Miskolc Mathematical Notes
Vol. 20 (2019), No. 2, pp. 1157–1166

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2019.2334

# BOUNDING THE CONVEX COMBINATION OF ARITHMETIC AND INTEGRAL MEANS IN TERMS OF ONE-PARAMETER HARMONIC AND GEOMETRIC MEANS

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*Received 22 May, 2017*

*Abstract.* In the article, we find the best possible parameters $\lambda_1$, $\mu_1$, $\lambda_2$ and $\mu_2$ on the interval $[0,1/2]$ such that the double inequalities

$$H(a,b;\lambda_1) < \alpha A(a,b) + (1-\alpha)T(a,b) < H(a,b;\mu_1),$$

$$G(a,b;\lambda_2) < \alpha A(a,b) + (1-\alpha)T(a,b) < G(a,b;\mu_2)$$

hold for all $\alpha \in [0,1]$ and $a,b > 0$ with $a \neq b$, where $A(a,b) = (a+b)/2$, $T(a,b) = 2\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta/\pi$, $H(a,b;\lambda) = 2[\lambda a + (1-\lambda)b][\lambda b + (1-\lambda)a]/(a+b)$, $G(a,b;\mu) = \sqrt{[\mu a + (1-\mu)b][\mu b + (1-\mu)a]}$ are the arithmetic, integral, one-parameter harmonic and one-parameter geometric means of $a$ and $b$, respectively.

2010 *Mathematics Subject Classification:* 26E60; 33C10

*Keywords:* integral mean, modified Bessel function, arithmetic mean, harmonic mean, geometric mean

## 1. Introduction

For $\lambda, \mu \in [0,1]$, the arithmetic mean $A(a,b)$, harmonic mean $H(a,b)$, geometric mean $G(a,b)$, integral mean $T(a,b)$ [25], one-parameter harmonic mean $H(a,b;\lambda)$ and one-parameter geometric mean $G(a,b;\mu)$ of two distinct positive real numbers $a$ and $b$ are given by

$$A(a,b) = \frac{a+b}{2}, \quad H(a,b) = \frac{2ab}{a+b}, \tag{1.1}$$

$$G(a,b) = \sqrt{ab}, \quad T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta, \tag{1.2}$$

$$H(a,b;\lambda) = H[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)b], \tag{1.3}$$

$$G(a,b;\mu) = G[\mu a + (1-\mu)b, \mu b + (1-\mu)b], \tag{1.4}$$

The first author was supported in part by the Natural Science Fund, Grant No. 2018YZ07.

respectively. The integral mean $T(a,b)$ has been the subject of intensive research in recent years due to it has been widely applied in pure and applied mathematics, physics and other natural sciences [2,4–12,14–16,19,21,22,24,26–30,36–38,40].

The identity

$$T(a,b)=\sqrt{ab}\,I_0\left(\log\sqrt{b/a}\right) \tag{1.5}$$

and inequalities

$$L(a,b)<T(a,b)<\frac{A(a,b)+G(a,b)}{2}<\frac{2A(a,b)+G(a,b)}{3}<I(a,b) \tag{1.6}$$

for all $a,b>0$ with $a\neq b$ were established by Qi, Shi, Liu and Yang [18], where

$$I_\nu(t)=\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+\nu+1)}\left(\frac{t}{2}\right)^{2n+\nu} \tag{1.7}$$

is the modified Bessel function of the first kind [1], $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is the classical gamma function [13,39], and $L(a,b)=(b-a)/(\log b-\log a)$ and $I(a,b)=(b^b/a^a)^{1/(b-a)}/e$ are respectively the logarithmic and inentric means of $a$ and $b$.

Yang and Chu [31,32], and Yang, Chu and Song [33] proved that the inequalities

$$\lambda_1\sqrt{L(a,b)A(a,b)}<T(a,b)<\mu_1\sqrt{L(a,b)A(a,b)},$$
$$L^{\lambda_2}(a,b)A^{1-\lambda_2}(a,b)<T(a,b)<\mu_2L(a,b)+(1-\mu_2)A(a,b),$$
$$T(a,b)>L_p(a,b)$$
$$\lambda_3\sqrt{L(a,b)I(a,b)}<T(a,b)<\mu_3\sqrt{L(a,b)I(a,b)},$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1\leq\sqrt{2/\pi}$, $\mu_1\geq 1$, $\lambda_2\geq 3/4$, $\mu_2\leq 3/4$, $p\leq 3/2$, $\lambda_3\leq\sqrt{e/\pi}$ and $\mu_3\geq 1$, where $L_p(a,b)=[(b^p-a^p)/(p(\log b-\log a))]^{1/p}$ is the $p$-order generalized logarithmic mean of $a$ and $b$.

In [20], the authors proved that $p_1=0$, $q_1=1/4$, $p_2=0$ and $q_2=1/2-\sqrt{2}/4$ are the best possible parameters on the interval $[0,1/2]$ such that the double inequalities

$$H(a,b;p_1)<T(a,b)<H(a,b;q_1), \tag{1.8}$$

$$G(a,b;p_2)<T(a,b)<G(a,b;q_2) \tag{1.9}$$

hold for all $a,b>0$ with $a\neq b$.

Let $\alpha\in[0,1]$, $x\in[0,1/2]$, $a,b>0$ with $a\neq b$, $f(x)=H(a,b;x)$, $g(x)=G(a,b;x)$. Then we clearly see that both the functions $f(x)$ and $g(x)$ are strictly increasing on $[0,1/2]$. Inequality (1.6) and the well known inequalities

$$H(a,b)<G(a,b)<L(a,b)<I(a,b)<A(a,b)$$

lead to the conclusion that

$$f(0)=H(a,b)<\alpha A(a,b)+(1-\alpha)T(a,b)<A(a,b)=f(1/2), \tag{1.10}$$

$$g(0)=G(a,b)<\alpha A(a,b)+(1-\alpha)T(a,b)<A(a,b)=g(1/2). \tag{1.11}$$

From inequalities (1.10) and (1.11) together with the monotonicity of the functions $f(x)$ and $g(x)$ on the interval $[0,1/2]$, it is necessary to discover the best possible parameters $\lambda_1$, $\mu_1$, $\lambda_2$ and $\mu_2$ on the interval $[0,1/2]$ such that the double inequalities

$$H(a,b;\lambda_1) < \alpha A(a,b) + (1-\alpha)T(a,b) < H(a,b;\mu_1),$$

$$G(a,b;\lambda_2) < \alpha A(a,b) + (1-\alpha)T(a,b) < G(a,b;\mu_2)$$

hold for all $\alpha \in [0,1]$ and $a,b > 0$ with $a \neq b$.

## 2. Lemmas

**Lemma 1** (Theorem 2.18 in [3]). *The identity*

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$$

*holds for all $n \in \mathbb{N}$.*

**Lemma 2** ([17]). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real sequences with $b_n > 0$ and* $\lim_{n\to\infty} a_n/b_n = s$. *Then the power series $\sum_{n=0}^{\infty} a_n t^n$ is convergent for all $t \in \mathbb{R}$ and*

$$\lim_{t\to\infty} \frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s$$

*if the power series $\sum_{n=0}^{\infty} b_n t^n$ is convergent for all $t \in \mathbb{R}$.*

**Lemma 3** (Lemma 2.2 in [35]). *The double inequality*

$$\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}}$$

*holds for all $x > 0$ and $a \in (0,1)$.*

**Lemma 4** ([34]). *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r,r)$ $(r > 0)$ with $b_k > 0$ for all $k$. If the non-constant sequence $\{a_k/b_k\}_{k=0}^{\infty}$ is increasing (decreasing) for all $k$, then the function $t \mapsto A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$.*

**Lemma 5** ((3.5) in [23]). *The identity*

$$I_\lambda(t)I_\mu(t) = \sum_{n=0}^{\infty} \frac{\Gamma(2n+\lambda+\mu+1)}{n!\Gamma(n+\lambda+\mu+1)\Gamma(n+\lambda+1)\Gamma(n+\mu+1)}\left(\frac{t}{2}\right)^{2n+\lambda+\mu}$$

*holds for all $\lambda,\mu > -1$ and $t \in \mathbb{R}$.*

**Lemma 6.** *The identity*

$$\cosh(t)I_0(t) = \sum_{n=0}^{\infty} \frac{(4n)!}{2^{2n}[(2n)!]^3} t^{2n}$$

*holds for all $t \in \mathbb{R}$, where $\cosh(t) = (e^t + e^{-t})/2$ is the hyperbolic cosine functions.*

*Proof.* It follows from (1.7) and Lemmas 1 and 5 that

$$I_{-1/2}(t)=\sqrt{\frac{2}{\pi t}}\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}=\sqrt{\frac{2}{\pi t}}\cosh(t),$$

$$\begin{aligned}\cosh(t)I_0(t)&=\sqrt{\frac{\pi t}{2}}I_{-1/2}(t)I_0(t)\\&=\sqrt{\frac{\pi t}{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(2n+\frac{1}{2}\right)}{\left[n!\Gamma\left(n+\frac{1}{2}\right)\right]^2}\left(\frac{t}{2}\right)^{2n-1/2}=\sum_{n=0}^{\infty}\frac{(4n)!}{2^{2n}[(2n)!]^3}t^{2n}.\end{aligned}$$

□

**Lemma 7.** *The function*

$$f(t)=\frac{\cosh^2(t)-\cosh(t)I_0(t)}{\sinh^2(t)} \tag{2.1}$$

*is strictly increasing from* $(0,\infty)$ *onto* $(1/4,1)$, *where* $\sinh(t)=(e^t-e^{-t})/2$ *is the hyperbolic sine function.*

*Proof.* Let $n\in\mathbb{N}$, and $\{a_n\}$ and $\{b_n\}$ be defined by

$$a_n=\frac{(4n+4)!}{2^{2n+2}[(2n+2)!]^3},\quad b_n=\frac{2^{2n+2}}{(2n+2)!}, \tag{2.2}$$

respectively.

Then simple computations lead to

$$\frac{a_0}{b_0}=\frac{3}{8}, \tag{2.3}$$

$$\frac{a_n}{b_n}=\frac{(4n+4)!}{2^{4n+4}[(2n+2)!]^2}, \tag{2.4}$$

$$\frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n}=-\frac{(n+2)(2n+3)(8n+13)(4n+4)!}{2^{4n+5}[(2n+4)!]^2}<0 \tag{2.5}$$

for all $n\in\mathbb{N}$.

It follows from Lemmas 1, 2, 3 and 6 together with (2.1)-(2.4) that

$$\begin{aligned}f(t)&=1-\frac{2[\cosh(t)I_0(t)-1]}{\cosh(2t)-1}\\&=1-\frac{2\sum_{n=1}^{\infty}\frac{(4n)!}{2^{2n}[(2n)!]^3}t^{2n}}{\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n}}=1-\frac{2\sum_{n=0}^{\infty}a_nt^{2n}}{\sum_{n=0}^{\infty}b_nt^{2n}},\end{aligned} \tag{2.6}$$

$$f(0^+)=1-\frac{2a_0}{b_0}=\frac{1}{4}, \tag{2.7}$$

$$\frac{1}{\sqrt{\pi(2n+5/2)}} < \frac{a_n}{b_n} = \frac{\Gamma(2n+5/2)}{\sqrt{\pi}\,\Gamma(2n+3)} < \frac{1}{\sqrt{\pi(2n+2)}},$$
$$f(\infty) = 1 - 2\lim_{n\to\infty}\frac{a_n}{b_n} = 1. \tag{2.8}$$

From Lemma 4, (2.5) and (2.6) we clearly see that the function $f(t)$ is strictly increasing on $(0,\infty)$. Therefore, Lemma 7 follows from (2.7) and (2.8) together with the monotonicity of $f(t)$ on the interval $(0,\infty)$. □

**Lemma 8.** *The function*

$$g(t) = \frac{\cosh^2(t) - I_0^2(t)}{\sinh^2(t)} \tag{2.9}$$

*is strictly increasing from* $(0,\infty)$ *onto* $(1/2,1)$.

*Proof.* Let $n \in \mathbb{N}$, and $\{c_n\}$ and $\{d_n\}$ be defined by

$$c_n = \frac{(2n+2)!}{2^{2n+2}[(n+1)!]^4}, \quad d_n = \frac{2^{2n+2}}{(2n+2)!}, \tag{2.10}$$

respectively. Then simple computations lead to

$$\frac{c_0}{d_0} = \frac{1}{4}, \tag{2.11}$$

$$\frac{c_n}{d_n} = \frac{[(2n+2)!]^2}{2^{4n+4}[(n+1)!]^4}, \tag{2.12}$$

$$\frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = -\frac{(4n+7)(n+2)^2[(2n+2)!]^2}{2^{4n+6}[(n+1)!]^4} < 0 \tag{2.13}$$

for all $n \in \mathbb{N}$.

From Lemmas 1, 2, 3 and 5 together with (2.9)-(2.12) one has

$$I_0^2(t) = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^4}t^{2n},$$

$$g(t) = 1 - \frac{2[I_0^2(t)-1]}{\cosh(2t)-1}$$
$$= 1 - \frac{2\sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}(n!)^4}t^{2n}}{\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n}} = 1 - \frac{2\sum_{n=0}^{\infty}c_n t^{2n}}{\sum_{n=0}^{\infty}d_n t^{2n}}, \tag{2.14}$$

$$g(0^+) = 1 - \frac{2c_0}{d_0} = \frac{1}{2}, \tag{2.15}$$

$$\frac{1}{\pi(n+3/2)} < \frac{c_n}{d_n} = \frac{\Gamma^2(n+3/2)}{\pi\Gamma^2(n+2)} < \frac{1}{\pi(n+1)},$$

$$g(\infty) = 1 - 2\lim_{n\to\infty}\frac{c_n}{d_n} = 1. \tag{2.16}$$

Therefore, Lemma 8 follows easily from Lemma 4 and (2.13)-(2.16). □

## 3. Main results

**Theorem 1.** *Let $\lambda_1, \mu_1, \lambda_2, \mu_2 \in [0, 1/2]$. Then the double inequalities*

$$H(a,b;\lambda_1) < \alpha A(a,b) + (1-\alpha)T(a,b) < H(a,b;\mu_1), \tag{3.1}$$

$$G(a,b;\lambda_2) < \alpha A(a,b) + (1-\alpha)T(a,b) < G(a,b;\mu_2) \tag{3.2}$$

*hold for all $\alpha \in [0,1]$ and $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 1/2 - \sqrt{1-\alpha}/2$, $\mu_1 \geq 1/2 - \sqrt{1-\alpha}/4$, $\lambda_2 \leq 1/2 - \sqrt{1-\alpha^2}/2$ and $\mu_2 \geq 1/2 - \sqrt{2(1-\alpha)}/4$.*

*Proof.* Let $p, q \in [0, 1/2]$. Without loss of generality, we assume that $b > a > 0$ and $t = \log\sqrt{b/a} > 0$ due to $A(a,b)$, $T(a,b)$, $H(a,b;p)$ and $G(a,b;q)$ are symmetric and homogeneous of degree one with respect to $a$ and $b$. From (1.1)-(1.5) one has

$$A(a,b) = \sqrt{ab}\cosh(t),$$
$$T(a,b) = \sqrt{ab}I_0(t),$$
$$H(a,b;p) = \sqrt{ab}\cosh(t)\left[1-(1-2p)^2\frac{\sinh^2(t)}{\cosh^2(t)}\right],$$
$$G(a,b;q) = \sqrt{ab}\cosh(t)\sqrt{1-(1-2q)^2\frac{\sinh^2(t)}{\cosh^2(t)}},$$

$$H(a,b;p) - [\alpha A(a,b) + (1-\alpha)T(a,b)] = \frac{\sqrt{ab}\sinh^2(t)}{\cosh(t)}\left[(1-\alpha)f(t) - (1-2p)^2\right], \tag{3.3}$$

$$\begin{aligned} &G(a,b;q) - [\alpha A(a,b) + (1-\alpha)T(a,b)] \\ &= \frac{\sqrt{ab}\left[(1-\alpha)^2(\cosh^2(t) - I_0^2(t)) + 2\alpha(1-\alpha)(\cosh^2(t) - \cosh(t)I_0(t)) - (1-2q)^2\right]}{\cosh(t)\sqrt{1-(1-2q)^2\frac{\sinh^2(t)}{\cosh^2(t)}} + \alpha\cosh(t) + (1-\alpha)I_0(t)} \\ &= \frac{\sqrt{ab}\sinh^2(t)\left[2\alpha(1-\alpha)f(t) + (1-\alpha)^2 g(t) - (1-2q)^2\right]}{\cosh(t)\sqrt{1-(1-2q)^2\frac{\sinh^2(t)}{\cosh^2(t)}} + \alpha\cosh(t) + (1-\alpha)I_0(t)}, \end{aligned} \tag{3.4}$$

where $f(t)$ and $g(t)$ are defined by Lemmas 7 and 8, respectively.

Therefore, Theorem 1 follows easily from (3.3) and (3.4) together with Lemmas 7 and 8. □

*Remark* 1. Let $\alpha = 0$. Then inequalities (3.1) and (3.2) reduce to inequalities (1.8) and (1.9), respectively.

**Corollary 1.** *Let* $\lambda_1 = 1/2 - \sqrt{1-\alpha}/2$ *and* $\mu_1 = 1/2 - \sqrt{1-\alpha}/4$. *Then inequality* (3.1) *leads to*

$$\frac{1}{\cosh(t)} < I_0(t) < \frac{3\cosh(t)}{4} + \frac{1}{4\cosh(t)}$$

*for all* $t > 0$.

**Corollary 2.** *Let* $\lambda_2 = 1/2 - \sqrt{1-\alpha^2}/2$ *and* $\mu_2 = 1/2 - \sqrt{2(1-\alpha)}/4$. *Then inequality* (3.2)) *leads to*

$$\frac{\sqrt{1-\alpha^2+\alpha^2\cosh^2(t)}-\alpha\cosh(t)}{1-\alpha} < I_0(t) < \frac{\sqrt{\frac{1-\alpha}{2}+\frac{1+\alpha}{2}\cosh^2(t)}-\alpha\cosh(t)}{1-\alpha}$$

*for all* $\alpha \in (0,1)$ *and* $t > 0$. *In particular, if* $\alpha = 1/2$, *then one has*

$$\sqrt{3+\cosh^2(t)}-\cosh(t) < I_0(t) < \sqrt{1+3\cosh^2(t)}-\cosh(t)$$

*for all* $t > 0$.

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