Local approximation behavior of modified SMK operators

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Abstract. In this paper, for a general modification of the classical Szász–Mirakjan–Kantorovich operators, we obtain many local approximation results including the classical cases. In particular, we obtain a Korovkin theorem, a Voronovskaya theorem, and some local estimates for these operators.

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1. INTRODUCTION

As usual, let \( C[0, \infty) \) denote the space of all continuous functions on \([0, \infty)\). The classical Szász–Mirakjan–Kantorovich (SMK) operators \([4]\) are given by the relation

\[
K_n(f; x) := ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{I_{n,k}} f(t) dt, \quad n \in \mathbb{N}; x \in [a, 1];
\]  

where \( I_{n,k} = \left[ \frac{k}{n}, \frac{k+1}{n} \right] \) and \( f \) belongs to an appropriate subspace of \( C[0, \infty) \) for which the above series is convergent. Among of these subspaces, we can take the space \( C_B(0, \infty) \) of all bounded and continuous functions on \([0, \infty)\), or, the weighted space \( C_\gamma[0, +\infty), \gamma > 0, \) defined by the equality

\[
C_\gamma[0, +\infty) := \{ f \in C[0, +\infty) : |f(t)| \leq M(1 + t)^\gamma \text{ for some } M > 0 \}. 
\]

Assume now that \((u_n)\) is a sequence of functions on \([0, \infty)\) such that, for a fixed \( a \geq 0,\)

\[
0 \leq u_n(x) \leq x \quad \text{for every } x \in [a, \infty) \text{ and } n \in \mathbb{N}. 
\]  

Then, we consider the following modification of SMK operators:

\[
L_n(f; x) := \sum_{k=0}^{\infty} p_{k,n}(x) \int_{I_{n,k}} f(t) dt, \quad n \in \mathbb{N}, \ x \in [a, \infty),
\]  

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where
\[ p_{k,n}(x) := n e^{-nu(x)} \frac{(nu(x))^k}{k!}, \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \quad (1.4) \]

Throughout the paper we use the following test functions
\[ e_i(x) = x^i, \quad i = 0, 1, 2, 3, 4, \]
and the moment function
\[ \psi_x(y) = y - x. \]
So, using the fundamental properties of the classical SMK operators, one can get the following lemmas.

**Lemma 1.** For the operators \( L_n \), we have

(i) \( L_n(e_0; x) = 1 \),

(ii) \( L_n(e_1; x) = u_n(x) + \frac{1}{2n} \),

(iii) \( L_n(e_2; x) = u_n^2(x) + \frac{2u_n(x)}{n} + \frac{1}{3n^2} \),

(iv) \( L_n(e_3; x) = u_n^3(x) + \frac{9u_n^2(x)}{2n} + \frac{7u_n(x)}{2n^2} + \frac{1}{4n^3} \),

(v) \( L_n(e_4; x) = u_n^4(x) + \frac{8u_n^3(x)}{n} + \frac{15u_n^2(x)}{n^2} + \frac{6u_n(x)}{n^3} + \frac{1}{5n^4} \).

**Lemma 2.** For the operators \( L_n \), we have

(i) \( L_n(\psi_1; x) = u_n(x) - x + \frac{1}{2n} \),

(ii) \( L_n(\psi_2; x) = (u_n(x) - x)^2 + \frac{2u_n(x) - x}{n} + \frac{1}{3n^2} \),

(iii) \( L_n(\psi_3; x) = (u_n(x) - x)^3 + \frac{3(3u_n(x) - x)(u_n(x) - x)}{2n} + \frac{7u_n(x) - 2x}{2n^2} + \frac{1}{4n^3} \),

(iv) \( L_n(\psi_4; x) = (u_n(x) - x)^4 + \frac{2(4u_n(x) - x)(x - u_n(x))^2}{n} + \frac{15u_n^2(x) - 14ux_u_n + 2x^2}{n^2} + \frac{6u_n(x) - x^3}{n^3} + \frac{1}{5n^4} \).

Then, we see from Lemma 1 that, with some suitable choices of \( u_n \), our operators \( L_n \) may preserve the linear functions or the test function \( e_2 \). For example, taking \( a = \frac{1}{2} \), if we choose \( u_n(x) = x - \frac{1}{2n} \) for \( x \in \left[ \frac{1}{2}, \infty \right) \) and \( n \in \mathbb{N} \), then the corresponding operators \( L_n \) preserve the linear functions, i.e., they preserve the test functions \( e_0 \) and \( e_1 \) (see [8]). In this case, we know from [8] that the operators \( L_n \) have a better error estimation than the classical SMK operators on \( \left[ \frac{1}{2}, \infty \right) \). Also, taking \( a = \frac{1}{2} \) and

\[ u_n(x) := \frac{\sqrt{3n^2x^2 + 2} - \sqrt{3}}{n\sqrt{3}} \quad \text{for} \quad x \in \left[ \frac{1}{\sqrt{3}}, \infty \right) \quad \text{and} \quad n \in \mathbb{N}, \]
we see that the corresponding operators $L_n$ preserve the test functions $e_0$ and $e_2$. Finally, for $a = 0$ and
\[ u_n(x) := \frac{-1 + \sqrt{4n^2x^2 + 1}}{2n}, \quad x \geq 0 \text{ and } n \in \mathbb{N}, \]
the corresponding operators $L_n$ becomes the Kantorovich variant of the modified Szász–Mirakjan operators (see [6, 11]).

The first study regarding the preservation of $e_0$ and $e_2$ for the linear positive operators in order to get better error estimation, was first presented by King. In [9], King introduced a modification of the classical Bernstein polynomials and had a better error estimation than the classical ones on the interval $[0, \frac{1}{4}]$. Later, similar problems were accomplished for Szász–Mirakjan operators [6], Szász–Mirakjan–Beta operators [7], Meyer–König and Zeller operators [11], Bernstein–Chlodovsky operators [1], $q$-Bernstein operators [10], Baskakov operators and Stancu operators [12], and some other kinds of summation-type positive linear operators [2].

However, in the present paper, for a general sequence $(u_n)$ satisfying (1.2), we study the local approximation behavior of the operators $L_n$ defined by (1.3) and (1.4).

First of all, we get the following Korovkin-type approximation theorem for these operators.

**Theorem 1.** Let $(u_n)$ be a sequence of functions on $[0, \infty)$ satisfying (1.2) for a fixed $a \geq 0$. If
\[ \lim_{n \to \infty} u_n(x) = x \quad (1.5) \]
uniformly with respect to $x \in [a, b]$ with $b > a$, then, for all $f \in C_\gamma[0, +\infty)$ with $\gamma \geq 2$, we have
\[ \lim_{n \to \infty} L_n(f \cdot x) = f(x) \]
uniformly with respect to $x \in [a, b]$.

**Proof.** For a fixed $b > 0$, consider the lattice homomorphism $H_b: C[0, +\infty) \to C[a, b]$ defined by $H_b(f) := f|_{[a,b]}$ for every $f \in C[0, +\infty)$. In this case, from (1.5), we see that, for each $i = 0, 1, 2$,
\[ \lim_{n \to \infty} H_b(T_n(e_i)) = H_b(e_i) \]
uniformly on $[a, b]$. Hence, using the universal Korovkin-type property with respect to monotone operators (see Theorem 4.1.4(vi) of [3, p. 199]), we obtain that, for all $f \in C_\gamma[0, +\infty)$, $\gamma \geq 2$,
\[ \lim_{n \to \infty} L_n(f \cdot x) = f(x) \]
uniformly with respect to $x \in [a, b]$. $\square$

Finally, using Proposition 4.2.5(2) of [3], we can state the following approximation result in the space $L_p$:
Corollary 1. Let $1 \leq p < \infty$. Then for all $f \in L_p$, we have
$$\lim_{n \to \infty} L_n(f; x) = f(x)$$
uniformly with respect to $x \in [a, \infty)$ with $a \geq 0$.

2. A Voronovskaya-type theorem

In order to get a Voronovskaya-type theorem for the operators $L_n$ given by (1.3) and (1.4), we need the following lemma.

Lemma 3. Let $(u_n)$ be a sequence of functions on $[0, \infty)$ satisfying (1.2) for a fixed $a \geq 0$. If
$$\lim_{n \to \infty} \sqrt{n}(x - u_n(x)) = 0$$
(2.1)
uniformly with respect to $x \in [a, b]$, $b > a$, then we have
$$\lim_{n \to \infty} n^2 L_n(\psi_x^4; x) = 3x^2$$
(2.2)
uniformly with respect to $x \in [a, b]$.

Proof. Let $x \in [a, b]$, $b > a$, be fixed. Then, by (1.2), since
$$0 \leq x - u_n(x) \leq \sqrt{n}(x - u_n(x))$$
for every $n \in \mathbb{N}$, it follows from (2.1) that
$$\lim_{n \to \infty} u_n(x) = x$$
(2.3)
uniformly with respect to $x \in [a, b]$. Also, since
$$0 \leq \frac{u_n(x)}{n} = \frac{u_n(x) - x}{n} + \frac{x}{n} \leq x - u_n(x) + \frac{x}{n},$$
we obtain from (2.1) that
$$\lim_{n \to \infty} \frac{u_n(x)}{n} = 0$$
(2.4)
uniformly with respect to $x \in [a, b]$. Observe now that, by Lemma 2(iv),
$$n^2 L_n(\psi_x^4; x) = \left\{ \sqrt{n}(x - u_n(x)) \right\}^4 + 2 \left\{ \sqrt{n}(x - u_n(x)) \right\}^2 (4u_n(x) - x) + 15u_n^2(x) - 14xu_n(x) + 2x^2 + 6 u_n(x) - \frac{x}{n} + \frac{1}{5n^2}.$$  
Taking limit as $n \to \infty$ in the both sides of the last equality and also using (2.1), (2.3), (2.4), we immediately see that
$$\lim_{n \to \infty} n^2 L_n(\psi_x^4; x) = 3x^2$$
uniformly with respect to $x \in [a, b]$. The proof is complete. \(\square\)

We now get the following result.
Theorem 2. Let \((u_n)\) be a sequence of functions on \([0, \infty)\) satisfying (1.2) and (2.1) for a fixed \(a \geq 0\). Assume further that there exists a function \(\xi\) defined on \([a, \infty)\) such that
\[
\lim_{n \to \infty} n (x - u_n(x)) = \xi(x)
\] (2.5)
uniformly with respect to \(x \in [a, b]\), \(b > a\). Then, for every \(f \in C_\gamma[0, +\infty)\), \(\gamma \geq 4\), such that \(f', f'' \in C_\gamma[0, +\infty)\), we have
\[
\lim_{n \to \infty} n \{L_n(f : x) - f(x)\} = \frac{1}{2} x f''(x) + \left(\frac{1}{2} - \xi(x)\right) f'(x)
\] uniformly with respect to \(x \in [a, b]\).

Proof. Let \(f, f', f'' \in C_\gamma[0, +\infty)\) with \(\gamma \geq 4\). Define
\[
\Psi(y, x) = \begin{cases} 
   \frac{f(y) - f(x) - (y-x)f'(x) - \frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2} & \text{for } y \neq x, \\
   0 & \text{for } y = x.
\end{cases}
\]
Then, it is clear that \(\Psi(x, x) = 0\) and that the function \(\Psi(\cdot, x)\) belongs to \(C_\gamma[0, +\infty)\). Hence, it follows from the Taylor theorem that
\[
f(y) - f(x) = \psi_x(y) f'(x) + \frac{1}{2} \psi_x^2(y) f''(x) + \psi_x^2(y) \Psi(y, x).
\]
Now, by Lemma 2(ii) and (iii), we get
\[
n \{L_n(f : x) - f(x)\} = nf'(x)L_n(\psi_x ; x) + n \frac{f''(x)}{2} L_n(\psi_x^2 ; x)
+ n L_n(\psi_x^2(y) \Psi(y, x); x),
\]
which gives
\[
n \{L_n(f : x) - f(x)\} = f'(x) \left\{n (u_n(x) - x) + \frac{1}{2}\right\}
+ \frac{f''(x)}{2} \left\{(\sqrt{n} (u_n(x) - x))^2 + 2u_n(x) - x\right\}
+ n L_n(\psi_x^2(y) \Psi(y, x); x).
\] (2.6)
If we apply the Cauchy–Schwarz inequality for the last term on the right-hand side of (2.6), then we conclude that
\[
n |L_n(\psi_x^2(y) \Psi(y, x); x)| \leq (n^2 L_n(\psi_x^2(y); x))^{1/2} (L_n(\psi_x^2(y); x))^{1/2}.
\] (2.7)
Let \(\eta(y, x) := \psi_x^2(y, x)\). In this case, observe that \(\eta(x, x) = 0\) and \(\eta(\cdot, x) \in C_\gamma[0, +\infty)\). Then it follows from Theorem 1 that
\[
\lim_{n \to \infty} L_n(\psi_x^2(y, x); x) = \lim_{n \to \infty} L_n(\eta(y, x); x) = \eta(x, x) = 0
\] (2.8)
uniformly with respect to \( x \in [a, b] \), \( b > a \). So, considering (2.5), (2.7) and (2.8), and also using Lemma 3, we immediately see that
\[
\lim_{n \to \infty} nL_n \left( \psi^2(x) \Psi(y, x) ; x \right) = 0
\] (2.9)
uniformly with respect to \( x \in [a, b] \). Taking limit as \( n \to \infty \) in (2.6) and also using (2.1), (2.3), (2.5), (2.9) we have
\[
\lim_{n \to \infty} n \{ L_n(f; x) - f(x) \} = \frac{1}{2} x f''(x) + \left( \frac{1}{2} - \xi(x) \right) f'(x)
\]
uniformly with respect to \( x \in [a, b] \). The proof is complete. \( \square \)

We should note that one can find a sequence \( (u_n) \) satisfying all assumptions (1.2), (2.1) and (2.5) in Theorem 2. For example, if we take \( a = 0 \) and \( u_n(x) = x \), then our operators in (1.3) turn out to be the classical SMK operators \( K_n \) defined by (1.1). In this case, we have \( \xi(x) = 0 \). Hence, we obtain the following result.

**Corollary 2.** For the operators (1.1), if \( f \in C_{\gamma}[0, +\infty), \gamma \geq 4 \), such that \( f', f'' \in C_{\gamma}[0, +\infty) \), then we have
\[
\lim_{n \to \infty} n \{ K_n(f; x) - f(x) \} = \frac{1}{2} x f''(x) + \frac{1}{2} f'(x)
\]
uniformly with respect to \( x \in [0, b] \), \( b > 0 \).

Now, if take \( a = 0 \) and
\[
u_n(x) := u_n^{[1]}(x) = \frac{-1 + \sqrt{4n^2x^2 + 1}}{2n}, \quad x \in [0, \infty), \quad n \in \mathbb{N},
\] (2.10)
then our operators \( L_n \) in (1.3) turn out to be
\[
L_n^{[1]}(f; x) := ne^{-(-1+\sqrt{4n^2x^2+1})/2} \sum_{k=0}^\infty \left( \frac{-1 + \sqrt{4n^2x^2 + 1}}{2k!} \right)^k I_{n,k} f(t)dt.
\] (2.11)
In this case, observe that
\[
\xi(x) = \lim_{n \to \infty} n \left( x - u_n^{[1]}(x) \right) = \begin{cases} 0 & \text{if } x = 0, \\ 1/2 & \text{if } x > 0. \end{cases}
\]
So, the next result immediately follows from Theorem 2.

**Corollary 3.** For the operators (2.11), if \( f \in C_{\gamma}[0, +\infty), \gamma \geq 4 \), such that \( f', f'' \in C_{\gamma}[0, +\infty) \), then we have
\[
\lim_{n \to \infty} n \{ L_n^{[1]}(f; x) - f(x) \} = \begin{cases} f'(0)/2 & \text{if } x = 0, \\ xf''(x)/2 & \text{if } x > 0. \end{cases}
\]
Furthermore, if we choose $a = \frac{1}{2}$ and
\[ u_n(x) := u_n^{[2]}(x) = x - \frac{1}{2n}, \quad x \in \left[ \frac{1}{2}, \infty \right), \quad n \in \mathbb{N}, \]
then the operators in (1.3) reduce to the following operators (see [8]):
\[ L_n^{[2]}(f; x) := ne^{-\frac{3}{2}x} \sum_{k=0}^{\infty} \frac{(2nx - 1)^k}{2^k k!} \int_{I_{n,k}} f(t) dt. \quad (2.12) \]
Then, we observe that
\[ \xi(x) = \lim_{n \to \infty} n \left( x - u_n^{[2]}(x) \right) = \frac{1}{2}. \]
Therefore, we get the next result at once.

**Corollary 4 ([8]).** For the operators (2.12), if $f \in C_{\gamma}[0, +\infty), \gamma \geq 4$, such that $f', f'' \in C_{\gamma}[0, +\infty)$, then we have
\[ \lim_{n \to \infty} n \left\{ L_n^{[2]}(f; x) - f(x) \right\} = \frac{1}{2} xf''(x) \]
uniformly with respect to $x \in [1/2, b], b > 1/2$.

Finally, taking $a = \frac{1}{\sqrt{3}}$ and
\[ u_n(x) := u_n^{[3]}(x) = \frac{\sqrt{3n^2 x^2 + 2} - \sqrt{3}}{n \sqrt{3}}, \quad x \in \left[ \frac{1}{\sqrt{3}}, \infty \right), \quad n \in \mathbb{N}, \quad (2.13) \]
we get the following positive linear operators:
\[ L_n^{[3]}(f; x) := ne^{-\frac{\sqrt{3} - \sqrt{3n^2 x^2 + 2}}{\sqrt{3}}} \sum_{k=0}^{\infty} \frac{\left( \sqrt{3n^2 x^2 + 2} - \sqrt{3} \right)^k}{3^k / k!} \int_{I_{n,k}} f(t) dt. \quad (2.14) \]
In this case, we find that
\[ \xi(x) = \lim_{n \to \infty} n \left( x - u_n^{[3]}(x) \right) = 1. \]
Then, for the corresponding operators, we have the following

**Corollary 5.** For the operators (2.14), if $f \in C_{\gamma}[0, +\infty), \gamma \geq 4$, such that $f', f'' \in C_{\gamma}[0, +\infty)$, then we have
\[ \lim_{n \to \infty} n \left\{ L_n^{[3]}(f; x) - f(x) \right\} = \frac{1}{2} xf''(x) - \frac{1}{2} f'(x) \]
uniformly with respect to $x \in \left[ \frac{1}{\sqrt{3}}, b \right], b > \frac{1}{\sqrt{3}}$. 

3. LOCAL APPROXIMATION RESULTS FOR THE OPERATORS $L_n$

In order to study various local approximation properties of the operators $L_n$ we mainly use the (usual) modulus of continuity, the second modulus of smoothness, and Peetre’s $K$-functional.

By $C^2_B[0,\infty)$ we denote the space of all functions $f \in C_B[0,\infty)$ such that $f', f'' \in C_B[0,\infty)$. Let $\|f\|$ denote the usual supremum norm of a bounded function $f$. Then, the classical Peetre’s $K$-functional and the second modulus of smoothness of a function $f \in C_B[0,\infty)$ are defined respectively by

$$K(f, \delta) := \inf_{g \in C^2_B[0,\infty)} \{\|f - g\| + \delta \|g''\|\}$$

and

$$\omega_2(f, \delta) := \sup_{0 < h \leq \delta, x \in [0, \infty)} |f(x + 2h) - 2f(x + h) + f(x)|,$$

where $\delta > 0$. Then, by Theorem 2.4 of [5, p. 177], there exists a constant $C > 0$ such that

$$K(f, \delta) \leq C \omega_2(f, \sqrt{\delta}).$$

Also, as usual, by $\omega(f, \delta), \delta > 0$, we denote the usual modulus of continuity of $f$.

Then, we first get the following local approximation result.

**Theorem 3.** Let $(u_n)$ be a sequence of functions on $[0, \infty)$ satisfying (1.2) for a fixed $a \geq 0$. For any $f \in C_B[0,\infty)$ and for every $x \in [a, \infty)$, $n \in \mathbb{N}$, we have

$$|L_n(f; x) - f(x)| \leq C \omega_2(f, \sqrt{\delta_n(x)}) + \omega(f, u_n(x) - x + \frac{1}{2n})$$

for some constant $C > 0$, where

$$\delta_n(x) := (u_n(x) - x)^2 + \frac{2u_n(x) - x}{n} + \frac{1}{3n^2}.$$  \hspace{1cm} (3.2)

**Proof.** Define an operator $\Omega_n : C_B[0,\infty) \to C_B[0,\infty)$ by

$$\Omega_n(f; x) := L_n(f; x) - f\left(u_n(x) + \frac{1}{2n}\right) + f(x).$$  \hspace{1cm} (3.3)

So, by Lemma 2(ii), we get

$$\Omega_n(\psi_x; x) = L_n(\psi_x; x) - u_n(x) - \frac{1}{2n} + x = 0.$$  \hspace{1cm} (3.4)

Let $g \in C^2_B[0,\infty)$, the space of all functions having the second continuous derivative on $[0, \infty)$, and let $x \in [0, \infty)$. Then, it follows from the well-known Taylor formula that

$$g(y) - g(x) = \psi_x(y)g'(x) + \int_x^y \psi_t(y)g''(t) dt, \quad y \in [0, \infty).$$
By (3.4), we get
\[ |\Omega_n(g; x) - g(x)| = |\Omega_n(g(y) - g(x); x)| \]
\[ = \left| \Omega_n \left( \int_x^y \psi_t(y) g''(t) dt; x \right) \right| \]
\[ = \left| L_n \left( \int_x^y \psi_t(y) g''(t) dt; x \right) \right| \]
\[ - \int_x^{u_n(x) + \frac{1}{2n}} \psi_t \left( u_n(x) + \frac{1}{2n} \right) g''(t) dt \right|. \]

Using (3.3) and Lemma 2(ii), we obtain that
\[ |\Omega_n(g; x) - g(x)| \leq \frac{\|g''\|}{2} L_n \left( \psi^2_x; x \right) + \frac{\|g''\|}{2} \psi^2_x \left( u_n(x) + \frac{1}{2n} \right) \]
\[ = \frac{\|g''\|}{2} \left\{ \left( u_n(x) - x \right)^2 + \frac{2u_n(x) - x}{n} + \frac{1}{3n^2} \right\} + \left( u_n(x) - x + \frac{1}{2n} \right)^2 \]
which implies that
\[ |\Omega_n(g; x) - g(x)| \leq \|g''\| \delta_n(x), \quad (3.5) \]
where \( \delta_n(x) \) is given by (3.2). Then, for any \( f \in C_B[0, \infty) \), it follows from (3.5) that
\[ |L_n(f; x) - f(x)| \leq |\Omega_n(f - g; x) - (f - g)(x)| \]
\[ + |\Omega_n(g; x) - g(x)| + \left| f \left( u_n(x) + \frac{1}{2n} \right) - f(x) \right| \]
\[ \leq 2 \|f - g\| + \|g''\| \| \delta_n(x)\| + \left| f \left( u_n(x) + \frac{1}{2n} \right) - f(x) \right| \]
\[ \leq 2 \left\{ \|f - g\| + \delta_n(x) \|g''\| \right\} + \left| f \left( u_n(x) + \frac{1}{2n} \right) - f(x) \right|. \]
Hence, by (3.1), we deduce that
\[ |L_n(f; x) - f(x)| \leq 2 \left\{ \|f - g\| + \delta_n(x) \|g''\| \right\} + \omega \left( f, \left| u_n(x) - x + \frac{1}{2n} \right| \right) \]
\[ \leq 2K(f, \delta_n(x)) + \omega \left( f, \left| u_n(x) - x + \frac{1}{2n} \right| \right) \]
\[ \leq C \omega_2 \left( f, \sqrt{\delta_n(x)} \right) + \omega \left( f, \left| u_n(x) - x + \frac{1}{2n} \right| \right) \]
which completes the proof. \( \Box \)
Using Theorem 3, one can get the following special cases.

**Corollary 6.** For the classical SMK operators (1.1), we have, for any \( x \geq 0, n \in \mathbb{N} \) and \( f \in C_B[0, \infty) \),

\[
|K_n(f; x) - f(x)| \leq C \omega_2 \left( f, \sqrt{\frac{x}{n}} + \frac{1}{3n^2} \right) + \omega \left( f, \frac{1}{2n} \right).
\]

**Corollary 7.** For the operators (2.11), we have, for any \( f \in C_B[0, \infty), x \geq 0 \) and \( n \in \mathbb{N} \),

\[
\left| L_n^{(1)}(f; x) - f(x) \right| \leq C \omega_2 \left( f, \sqrt{\delta_n^{(1)}(x)} \right) + \omega \left( f, \sqrt{\frac{4n^2x^2 + 1 - 2nx}{2n}} \right),
\]

where \( \delta_n^{(1)}(x) := 2x^2 - \frac{1}{6n^2} + \frac{(1-2nx)\sqrt{4n^2x^2 + 1}}{2n^2} \).

**Corollary 8.** For the operators (2.12), we have, for any \( f \in C_B[0, \infty), x \geq \frac{1}{2} \) and \( n \in \mathbb{N} \),

\[
\left| L_n^{(2)}(f; x) - f(x) \right| \leq C \omega_2 \left( f, \sqrt{\delta_n^{(2)}(x)} \right),
\]

where \( \delta_n^{(2)}(x) := \frac{x}{n} - \frac{5}{12n^2} \).

**Corollary 9.** For the operators (2.14) we have, for any \( f \in C_B[0, \infty), x \geq \frac{1}{\sqrt{3}} \) and \( n \in \mathbb{N} \),

\[
\left| L_n^{(3)}(f; x) - f(x) \right| \leq C \omega_2 \left( f, \sqrt{\delta_n^{(3)}(x)} \right)
+ \omega \left( f, \frac{2\sqrt{3}\sqrt{3n^2x^2 + 2} - 6(nx + 1) + 3}{6n} \right),
\]

where \( \delta_n^{(3)}(x) := 2x^2 + \frac{x(3 - 2\sqrt{3}\sqrt{3n^2x^2 + 2})}{3n} \).

4. **Estimates for Lipschitz-type functions**

In this section, for a fixed \( a \geq 0 \), we consider the following Lipschitz-type space

\[
\text{Lip}_M^+(r) := \left\{ f \in C_B[0, \infty) : |f(t) - f(x)| \leq M \frac{|t-x|^r}{(t+x)^{r/2}}; \ x, t \in (a, \infty) \right\},
\]

where \( M \) is any positive constant and \( 0 < r \leq 1 \).

In order to give an estimation in approximating the functions in \( \text{Lip}_M^+(r) \) we need the next lemma.
Lemma 4. Let \((u_n)\) be a sequence of functions on \([0, \infty)\) satisfying (1.2) for a fixed \(a \geq 0\). For every \(x > a\) and \(n \in \mathbb{N}\), we have
\[
\sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x| dt \leq \sqrt{\frac{\delta_n(x)}{n}},
\] (4.1)
where \(p_{n,k}(x)\) and \(\delta_n(x)\) are given by (1.4) and (3.2), respectively.

Proof. Applying the Cauchy–Schwarz inequality to the series in the left hand side of (4.1), we get
\[
\sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x| dt \leq \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \left( \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x| dt \right)^2 \right\}^{1/2}.
\]
If we again apply the Cauchy–Schwarz inequality to the integral in the right-hand side of the last inequality and also use Lemma 2(ii), then we see that
\[
\sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x| dt \leq \frac{1}{\sqrt{n}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2 dt \right\}^{1/2}
= \frac{1}{\sqrt{n}} \sqrt{L_n(\psi^2; x)}
= \frac{\delta_n(x)}{n},
\]
whence the result follows. \(\square\)

Now we are in position to give our result.

Theorem 4. Let \((u_n)\) be a sequence of functions on \([0, \infty)\) satisfying (1.2) for a fixed \(a \geq 0\). Then, for any \(f \in \text{Lip}_{M}^*(r),\ r \in (0, 1],\) and for every \(n \in \mathbb{N}\) and \(x \in (a, \infty),\) we have
\[
|L_n(f;x) - f(x)| \leq \frac{M s_{n}^{r/2}(x)}{n^{1-r+2/r}}.
\] (4.2)
where \(\delta_n(x)\) is given by (3.2).

Proof. We first assume that \(r = 1\). So, let \(f \in \text{Lip}_{M}^*(1)\) and \(x \in (a, \infty)\). Then, we get
\[
|L_n(f;x) - f(x)| \leq \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)| dt
\]
\[
\leq M \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{|t-x|}{\sqrt{t+x}} dt.
\]
Since \( \frac{1}{\sqrt{1+x}} \leq \frac{1}{\sqrt{x}} \), we may write that

\[
|L_n(f; x) - f(x)| \leq M \frac{1}{\sqrt{1+x}} \sum_{k=0}^{\infty} p_{n,k}(x) \int_{k/n}^{k+1/n} |t-x| \, dt.
\]

Now, by Lemma 4, we conclude that

\[
|L_n(f; x) - f(x)| \leq M \frac{\sqrt{d_n(x)}}{n \sqrt{x}}, \tag{4.3}
\]

which gives the desired result for \( r = 1 \). Assume now that \( r \in (0, 1) \). Then, taking \( p = \frac{1}{r} \) and \( q = \frac{1}{1-r} \), for any \( f \in \text{Lip}_{1, M}(r) \), if we apply the Hölder inequality two times, then we obtain that

\[
|L_n(f; x) - f(x)| \leq \sum_{k=0}^{\infty} p_{n,k}(x) \left( \int_{k/n}^{k+1/n} |f(t) - f(x)| \, dt \right)^{1/r} \left( \int_{k/n}^{k+1/n} \frac{1}{|t-x|^{1/r}} \, dt \right)^{r}
\]

Using the definition of the space \( \text{Lip}_{1, M}(r) \) and also considering Lemma 4, we get

\[
|L_n(f; x) - f(x)| \leq \frac{M}{n^{1-r}} \sum_{k=0}^{\infty} p_{n,k}(x) \left( \int_{k/n}^{k+1/n} \frac{1}{|t-x|^{1/r}} \, dt \right)^{r}
\]

\[
\leq \frac{M}{n^{1-r}x^{r/2}} \sum_{k=0}^{\infty} p_{n,k}(x) \left( \int_{k/n}^{k+1/n} \frac{1}{|t-x|} \, dt \right)^{r}
\]

\[
\leq \frac{M \delta_n^{r/2}(x)}{n^{1-r+r/2}x^{r/2}}.
\]

Thus, the proof is complete. \( \square \)

Finally, it should be noted that our Theorem 4 includes many special cases as in the previous sections. However, we omit the details.

References


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