u_1) + \Psi_1(u_1)} < \frac{\sup_{(\Phi - \Psi_1)^{-1}([-\infty, \rho])} \Psi_2 - \Psi_2(v_1)}{\rho - \Phi(v_1) + \Psi_1(v_1)}.$$

Likewise, condition (2.2) is equivalent to the existence of $u_1, v_1 \in X$ satisfying

$$\Psi_2(v_1) < \rho < \Psi_2(u_1)$$

and

$$\frac{\Phi(u_1) - \Psi_1(u_1) - \inf_{\Psi_2^{-1}([\rho, +\infty])} (\Phi - \Psi_1)}{\Psi_2(u_1) - \rho} < \frac{\Phi(v_1) - \Psi_1(v_1) - \inf_{\Psi_2^{-1}([\rho, +\infty])} (\Phi - \Psi_1)}{\Psi_2(v_1) - \rho}.$$

Here and in the sequel, X will denote the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_{\Omega} (|\Delta u(x)|^p + |\nabla u(x)|^p + |u(x)|^p) dx \right)^{\frac{1}{p}}.$$

Put

$$c = \sup_{u \in X \setminus \{0\}} \frac{\max_{x \in \bar{\Omega}} |u(x)|}{\|u\|}.$$

For $p > \max\{1, \frac{N}{2}\}$, since the embedding $X \hookrightarrow C^0(\bar{\Omega})$ is compact, one has $c < +\infty$. Let $M : [0, +\infty[\rightarrow \mathbb{R}$ be a continuous function such that there is a positive constant m with $M(t) \geq m$ for all $t \geq 0$. Set

$$\begin{aligned} \tilde{M}(t) &= \int_0^t (M(s))^{p-1} ds \quad \text{for all } t \geq 0, \\ M^- &:= \min\{1, m^{p-1}, \eta\}. \end{aligned}$$

The following lemma is useful in the proof of our results.

Lemma 1. *If $\gamma : \Omega \rightarrow [0, +\infty[$ be a non-zero function in $L^1(\Omega)$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ a continuous non-zero function, we denote by $T_{\gamma,h}$ the functional defined on X by putting*

$$T_{\gamma,h}(u) = \int_{\Omega} \gamma(x)h(u(x))dx$$

for all $u \in X$. Then, one has

$$\limsup_{u \rightarrow 0} \frac{T_{\gamma,h}(u)}{\|u\|^p} \leq c^p \|\gamma\|_{L^1(\Omega)} \max \left\{ 0, \limsup_{\xi \rightarrow 0} \frac{h(\xi)}{|\xi|^p} \right\} \quad (2.5)$$

and

$$\limsup_{\|u\| \rightarrow +\infty} \frac{T_{\gamma,h}(u)}{\|u\|^p} \leq c^p \|\gamma\|_{L^1(\Omega)} \max \left\{ 0, \limsup_{|\xi| \rightarrow +\infty} \frac{h(\xi)}{|\xi|^p} \right\}. \quad (2.6)$$

Also, the functional $T_{\gamma,H}$, where H is defined by $H(\xi) = \int_0^\xi h(t) dt$ for all $\xi \in \mathbb{R}$, turns out to be in $C^1(X, \mathbb{R})$ and its derivative is given by

$$T'_{\gamma,H}(u)(v) = \int_{\Omega} \gamma(x)h(u(x))v(x)dx$$

for all $u, v \in X$. Moreover, the compact embedding of X in $C^0(\bar{\Omega})$ implies that $T'_{\gamma,H} : X \rightarrow X^*$ is compact.

Proof of the previous lemma is similar to proof of Lemma 2.1 in [5]. The following lemma is Proposition 2.3 in [8], it is needed in the proof of Theorem 2.

Lemma 2. *Let $\Lambda : X \rightarrow X^*$ be the operator defined by*

$$\Lambda(u)v = \int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x) dx$$

$$\begin{aligned}
 &+ \left[M \left(\int_{\Omega} |\nabla u|^p dx \right) \right]^{p-1} \int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx \\
 &+ \eta \int_{\Omega} |u(x)|^{p-2} u(x) v(x) dx
 \end{aligned}$$

for every $u, v \in X$. Then Λ admits a continuous inverse on X^* .

We can deduce that $u \in X$ is a weak solution to problem (1.1) if and only if u is a critical point of the functional

$$\frac{1}{p} \int_{\Omega} |\Delta u(x)|^p dx + \frac{1}{p} \tilde{M} \left(\int_{\Omega} |\nabla u|^p dx \right) + \frac{\eta}{p} \int_{\Omega} |u(x)|^p dx - \lambda T_{a,F} - T_{b,G}$$

which represents, therefore, the energy functional related to problem (1.1).

3. MAIN RESULTS

Now, we present our main result.

Theorem 2. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two continuous non-constant functions and let $a, b : \Omega \rightarrow [0, +\infty[$ be two non-constant functions in $L^1(\Omega)$. Assume that*

$$(f_1) \max \left\{ \limsup_{|\xi| \rightarrow +\infty} \frac{\int_0^{\xi} f(t) dt}{|\xi|^p}, \limsup_{\xi \rightarrow 0} \frac{\int_0^{\xi} f(t) dt}{|\xi|^p} \right\} \leq 0,$$

$$(g_1) \sup_{\xi \in \mathbb{R}} \int_0^{\xi} g(t) dt < +\infty, \quad \limsup_{\xi \rightarrow 0} \frac{\int_0^{\xi} g(t) dt}{|\xi|^p} < \frac{M^-}{pc^p \|b\|_{L^1(\Omega)}}.$$

Finally, suppose that there exist $\sigma > c^p \sqrt{\frac{p}{M^-}} \max \left\{ 1, (\|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G)^{\frac{1}{p}} \right\}$ and $\xi_1 \in \mathbb{R}$ such that

$$(f_2) 0 < \int_0^{\xi_1} f(t) dt = \sup_{|\xi| \leq \sigma} \int_0^{\xi} f(t) dt < \sup_{\xi \in \mathbb{R}} \int_0^{\xi} f(t) dt,$$

$$(g_2) |\xi_1|^p \leq \frac{p}{\eta |\Omega|} \|b\|_{L^1(\Omega)} \int_0^{\xi_1} g(t) dt.$$

Under such hypotheses, there exists $\lambda^* > 0$ such that the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2} \Delta u) - \left[M \left(\int_{\Omega} |\nabla u|^p dx \right) \right]^{p-1} \Delta_p u + \eta |u|^{p-2} u \\ \quad = \lambda^* a(x) f(u) + b(x) g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least three non-zero solutions, two of which are global minima of the associated energy functional.

Proof. Our aim is to apply Theorem 1. Take Ψ_1, Ψ_2 equal, respectively, to $T_{b,G}, T_{a,F}$ defined in Section 2. For each $u \in X$, let the functional $\Phi : X \rightarrow \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{p} \int_{\Omega} |\Delta u(x)|^p dx + \frac{1}{p} \tilde{M} \left(\int_{\Omega} |\nabla u|^p dx \right) + \frac{\eta}{p} \int_{\Omega} |u(x)|^p dx.$$

The function Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned} \Phi'(u)h &= \int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x) dx \\ &\quad + \left[M \left(\int_{\Omega} |\nabla u|^p dx \right) \right]^{p-1} \int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx \\ &\quad + \eta \int_{\Omega} |u(x)|^{p-2} u(x) v(x) dx \end{aligned}$$

for every $v \in X$, while Lemma 2 gives that Φ' admits a continuous inverse on X^* . Furthermore, Φ is sequentially weakly lower semicontinuous.

Now, take $u_0 = 0$ and $v_0 = \xi_1$; of course, assumption (i) of Theorem 1 is evident. Moreover, thanks to (f_2) and (g_2) , one has

$$\Phi(v_0) = \frac{\eta}{p} |\xi_1|^p |\Omega| \leq \|b\|_{L^1(\Omega)} \int_0^{\xi_1} g(t) dt = T_{b,G}(v_0)$$

and

$$T_{a,F}(v_0) = \|a\|_{L^1(\Omega)} \int_0^{\xi_1} f(t) dt > 0,$$

so assumption (ii) of Theorem 1 is satisfied. Now, since $\frac{M^-}{p} \|u\|^p \leq \Phi(u)$ for each $u \in X$, we see that

$$\frac{T_{b,G}(u)}{\Phi(u)} \leq \frac{pT_{b,G}(u)}{M^- \|u\|^p},$$

and by Lemma 1, we get

$$\limsup_{\|u\| \rightarrow +\infty} \frac{T_{b,G}(u)}{\Phi(u)} \leq \frac{pc^p}{M^-} \|b\|_{L^1(\Omega)} \max \left\{ 0, \limsup_{|\xi| \rightarrow +\infty} \frac{\int_0^{\xi} g(t) dt}{|\xi|^p} \right\} \leq 0.$$

Also, in view of Lemma 1 and (g_1) , we have

$$\limsup_{u \rightarrow 0} \frac{T_{b,G}(u)}{\Phi(u)} \leq \frac{pc^p}{M^-} \|b\|_{L^1(\Omega)} \max \left\{ 0, \limsup_{\xi \rightarrow 0} \frac{\int_0^{\xi} g(t) dt}{|\xi|^p} \right\} < 1.$$

So, (2.3) is fulfilled. Likewise, from Lemma 1 and (f_1) , one has

$$\limsup_{\|u\| \rightarrow +\infty} \frac{T_{a,F}(u)}{\Phi(u)} \leq \frac{pc^p}{M^-} \|a\|_{L^1(\Omega)} \max \left\{ 0, \limsup_{|\xi| \rightarrow +\infty} \frac{\int_0^{\xi} f(t) dt}{|\xi|^p} \right\} \leq 0$$

and

$$\limsup_{u \rightarrow 0} \frac{T_{a,F}(u)}{\Phi(u)} \leq \frac{pc^p}{M^-} \|a\|_{L^1(\Omega)} \max \left\{ 0, \limsup_{\xi \rightarrow 0} \frac{\int_0^\xi f(t) dt}{|\xi|^p} \right\} \leq 0,$$

which satisfy (2.4). Finally, let us check that (2.1) holds. Since $\sigma > c \sqrt[p]{\frac{p}{M^-}}$, we see that

$$1 - \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G < \frac{\sigma^p M^-}{pc^p} - \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G$$

and so it is possible to choose $\rho \in \mathbb{R}$ such that

$$\max \left\{ 0, 1 - \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G \right\} < \rho < \frac{\sigma^p M^-}{pc^p} - \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G.$$

Let $u \in X$ such that $\Phi(u) - T_{b,G}(u) \leq \rho$. For each $u \in X$, by the definition of σ and the choice of ρ , we get

$$\frac{M^-}{p} \|u\|^p \leq \Phi(u) \leq \rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G$$

and then, being $\rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G > 1$, we obtain

$$\|u\| \leq \left(\frac{p}{M^-} (\rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G) \right)^{\frac{1}{p}} \leq \frac{\sigma}{c}.$$

Thus, owing to the embedding of X in $C^0(\bar{\Omega})$ we have the inclusion

$$\{u \in X : \Phi(u) - T_{b,G}(u) \leq \rho\} \subseteq \left\{ u \in X : \sup_{x \in \bar{\Omega}} |u(x)| \leq \sigma \right\}. \tag{3.1}$$

Now, in order to fulfil the equivalent formulation of (2.1) recalled in Remark 1, choose $u_1 = v_0$ and take as v_1 any constant d such that $F(d) > \sup_{[-\sigma, \sigma]} F$. Such a d exists by (f_2) . Thanks to (g_2) , we have

$$\Phi(u_1) - T_{b,G}(u_1) \leq \frac{\eta|\Omega|}{p} |\xi_1|^p - \|b\|_{L^1(\Omega)} \int_0^{\xi_1} g(t) dt \leq 0 < \rho.$$

Moreover, $\Phi(v_1) - T_{b,G}(v_1)$ has to be necessarily strictly greater than ρ , otherwise, by (3.1), we would have $|d| \leq \sigma$ and $F(d) \leq \sup_{[-\sigma, \sigma]} F$, a contradiction. Then, due to (f_2) and to the choice of d , we easily obtain that

$$\sup_{(\Phi - T_{b,G})^{-1}([-\infty, \rho])} T_{a,F} \leq T_{a,F}(u_1)$$

and

$$\sup_{(\Phi - T_{b,G})^{-1}([-\infty, \rho])} T_{a,F} \leq T_{a,F}(v_1).$$

Thus, the following inequalities hold

$$\frac{\sup_{(\Phi - T_{b,G})^{-1}(\cdot - \infty, \rho]} T_{a,F} - T_{a,F}(u_1)}{\rho - \Phi(u_1) + T_{b,G}(u_1)} < 0 < \frac{\sup_{(\Phi - T_{b,G})^{-1}(\cdot - \infty, \rho]} T_{a,F} - T_{a,F}(v_1)}{\rho - \Phi(v_1) + T_{b,G}(v_1)}$$

and, each assumption of Theorem 1 being satisfied, our problem admits at least three non-zero weak solutions, two of which are global minima of the energy functional. \square

Remark 2. Suppose that $\alpha \geq 0$ and $\beta > 0$ be two real numbers and suppose $M : [\alpha, \beta] \subseteq [0, +\infty[\rightarrow \mathbb{R}$ be a function defined by $M(t) = \theta_1 + \theta_2 t$ for each $t \in [\alpha, \beta]$ where $\theta_1, \theta_2 > 0$. So, $M^- = \min\{1, (\theta_1 + \theta_2 \alpha)^{p-1}, \eta\}$ and Theorem 2 holds for the following problem

$$\begin{cases} \Delta(|\Delta u|^{p-2} \Delta u) - (\theta_1 + \theta_2 \int_{\Omega} |\nabla u|^p)^{p-1} \Delta_p u + \eta |u|^{p-2} u \\ \quad = \lambda a(x) f(u) + b(x) g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f, g, a and b satisfy in the conditions of Theorem 2.

Remark 3. It is worth pointing out that, if assumption (f_2) of Theorem 2 is replaced by the existence of

$$\sigma > c \sqrt[p]{\frac{P}{M^-}} \max \left\{ 1, (\|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G)^{\frac{1}{p}} \right\}$$

and $\xi_1, \xi_2 \in \mathbb{R}$, with $\xi_1 \xi_2 > 0$, such that

$$0 < \int_0^{\xi_1} f(t) dt = \sup_{|\xi| \leq \sigma} \int_0^{\xi} f(t) dt < \int_0^{\xi_2} f(t) dt, \quad (3.2)$$

under the additional assumption $|\Omega| \geq \frac{M^-}{\eta c^p}$, we can ensure that the three non-zero solutions of the thesis of Theorem 2 are non-negative (respectively non-positive) provided $\xi_1 > 0$ (respectively $\xi_2 < 0$). To see this, it suffices to apply Theorem 2 to the functions $f_0, g_0 : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \xi \geq 0, \\ 0 & \text{if } \xi < 0, \end{cases}$$

$$g_0(\xi) = \begin{cases} g(\xi) & \text{if } \xi \geq 0, \\ 0 & \text{if } \xi < 0, \end{cases}$$

when $\xi_1 > 0$ or by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \xi \leq 0, \\ 0 & \text{if } \xi > 0, \end{cases}$$

$$g_0(\xi) = \begin{cases} g(\xi) & \text{if } \xi \leq 0, \\ 0 & \text{if } \xi > 0, \end{cases}$$

when $\xi_1 < 0$. In fact, conditions (f_1) and (g_1) ensure that $f(0) = 0$ and $g(0) = 0$, namely, that f_0 and g_0 are continuous functions. Therefore, condition $|\Omega| \geq \frac{M^-}{\eta c^p}$, guarantees that $|\xi_1| \leq \sigma$, in fact by (g_2) one has

$$|\xi_1|^p \leq \frac{p \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G}{\eta |\Omega|} \leq \frac{M^- \sigma^p}{\eta c^p |\Omega|} \leq \sigma^p.$$

Hence, the function f_0 satisfies in the assumption (f_2) .

From Theorem 2 and Remark 3, we get:

Corollary 1. Let $|\Omega| \geq \frac{M^-}{\eta c^p}$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that

$$(f_3) \sup_{\xi \in \mathbb{R}} \int_0^\xi f(t) dt < +\infty, \quad \limsup_{\xi \rightarrow 0} \frac{\int_0^\xi f(t) dt}{|\xi|^p} \leq 0.$$

Moreover, suppose that there exist $\sigma > 0$ and $\xi_1, \xi_2 \in \mathbb{R}$, with $\xi_1 \xi_2 > 0$, such that

$$(f_4) 0 < \int_0^{\xi_1} f(t) dt = \sup_{|\xi| \leq \sigma} \int_0^\xi f(t) dt < \int_0^{\xi_2} f(t) dt$$

and

$$(f_5) \frac{\eta |\Omega| |\xi_1|^p}{\int_0^{\xi_1} f(t) dt} < \frac{M^- \sigma^p}{c^p \sup_{\xi \in \mathbb{R}} \int_0^\xi f(t) dt}.$$

Under such hypotheses, for every non-constant function $a : \Omega \rightarrow [0, +\infty[$ in $L^1(\Omega)$ satisfying

$$(f_6) \max \left\{ \frac{1}{\sup_{\mathbb{R}} F}, \frac{\eta |\Omega| |\xi_1|^p}{p \int_0^{\xi_1} f(t) dt} \right\} \leq \|a\|_{L^1(\Omega)} < \frac{M^- \sigma^p}{p c^p \sup_{\xi \in \mathbb{R}} \int_0^\xi f(t) dt},$$

there exists $\widehat{\lambda} > 1$ such that the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2} \Delta u) - \left[M \left(\int_{\Omega} |\nabla u|^p dx \right) \right]^{p-1} \Delta_p u + \eta |u|^{p-2} u \\ \quad = \widehat{\lambda} a(x) f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three non-zero solutions which are non-negative or non-positive according to whether $\xi_1 > 0$ or $\xi_1 < 0$.

Another consequence of Theorem 2 is as follows:

Proposition 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and z_1, z_2, z_3, z_4 four positive constants, with $z_1 < z_2 < z_3 < z_4$ such that $f(\xi) \geq 0$ for all $\xi \in]-\infty, -z_4] \cup [-z_3, 0] \cup [z_1, z_2]$, while $f(\xi) \leq 0$ for all $\xi \in]-z_4, -z_3] \cup [0, z_1] \cup [z_2, +\infty[$, and

$$0 < \int_0^{z_2} f(t) dt < \int_0^{-z_4} f(t) dt.$$

Moreover, let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and let $b : \Omega \rightarrow [0, +\infty[$ be a function in $L^1(\Omega)$ such that

$$\int_0^{z_2} g(t) dt > 0, \quad \sup_{\xi \in \mathbb{R}} \int_0^{\xi} g(t) dt < +\infty, \quad \limsup_{\xi \rightarrow 0} \frac{\int_0^{\xi} g(t) dt}{|\xi|^p} < \frac{M^-}{pc^p \|b\|_{L^1(\Omega)}}$$

and

$$\max \left\{ \frac{1}{\sup_{\mathbb{R}} G}, \frac{\eta |\Omega| |z_2|^p}{p \int_0^{z_2} g(t) dt} \right\} \leq \|b\|_{L^1(\Omega)} < \frac{M^- z_3^p}{pc^p \sup_{\xi \in \mathbb{R}} \int_0^{\xi} g(t) dt}.$$

Under such hypotheses, for each non-zero function $a : \Omega \rightarrow [0, +\infty[$ in $L^1(\Omega)$, there exists $\lambda^* > 0$ such that the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2} \Delta u) - \left[M \left(\int_{\Omega} |\nabla u|^p dx \right) \right]^{p-1} \Delta_p u + \eta |u|^{p-2} u \\ \quad = \lambda^* a(x) f(u) + b(x) g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three non-zero solutions, two of which are global minima of the associated energy functional.

Proof. The same assumption also implies that $\int_0^{\xi} f(t) dt \leq 0$ for all $\xi \in]-z_3, z_1]$, and so (f_1) holds. Consequently, if we take $\xi_1 = z_2$, the assumptions of Theorem 2 are satisfied and the conclusion follows. \square

We now exhibit an example in which the hypotheses of Proposition 1 are satisfied.

Example 1. Suppose that $M : [0, +\infty[\rightarrow \mathbb{R}$ is a continuous function such that $M^- > 162$ and suppose $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 9\}$. Define $f, g : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$f(t) = -t^5 - 7t^4 + 7t^3 + 43t^2 - 42t,$$

$$g(t) = \begin{cases} 0 & \text{if } t < 0, \\ t^2 & \text{if } 0 \leq t \leq 2, \\ -2t + 8 & \text{if } 2 < t \leq 4, \\ 0 & \text{if } t > 4. \end{cases}$$

Also, put $a(x, y) = b(x, y) = x^2 + y^2$ for all $(x, y) \in \Omega$.

Then, by choosing $z_1 = 1$, $z_2 = 2$, $z_3 = 3$, $z_4 = 7$, $p = 3$, and since $c = \sqrt[3]{\frac{4}{\pi}}$, the conclusion of Proposition 1 holds for the following problem

$$\begin{cases} \Delta(|\Delta u| \Delta u) - \left[M \left(\int_{\Omega} |\nabla u|^3 dx dy \right) \right]^2 \Delta_3 u + \eta |u| u \\ \quad = \lambda^* a(x, y) f(u) + b(x, y) g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

REFERENCES

- [1] G. Afrouzi and S. Shokooh, "Existence of infinitely many solutions for quasilinear problems with a $p(x)$ -biharmonic operator," *Electron. J. Diff. Eqs.*, vol. 2015, no. 317, pp. 1–14, 2015.
- [2] C. Alves, F. Corrêa, and T. Ma, "Positive solutions for a quasilinear elliptic equation of Kirchhoff type," *Comput. Math. Appl.*, vol. 49, pp. 85–93, 2005, doi: [10.1016/j.camwa.2005.01.008](https://doi.org/10.1016/j.camwa.2005.01.008).
- [3] A. Bensedik and M. Boucekif, "On an elliptic equation of Kirchhoff-type with a potential asymptotically linear at infinity," *Math. Comput. Modelling*, vol. 49, pp. 1089–1096, 2009, doi: [10.1016/j.mcm.2008.07.032](https://doi.org/10.1016/j.mcm.2008.07.032).
- [4] H. Berger, "A new approach to the analysis of large deflections of plates," *J. Appl. Mech.*, vol. 22, pp. 465–472, 1995.
- [5] F. Cammaroto and L. Vilasi, "Multiplicity results for a Neumann boundary value problem involving the $p(x)$ -Laplacian," *Taiwanese J. Math.*, vol. 16, no. 2, pp. 621–634, 2012, doi: [10.11650/twjmath/1500406606](https://doi.org/10.11650/twjmath/1500406606).
- [6] G. Cordaro, "On a minimax problem of Ricceri," *J. Inequal. Appl.*, vol. 6, pp. 261–285, 2001, doi: [10.1155/S1025583401000169](https://doi.org/10.1155/S1025583401000169).
- [7] F. Corrêa and G. Figueiredo, "On an elliptic equation of p -Kirchhoff type via variational methods," *Bull. Aust. Math. Soc.*, vol. 74, no. 02, pp. 263–277, 2006, doi: [10.1017/S000497270003570X](https://doi.org/10.1017/S000497270003570X).
- [8] S. Heidarkhani, S. Khademloo, and A. Solimaninia, "Multiple solutions for a perturbed fourth-order Kirchhoff type elliptic problem," *Portugal. Math.*, vol. 1, no. 1, pp. 39–61, 2014, doi: [10.4171/PM/1940](https://doi.org/10.4171/PM/1940).
- [9] G. Kirchhoff, *Mechanik*. Leipzig, Germany: Teubner, 1876.
- [10] C. Li and C. Tang, "Three solutions for a Navier boundary value problem involving the p -biharmonic," *Nonlinear Anal.*, vol. 72, pp. 1339–1347, 2010, doi: [10.1016/j.na.2009.08.011](https://doi.org/10.1016/j.na.2009.08.011).
- [11] T. Ma, "Existence results for a model of nonlinear beam on elastic bearings," *Appl. Math. Lett.*, vol. 13, pp. 11–15, 2000, doi: [10.1016/S0893-9659\(00\)00026-4](https://doi.org/10.1016/S0893-9659(00)00026-4).
- [12] T. Ma, "Remarks on an elliptic equation of Kirchhoff type," *Nonlinear Anal.*, vol. 63, pp. 1967–1977, 2005, doi: [10.1016/j.na.2005.03.021](https://doi.org/10.1016/j.na.2005.03.021).
- [13] T. Ma, "Positive solutions for a nonlocal fourth order equation of Kirchhoff type," *Discrete Contin. Dyn. Syst.*, pp. 694–703, 2007, doi: [10.1016/S0893-9659\(00\)00026-4](https://doi.org/10.1016/S0893-9659(00)00026-4).
- [14] B. Ricceri, "A further three critical points theorem," *Nonlinear Anal.*, vol. 71, pp. 4151–4157, 2009, doi: [10.1016/j.na.2009.02.074](https://doi.org/10.1016/j.na.2009.02.074).
- [15] B. Ricceri, "A class of nonlinear eigenvalue problems with four solutions," *J. Nonlinear Convex Anal.*, vol. 11, no. 3, pp. 503–511, 2010.
- [16] B. Ricceri, "On an elliptic Kirchhoff-type problem depending on two parameters," *J. Global Optimization*, vol. 46, p. 543–549, 2010, doi: [10.1007/s10898-009-9438-7](https://doi.org/10.1007/s10898-009-9438-7).
- [17] B. Ricceri, "Another four critical points theorem," *J. Nonlinear Convex Anal.*, vol. 2, pp. 163–171, 2011.
- [18] S. Shokooh, "Multiplicity results for a non-homogeneous Neumann problem via a variational principle of Ricceri," *Minimax Theory Appl.*, vol. 2, no. 2, pp. 249–263, 2017.

- [19] J. Sun and C. Tang, "Existence and multiplicity of solutions for Kirchhoff type equations," *Non-linear Anal.*, vol. 74, no. 4, pp. 1212–1222, 2011, doi: [10.1016/j.na.2010.09.061](https://doi.org/10.1016/j.na.2010.09.061).
- [20] S. Woinowsky-Krieger, "The effect of an axial force on the vibration of hinged bars," *J. Appl. Mech.*, vol. 17, pp. 35–36, 1950.
- [21] M. Yang and Z. Han, "Existence and multiplicity results for Kirchhoff type problems with four superlinear potential," *Applicable Anal.*, vol. 91, no. 11, pp. 2045–2055, 2012, doi: [10.1080/00036811.2011.587808](https://doi.org/10.1080/00036811.2011.587808).
- [22] H. Yin and Y. Liu, "Existence of three solutions for a Navier boundary value problem involving the $p(x)$ -biharmonic," *Bull. Korean. Math. Soc.*, vol. 50, no. 6, pp. 1817–1826, 2013, doi: [10.4134/BKMS.2013.50.6.1817](https://doi.org/10.4134/BKMS.2013.50.6.1817).
- [23] H. Yin and M. Xu, "Existence of three solutions for a Navier boundary value problem involving the $p(x)$ -biharmonic operator," *Ann. Polon. Math.*, vol. 109, pp. 47–54, 2013, doi: [10.4064/ap109-1-4](https://doi.org/10.4064/ap109-1-4).
- [24] Z. Zhang and K. Perera, "Existence of three solutions for a Navier boundary value problem involving the $p(x)$ -biharmonic operator," *J. Math. Anal. Appl.*, vol. 317, no. 2, pp. 456–463, 2006, doi: [10.1016/j.jmaa.2005.06.102](https://doi.org/10.1016/j.jmaa.2005.06.102).

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