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MODULES WHOSE PROJECTION INVARIANT SUBMODULES HAVE PROJECTION INVARIANT CLOSURES

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Abstract. We elaborate the class of PC-modules in which every projection invariant submodules has projection invariant closures in this study. We provide examples that the class of PC-modules does not belong to the system of generalization of extending modules. Moreover, we clarify direct sums and direct summands properties for the former class of modules. It is proved that the aforementioned property is not closed under direct sums. Thereupon, we cope with when the direct sums of PC-modules enjoy with the property.

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1. Introduction

Let R be a ring with unity and M a unitary right R-module. A submodule K of M is called a *complement* in M if K has no proper essential extension in M. Recall that a module M is called *extending* (or CS) if every submodule N of M is essential in a direct summand M; or every complement submodule of M is a direct summand of M [4, 15]. A submodule N of M is called a *projection (fully) invariant* if $f(N) \subseteq N$ for all $f^2 = f \in End(M_R)$ ($f \in End(M_R)$). It is well known that every fully invariant submodule is projection invariant (see, [5, page 50]). There are many examples of projection invariant submodules in different algebraic structures. Motivating on this class of submodules, a module M is said to be *π-extending* [3], if every projection invariant submodule of M is essential in a direct summand of M. It is shown that the class of π-extending modules is closed under direct sums, but not direct summands (see, [3, Example 5.5], [14, Example 4]). Hence it is investigated some special subclasses of π-extending modules. To this end, a module M is called *strongly π-extending* [9], if every projection invariant submodule of M is essential in a fully invariant direct summand of M. It is proved that the class of strongly π-extending modules is a proper subclass of π-extending modules.

Let N be a submodule of M. A submodule T of M is called a *closure* (or *essential closure*) of N in M, if $N \leq_e T \leq_c M$. It is renowned that, every submodule has a closure (see, [15, Proposition 2.5]). Consequently, we concern with closure properties of projection invariant submodules. Thereby, we call a module M is *projection invariant closure module* (denoted, *PC-module*), if every projection invariant submodule of M has a projection invariant closure in M. We observe indecomposable modules, uniform modules and nonsingular modules are the examples of PC-modules. Moreover it is shown that the PC condition is more general than strongly π-extending property. Although strongly π-extending property implies π-extending condition, we provide by counter examples that the classes of PC-modules and π-extending modules are incomparable. Apart from that, we delve into the direct sums and direct summands properties of PC-modules. We prove that projection invariant direct summands of PC-modules are PC-modules. Even so the aforementioned property is not closed under direct sums. Therefore we show when the direct sums of PC-modules is PC-module. To this end, a module M has *complement sum property, CSP* [7], if the sum of every pair of complements of M is a complement of M. Furthermore we get the hang of CSP condition which is not Morita invariant. Additionally, we characterize that the extending and quasi-continuous conditions are equivalent for a module with CSP.

For notation, we use $M_n(R)$ and R^n for the full n-by-n matrix ring over R and the direct sum of n copies of R for any positive integer n, respectively. For a nonempty subset X of M, $X \leq M$, $X \leq_e M$, $X \leq_c M$, $X \leq_d M$ and $X \trianglelefteq_p M$ denote the submodule of M, the essential submodule of M, the complement submodule of M, the direct summand of M and the projection invariant submodule of M, respectively. For unknown terminology and notation, see [1, 4, 10, 11].

2. Preliminary Results

We locate PC-modules with the other renowned classes of modules (e.g., π-extending, strongly π-extending) in this section. Observe that indecomposable modules and uniform modules are the examples of PC-modules. Furthermore, nonsingular modules have the foregoing property as shown in the first result.

Lemma 1. *Every nonsingular module is a PC-module.*

Proof. Let M_R be a nonsingular module and $X \trianglelefteq_p M$. Then $X \leq_e T \leq_c M$ for some submodule T of M. Since M_R is nonsingular and $X \trianglelefteq_p M$, T is a projection invariant submodule of M by [3, Lemma 2.3]. Hence M is a PC-module. □

Let $M_{\mathbb{Z}} = \mathbb{Z}/\mathbb{Z}p$ for any prime p. It is clear that $M_{\mathbb{Z}}$ is not nonsingular, but it is a PC-module. It follows that the converse of above lemma is not true. Moreover, there are examples which show that the PC condition does not imply uniform or indecomposable properties. For example, let $M_{\mathbb{Z}} = (\mathbb{Z} \oplus \mathbb{Z})_{\mathbb{Z}}$. It is renowned that $M_{\mathbb{Z}}$ is nonsingular, and hence it is a PC-module by Lemma 1. However $M_{\mathbb{Z}}$ is neither uniform nor indecomposable. The next result identifies the connection between the classes of strongly π-extending modules and PC-modules.

Lemma 2. *If M_R is a strongly π-extending module, then M_R has PC condition. But the converse need not to be true.*

Proof. Let M_R be a strongly π-extending module and $N \trianglelefteq_p M$. Then $N \leq_e D \leq_d M$ for some fully invariant direct summand D of M. It follows that $N \leq_e D \leq_c M$ where $D \trianglelefteq_p M$. Therefore M_R is a PC-module. On the other hand, for the converse, let R be a domain which is not right Ore. Since R_R is a prime ring, every nonzero ideal of R_R is essential in R_R. Thus R_R is a PC-module, because R is an indecomposable R-module. However R_R is not uniform, so R is not π-extending and hence it is not strongly π-extending by [9, Corollary 2.4]. □

It is interpreted that every strongly π-extending module is π-extending in [9]. In connection with the above lemma one might ask that are there any implications between PC-modules and π-extending modules? Now, we get across the following counter examples for the former question.

Example 1. (i) Consider $M_{\mathbb{Z}} = \prod_{i \in I} A_i$ where $A_i = \mathbb{Z}$ for $i \in I$. Note that the Specker group $M_{\mathbb{Z}}$ is not π-extending by [5]. On the other hand, $M_{\mathbb{Z}}$ is a nonsingular by [6, Proposition 1.22]. Thereupon $M_{\mathbb{Z}}$ is a PC-module by Lemma 1.

(ii) Let $M_{\mathbb{Z}} = \mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}p)$ for any prime p. Then $M_{\mathbb{Z}}$ is π-extending which is not strongly π-extending by [9, Corollary 2.4 $(iv) \not\Rightarrow (i)$]. Hence $M_{\mathbb{Z}}$ is not a PC-module by Proposition 1(i).

Note that a ring R is called *Abelian* if every idempotent of R is central. The next fact provides the implications between the classes of π-extending modules, strongly π-extending modules and PC-modules under some additional conditions.

Proposition 1. (i) *Let M_R be a PC-module. Then M_R is a π-extending module if and only if M_R is a strongly π-extending module.*

(ii) *Let $S = End(M_R)$ be an Abelian ring. If M_R is a π-extending, then M_R is a PC-module.*

(iii) *If M_R be a PC-module and every projection invariant essentially closed submodule of M is a fully invariant direct summand, then M_R is strongly π-extending.*

Proof. (i) Let M_R be a π-extending module and $X \trianglelefteq_p M$. Then $X \leq_e K \leq_d M$ for some direct summand K of M. Since M_R is a PC-module, K is projection invariant in M. Hence K is a fully invariant submodule of M by [5, page 50]. Thus M is a strongly π-extending module. The converse follows by [9, Corollary 2.4].

(ii) Let $X \trianglelefteq_p M$. Then $X \leq_e K \leq_d M$ for some submodule K of M. Since S is Abelian, it can be easily seen that every direct summand of M is projection invariant in M. Therefore M is a PC-module.

(iii) Let M be a PC-module and $X \trianglelefteq_p M$. Then there exists a projection invariant submodule K of M such that $X \leq_e K \leq_c M$. By hypothesis, K is a fully invariant direct summand, so M is strongly π-extending. □

Notice that $M_{\mathbb{Z}} = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ is a PC-module for any prime p. Indeed, it is clear from [8, Example 2.14], $M_{\mathbb{Z}}$ is a π-extending module with an Abelian endomorphism ring. Thereby, $M_{\mathbb{Z}}$ is a PC-module by Proposition 1(ii). This example and Example 1(ii) show that any submodules of a PC-module need not to be a PC-module, in general.

One might wonder whether PC-condition belongs to the system of generalization of extending modules or not. Hence we supply examples which simplify that there is no implication between the classes of PC-modules and extending modules.

Example 2. (i) Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$ be the upper triangular 2-by-2 matrix ring over $\mathbb{Z}$. It is follows that R_R is not extending by [15, Example 3.84]. Since R_R is nonsingular, it is a PC-module by Lemma 1.

(ii) Let S_3 be the symmetric group on the letter $\{1, 2, 3\}$ and $\mathbb{Z}_3$ the ring of integers modulo 3. Let $R = \mathbb{Z}_3[S_3]$ be the group ring of the group S_3 over $\mathbb{Z}_3$. Then R is right self-injective. Thus R_R is extending, and hence it is π-extending. However R_R is not a strongly π-extending module by [2, Example 1.1] and [9, Corollary 2.4]. Now assume that R_R is a PC-module. Then R_R is not π-extending by Proposition 1(i), a contradiction. Therefore R_R is not a PC-module.

Recall from [12], a module M is called *UC-module*, if every submodule of M has a unique closure. Following the idea in [12], it is natural to think of *unique projection invariant closure module* (denoted, UPC-module) in which every projection invariant submodule has a unique projection invariant closure.

Observe that every nonsingular and PC-module are the examples of UPC-modules. However basically the results and their proofs arise out of just simple motifications of the result in [12]. To this end, Example 1(ii) and Example 2(ii) identify that projection invariant submodules need not to have a unique projection invariant closure.

3. Direct sums and direct summands

In this section, our main goal is to deal with the direct summand and direct sum properties of PC-module.

Proposition 2. *Let M be a PC-module such that $M = M_1 \oplus M_2$ where M_1 and M_2 are projection invariant submodules of M. Then M_1 and M_2 are PC-modules.*

Proof. Let $M = M_1 \oplus M_2$ for some projection invariant $M_1, M_2 \leq M$ and let X_1 be a projection invariant submodule of M_1. Then $X_1 \oplus M_2$ is a projection invariant submodule of M by [3, Lemma 4.13]. Hence $X_1 \oplus M_2 \leq_e K \leq_c M$ for some projection invariant submodule K of M. Since $K \trianglelefteq_p M$, $K = (K \cap M_1) \oplus (K \cap M_2)$ where $K_1 = K \cap M_1 \trianglelefteq_p M_1$ and $K_2 = K \cap M_2 \trianglelefteq_p M_2$ by [5, page 50]. Hence we obtain $X_1 = M_1 \cap (X_1 \oplus M_2) \leq_e K \cap M_1 = K_1$. It is easy to see that $K_1 \leq_c M$. It follows from [10, Proposition 6.24 (1)] that $K_1 \leq_c M_1$. Therefore M_1 is a PC-module. Similarly, it can be seen that M_2 is a PC-module. □

Example 1(ii) examines that the direct sums of PC-modules need not to be PC-module. To this end, we determine when the class of PC-modules is closed under direct sums.

Proposition 3. *Let $M = M_1 \oplus M_2$ be an extending module for some $M_1, M_2 \leq M$. If M_1 and M_2 are PC-modules, then M is a PC-module.*

Proof. Let $Y \trianglelefteq_p M$. Then $Y = (Y \cap M_1) \oplus (Y \cap M_2)$ where $Y_1 = Y \cap M_1 \trianglelefteq_p M_1$ and $Y_2 = Y \cap M_2 \trianglelefteq_p M_2$ by [5, page 50]. Hence there exist projection invariant submodules K_1 of M_1 and K_2 of M_2 such that $X_1 \leq_e K_1 \leq_c M_1$ and $X_2 \leq_e K_2 \leq_c M_2$. It follows that M_1 and M_2 are extending by [11, Proposition 2.7]. Thus K_1 and K_2 are direct summands of M_1 and M_2, respectively. Consequently $K_1 \oplus K_2$ is a direct summand of M, so $K_1 \oplus K_2$ is a complement in M. Therefore $X = X_1 \oplus X_2 \leq_e K_1 \oplus K_2 \leq_c M$ such that $K_1 \oplus K_2 \trianglelefteq_p M$. Thereby M is a PC-module. □

Corollary 1. *Let $M = M_1 \oplus M_2$ be a semisimple (or uniform, or injective) module for some $M_1, M_2 \leq M$. If M_1 and M_2 are PC-modules, then M is a PC-module.*

Proof. It is a consequence of Proposition 3. □

Recall that a module M_R has the *summand sum property, SSP*, if for all $D_1, D_2 \leq_d M_R$, $D_1 + D_2 \leq_d M$. Motivating SSP definition on complement submodules, M has *complement sum property, CSP* [7], if for all $K_1, K_2 \leq_c M_R$, $K_1 + K_2 \leq_c M$. Even though the authors in [7] proved that CSP condition implies SSP, we give the following example which shows that the reverse implication of the former property is not true, in general.

Example 3. Let M be the $\mathbb{Z}$-module $\mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}p)$ as in Example 1(ii). Note that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/\mathbb{Z}p) = \mathbb{Z}/\mathbb{Z}p \text{ and } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}p, \mathbb{Z}) = 0.$$

Thus M has SSP by [15, Exercise 2.41]. On the other hand, we claim $M_{\mathbb{Z}}$ does not have CSP. Assume the contrary. Since $\mathbb{Z}$ and $\mathbb{Z}/\mathbb{Z}p$ are PC-modules, $M_{\mathbb{Z}}$ is a PC-module by Theorem 3, a contradiction (see, Example 1(ii)). Therefore M does not satisfy CSP.

Now we compose some useful properties of modules with CSP which might help us to consider being Morita invariant property as well as the application to the full matrix rings for the former condition. Let us begin with an easy fact and an example for extending case.

Lemma 3. *SSP and CSP conditions are coincide for an extending module.*

Proof. It is routine to check. □

Example 4. Let K be a field and $R_R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$. Then R_R is an extending module which does not satisfy C_3. Hence R_R does not have SSP. It follows from Lemma 3 that R_R does not have CSP.

Theorem 1. *Let R be a ring such that $R = ReR$ and $S = eRe$ for some $e^2 = e \in R$. Then M_R has CSP if and only if the right S-module module Me has CSP.*

Proof. It is clear from [15, Lemma 2.76 and Proposition 2.77 (ii)]. □

Corollary 2. *Let R be a ring such that $R = ReR$ for some $e^2 = e \in R$. Then R_R has CSP if and only if the right eRe-module Re satisfies CSP.*

Proof. It is clear from Theorem 1. □

Theorem 2. *$M_n(R)$ has CSP condition if and only if the free right R-module R^n has CSP condition.*

Proof. Note that $M_n(R) = M_n(R)eM_n(R)$ where e is the matrix unit with 1 in the $(1,1)$-th position and zero elsewhere. Now apply the Theorem 1 and Corollary 2 to get the theorem. □

As an application of Theorem 2, it can be easily seen that $M_2(\mathbb{Z})$ does not have CSP condition. Indeed, it is renowned that $M_{\mathbb{Z}} = (\mathbb{Z} \oplus \mathbb{Z})_{\mathbb{Z}}$ is an extending module which does not satisfy SSP by [15, Example 2.82]. Hence $M_{\mathbb{Z}}$ does not have CSP by Lemma 3. Thus $M_2(\mathbb{Z})$ does not have CSP by Theorem 2. This example explains that CSP condition is not Morita invariant. Now we proceed our main aim of this section.

Theorem 3. *Let M be a right R-module with CSP such that $M = M_1 \oplus M_2$ for some $M_1, M_2 \leq M$. If M_1 and M_2 are PC-modules, then M is a PC-module.*

Proof. Let $X \trianglelefteq_p M$. Then $X = (X \cap M_1) \oplus (X \cap M_2)$ where $X_1 = X \cap M_1 \trianglelefteq_p M_1$ and $X_2 = X \cap M_2 \trianglelefteq_p M_2$ by [5, page 50]. Hence there exist projection invariant submodules K_1 of M_1 and K_2 of M_2 such that $X_1 \leq_e K_1 \leq_c M_1$ and $X_2 \leq_e K_2 \leq_c M_2$. Thus $X \leq_e K_1 \oplus K_2 \leq M$ where $K_1 \oplus K_2 \trianglelefteq_p M$. Note that $K_1 \oplus K_2 \leq_c M$ by CSP condition. Thus M is a PC-module. □

Recall from [13], a module M has (P_n) condition if for every submodule K of M such that T is a direct sum $T_1 \oplus \cdots \oplus T_n$ of complements T_i $(1 \leq i \leq n)$ in M, every homomorphism $\alpha_1 : T \to M$ can be lifted to a homomorphism $\alpha_2 : M \to M$. In [13], the authors proved that if M satisfies (P_n) then M satisfies (P_{n-1}) for all $n \geq 2$. Moreover, they engage a characterization of quasi-continuous modules in terms of (P_n) conditions for every positive integer n. Realize that (P_1) does not imply (P_2) (see, [13, Example 10]). The following result spells out (P_1) and (P_2) conditions are equivalent for a module with CSP.

Proposition 4. *Let M be a right R-module with CSP. Then M has (P_2) condition if and only if M has (P_1) condition.*

Proof. Let M be a right R-module with CSP. If M has (P_2), then M has (P_1) by [13, page 341]. Conversely, let M has (P_1). Consider $K_1, K_2 \leq_c M$ with α :

$K_1 \oplus K_2 \to M$ homomorphism. Since M has CSP, $K_1 \oplus K_2 \leq_c M$. Hence $\alpha : K_1 \oplus K_2 \to M$ can be lifted to a homomorphism $\theta : M \to M$ by the condition of (P_1). Therefore M has (P_2) condition. □

We strengthen the characterization of quasi-continuous modules, which is presented in [13], for a module with CSP condition.

Theorem 4. *The followings are equivalent for a module M_R with CSP.*
(1) *M is quasi-continuous.*
(2) *M has (P_n) for every positive integer n.*
(3) *M has (P_n) for some integer $n \geq 2$.*
(4) *M has (P_2).*
(5) *M has (P_1).*
(6) *M is extending.*

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) Clear from [13, Theorem 4].
(4) ⇔ (5) It follows from Proposition 4.
(5) ⇔ (6) Let M has (P_1). Then M has (P_2) by Proposition 4. Hence M is extending from the fact of (4) ⇔ (1). Conversely, let $K \leq_c M$ and $\alpha : K \to M$ be a homomorphism. Since M is extending, K is a direct summand of M. Consider $g = \pi\iota$ where $\pi : M \to K$ is projection and $\iota : K \to M$ is inclusion. It is easy to check that $g|_K = \alpha$, hence M has (P_1). □

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