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RATE OF CONVERGENCE OF $q-$ ANALOGUE OF A CLASS OF NEW BERNSTEIN TYPE OPERATORS

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Abstract. Sharma [32] introduced a q-analogue of a new sequence of classical Bernstein type operators defined by Deo et al. [14] for functions defined in the interval $[0, \frac{n}{n+1}]$. The purpose of this paper is to study the rate of convergence of these operators with the aid of the modulus of continuity and a Lipschitz type space. Subsequently, we define the bivariate case of these operators and discuss the approximation properties by means of the complete and partial modulus of continuity, Lipschitz class and the Peetre's K- functional. Some numerical results which show the rate of convergence of these operators to certain functions using Maple algorithms are given. Lastly, we construct the associated GBS operators and study the approximation of Bögel continuous and Bögel differentiable functions. The comparison of convergence of the bivariate operator and its GBS type operator is made considering numerical examples.

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1. INTRODUCTION

For $f \in C[0,1]$, Bernstein [8] constructed a sequence of polynomials

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \;\; n = 1,2\cdots, \quad x \in [0,1],$$

and proved that the sequence $B_n(f;x)$ converges to $f(x)$, as $n \to \infty$, uniformly in $x \in [0,1]$. These polynomials are called Bernstein polynomials and possess many remarkable properties. For $f \in L_1[0,1]$, the space of Lebesgue integrable functions in $[0,1]$, Durrmeyer [18] introduced an integral modification of Bernstein operators as

$$D_n(f;x)=(n+1)\sum_{k=0}^{n} b_{nk}(x)\int_0^1 b_{nk}(t)f(t)dt,$$

which was extensively studied by Derriennic [15].

In recent years, the applications of q-calculus in the area of approximation theory is one of the main areas of research (see [1], [2], [13]). Also, the reader should consult the monographies of A. Aral et al. [5], Gupta et al. [23] and G. Tachev et al. [24]. In 1987, for $f\in C[0,1]$ and $0<q<1$, Lupas [27] introduced a q-analogue of Bernstein polynomials. After a decade, another q-generalization of Bernstein polynomials was introduced by Phillips [31]. The q-analogue of Bernstein polynomials due to Phillips was studied by several researchers e.g Ostrovska [29, 30], Kim [26] and Wang [38] etc. In 2005, Derriennic [16] introduced the q-analogue of Bernstein-Durrmeyer polynomials with Jacobi weights and studied some approximation properties. Later, Gupta [21] introduced the q-analogue of the Bernstein-Durrmeyer operators which was investigated later by Finta and Gupta ([20], [22]) and several other researchers. Dalmanoglu [13] introduced the Kantorovich type modification of q-Bernstein polynomials and established some approximation results. Muraru [28] introduced Bernstein-Schurer polynomials based on q-integers and established the rate of convergence in terms of the modulus of continuity. Agrawal et al. [2] considered the Stancu variant of these operators and obtained some local and global direct results. Later, Agrawal et al. [4] proposed the Durrmeyer type modification of these operators and discussed some local direct results and the rate of convergence of the modified limit q-Bernstein-Schurer type operators.

Deo et al. [14] introduced a new sequence of Bernstein type operators V_n as

$$(V_nf)(x)=\sum_{k=0}^{n} p_{n,k}(x)f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$p_{n,k}(x)=\left(1+\frac{1}{n}\right)^n\binom{n}{k}x^k\left(\frac{n}{n+1}-x\right)^{n-k},\quad x\in\left[0,\frac{n}{n+1}\right].$$

In the same paper, to approximate Lebesgue integrable functions on the interval $[0,1]$, the authors defined an integral modification of the operators (1.1) as

$$(L_nf)(x)=\frac{(n+1)^2}{n}\sum_{k=0}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k}(t)f(t)dt,$$

and studied some approximation properties.
From [25, page 69], the definite q-integral in the interval $[0,a], a>0$ is defined as:

$$\int_0^a f(x)d_qx = (1-q)a\sum_{j=0}^{\infty} q^j f(q^j a).$$

Later, Sharma [32] investigated the q-analogue of these operators given by

$$(L_{n,q}f)(x) = \frac{[n+1]_q^2}{[n]_q}\sum_{k=0}^{n} q^{-k} p_{n,k,q}(x)\int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k,q}(qt)f(t)d_qt, \qquad (1.2)$$

where

$$p_{n,k,q}(x) := \frac{[n+1]_q^n}{[n]_q^n}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k\left(\frac{[n]_q}{[n+1]_q}-x\right)_q^{n-k}, \quad x\in\left[0,\frac{[n]_q}{[n+1]_q}\right].$$

In the present paper, first we obtain the order of approximation of the operators defined by (1.2) by means of the modulus of continuity and the Lipschitz class. Then we proceed to define the bivariate generalization of these operators and investigate their rate of convergence with the help of the moduli of continuity and the K- functional. Lastly, we introduce the associated GBS operators and discuss their degree of approximation by means of the mixed modulus of smoothness.

2. Direct results

Lemma 1 ([32]). *For the operators given by (1.2), the following equalities hold:*

i) $(L_{n,q}1)(x) = 1$;

ii) $(L_{n,q}t)(x) = \dfrac{[n]_q}{[n+1]_q[n+2]_q} + \dfrac{q[n]_q x}{[n+2]_q}$;

iii) $(L_{n,q}t^2)(x) = \dfrac{[n]_q}{[n+2]_q[n+3]_q}\left\{[n-1]_q q^4x^2 + \dfrac{q^3x[n]_q + q(1+2q)x[n]_q}{[n+1]_q} + \dfrac{[2]_q[n]_q}{[n+1]_q^2}\right\}.$

Consequently,

i) $(L_{n,q}(t-x))(x) = \dfrac{[n]_q}{[n+1]_q[n+2]_q} + \dfrac{q[n]_q-[n+2]_q}{[n+2]_q}x$;

ii)
$$(L_{n,q}(t-x)^2)(x) = \frac{[2]_q[n]_q^2}{[n+1]_q^2[n+2]_q[n+3]_q} + \left(\frac{q^3[n]_q^2 + q(1+2q)[n]_q^2 - 2[n]_q[n+3]_q}{[n+1]_q[n+2]_q[n+3]_q}\right)x + \left(\frac{q^4[n-1]_q[n]_q - 2q[n]_q[n+3]_q + [n+2]_q[n+3]_q}{[n+2]_q[n+3]_q}\right)x^2.$$

Lemma 2. *For the operators given by (1.2), the following equality holds*

$$(L_{n,q}(t-x)^4)(x) = \frac{1}{[n+1]_q^4[n+2]_q[n+3]_q[n+4]_q[n+5]_q} \cdot \Big\{ x^4 q^{14}(1-q)^4[n]_q^8$$
$$+q^9x^3(1-q)^2(8xq^6-11xq^5-q^4+3xq^4+4q^3-4xq^3+6q^2-6xq^2+4q-4xq+1)[n]_q^7$$
$$-q^5x^2(-1-3q+q^{12}x^3+3q^{11}x^3+3q^{10}x^3+12xq^2-36xq^8+4xq+15xq^3+8xq^6$$
$$+6xq^5+4xq^4-19q^5x^2-17q^6x^2+74q^{10}x^2+53q^8x^2-26q^{11}x^2-74q^9x^2-8q^3x^2$$
$$-6q^2x^2+9q^4x^2+2xq^9-3xq^{10}-7q^2+4q^8-7q^3+3q^6+6q^5-2q^4+2q^7)[n]_q^6+\mathcal{O}([n]_q^5)\Big\}.$$

In what follows, let $(q_n)_n, 0<q_n<1$ be a sequence satisfying the following condition

$$\lim_{n\to\infty} q_n = 1.$$

Remark 1. Let $(q_n)_n, 0<q_n<1$ be a sequence satisfying the following conditions

$$\lim_{n\to\infty} q_n = 1, \ \lim_{n\to\infty} q_n^n = c, \ c\in[0,1).$$

By simple computations, we have

$$\lim_{n\to\infty}[n]_{q_n}(L_{n,q_n}(t-x))(x) = 1-(c+1)x,$$
$$\lim_{n\to\infty}[n]_{q_n}(L_{n,q_n}(t-x)^2)(x) = 2x(1-x),$$
$$\lim_{n\to\infty}[n]_{q_n}^2(L_{n,q_n}(t-x)^4)(x) = x^2(1-x)(7x^2-7x+5).$$

3. Main results

Next we assume that $C[0,\frac{[n]_q}{[n+1]_q}]$ is the class of all real valued continuous functions on $[0,\frac{[n]_q}{[n+1]_q}]$ endowed with the norm $||.||$ defined as

$$||f|| = \sup_{x\in\left[0,\frac{[n]_q}{[n+1]_q}\right]} |f(x)|.$$

Lemma 3. *For each $f\in C[0,\frac{[n]_q}{[n+1]_q}]$, we have $||L_{n,q}(f)||\le||f||$.*

Proof. Using the definition (1.2) and Lemma 1, the proof of this lemma easily follows. Hence, the details are omitted. ☐

For $f\in C[0,\frac{[n]_q}{[n+1]_q}]$, the Steklov mean is defined as

$$f_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[2f(x+u+v)-f(x+2(u+v))]dudv. \tag{3.1}$$

The first and second order modulus of continuity are respectively defined as

$$\omega(f,\delta) = \sup_{x,u,v\in\left[0,\frac{[n]_q}{[n+1]_q}\right],|u-v|\le\delta} |f(x+u)-f(x+v)|$$

and

$$\omega_2(f,\delta) = \sup_{x,u,v\in\left[0,\frac{[n]_q}{[n+1]_q}\right],|u-v|\le\delta} |f(x+2u)-2f(x+u+v)+f(x+2v)|, \quad \delta>0.$$

Lemma 4. *The Steklov mean $f_h(x)$ satisfies the following properties:*

i) $||f_h - f|| \le \omega_2(f,h)$,

ii) *If f is continuous, then $f_h', f_h'' \in C\left[0,\frac{[n]_q}{[n+1]_q}\right]$ and*

$$||f_h'||_{C\left[0,\frac{[n]_q}{[n+1]_q}\right]} \le \frac{5}{h}\omega(f,h), \quad ||f_h''||_{C\left[0,\frac{[n]_q}{[n+1]_q}\right]} \le \frac{9}{h^2}\omega_2(f,h).$$

Theorem 1. *Let $f \in C[0,\frac{[n]_{q_n}}{[n+1]_{q_n}}]$. Then for each $x \in [0,\frac{[n]_{q_n}}{[n+1]_{q_n}}]$, we have*

$$|(L_{n,q_n}f)(x)-f(x)| \le 5\omega\left(f,\frac{1}{\sqrt{[n+2]_{q_n}}}\right)\left(\frac{1+2x}{\sqrt{[n+2]_{q_n}}}\right)+\omega_2\left(f,\frac{1}{\sqrt{[n+2]_{q_n}}}\right)$$
$$\left[2+\frac{9}{2}\left\{\left(\frac{{q_n}^4[n-1]_{q_n}[n]_{q_n}-2q_n[n]_{q_n}[n+3]_{q_n}+[n+2]_{q_n}[n+3]_{q_n}}{[n+3]_{q_n}}\right)x^2\right.\right.$$
$$\left.\left.+\left(\frac{q_n^3[n]_{q_n}^2+q_n(1+2q_n)[n]_{q_n}^2-2[n]_{q_n}[n+3]_{q_n}}{[n+1]_{q_n}[n+3]_{q_n}}\right)x+\frac{[2]_{q_n}[n]_{q_n}^2}{[n+1]_{q_n}^2[n+3]_{q_n}}\right\}\right].$$

Proof. Using the Steklov mean f_h defined by (3.1), we may write

$$\begin{aligned}&|(L_{n,q_n}f)(x)-f(x)| \le \\ &\le |(L_{n,q_n}(f-f_h))(x)|+|(L_{n,q_n}(f_h-f_h(x)))(x)|+|f_h(x)-f(x)|.\end{aligned} \tag{3.2}$$

Using Lemma 3 and Lemma 4, we have

$$|(L_{n,q_n}(f-f_h))(x)| \le ||f-f_h|| \le \omega_2(f,h).$$

Now, by Taylor's expansion, we have

$$f_h(t) = f_h(x)+(t-x)f_h'(x)+\int_x^t (t-u)f_h''(u)du$$

$$\begin{aligned}|(L_{n,q_n}(f_h-f_h(x)))(x)| &\le |(L_{n,q_n}((t-x)f_h'(x)))(x)| \\ &+\left|L_{n,q_n}\left(\int_x^t (t-u)f_h''(u)du\right)(x)\right| \\ &\le ||f_h'|||(L_{n,q_n}(t-x))(x)| \\ &+||f_h''||(L_{n,q_n}(|\int_x^t |t-u|)du|)(x) \\ &= ||f_h'|||(L_{n,q_n}(t-x))(x)|+\frac{1}{2}||f_h''||(L_{n,q_n}(t-x)^2)(x).\end{aligned}$$

Applying Lemma 4, Lemma 1 and choosing $h = \sqrt{\frac{1}{[n+2]_{q_n}}}$, we get the required result. □

Let us assume that $\delta_{n,q_n}(x) = \sqrt{L_{n,q_n}((t-x)^2)(x)}$.

Theorem 2. *If f has a continuous derivative f' and $\omega(f',\delta)$ is the modulus of continuity of f' on $[0, \frac{[n]_{q_n}}{[n+1]_{q_n}}]$, then*

$$|(L_{n,q_n}f)(x) - f(x)| \leq M\,|\mu_{n,q_n}(x)| + \omega(f', \delta^2_{n,q_n}(x))\Big(1+\delta_{n,q_n}(x)\Big),$$

where M is a positive constant such that $|f'(x)| \leq M, x \in \left[0, \frac{[n]_{q_n}}{[n+1]_{q_n}}\right]$ and

$$\mu_{n,q_n}(x) = \left(\frac{q_n[n]_{q_n} - [n+2]_{q_n}}{[n+2]_{q_n}}x + \frac{[n]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}}\right). \tag{3.3}$$

Proof. On applying the mean value theorem, we get

$$\begin{aligned} f(t) - f(x) &= (t-x)f'(\xi) \\ &= (t-x)f'(x) + (t-x)(f'(\xi) - f'(x)), \end{aligned}$$

where ξ lies between x and t. Using the definition (1.2) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |(L_{n,q_n}f)(x) - f(x)| &\leq |f'(x)||(L_{n,q_n}(t-x))(x)| + \frac{[n+1]^2_{q_n}}{[n]_{q_n}}\sum_{k=0}^{n} q_n^{-k} p_{n,k,q_n}(x) \\ &\times \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t)|t-x||f'(\xi) - f'(x)|d_{q_n}t \\ &\leq M|\mu_{n,q_n}(x)| + \frac{[n+1]^2_{q_n}}{[n]_{q_n}}\sum_{k=0}^{n} q_n^{-k} p_{n,k,q_n}(x) \\ &\times \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t)\omega(f',\delta)\left(\frac{|t-x|}{\delta}+1\right)|t-x|d_{q_n}t \\ &\leq M|\mu_{n,q_n}(x)| + \frac{\omega(f',\delta)}{\delta}(L_{n,q_n}(t-x)^2)(x) \\ &+ \omega(f',\delta)\sqrt{(L_{n,q_n}(t-x)^2)(x)}. \end{aligned}$$

On choosing $\delta := \delta^2_{n,q_n}(x)$, we get the desired result. □

Szász [36], considered the Lipschitz type space defined as:

$$Lip_M^*(\xi) := \left\{ f \in C[0,\infty) : |f(t)-f(x)| \le M_f \frac{|t-x|^{\xi}}{(t+x)^{\frac{\xi}{2}}}; \text{ where } M_f \right.$$
$$\left. \text{is a constant which depends on } \; f,\ t \in [0,\infty),\ x \in (0,\infty) \right\},$$

where $0 < \xi \le 1$ and $r \in (0,1]$, to establish the uniform convergence of the Szász operators for functions in this space.

For $r \in (0,1]$ and $M > 0$, we define an analogue of this space in our case as follows:

$$Lip_M^*(r) := \left\{ f \in C\left[0, \frac{[n]_q}{[n+1]_q}\right] : |f(t)-f(x)| \le M \frac{|t-x|^r}{(t+x)^{r/2}}; \right.$$
$$\left. x \in \left(0, \frac{[n]_{q_n}}{[n+1]_{q_n}}\right],\ t \in \left[0, \frac{[n]_{q_n}}{[n+1]_{q_n}}\right] \right\}.$$

We observe that we get only pointwise approximation due to the presence of x in the error estimate of $(L_{n,q_n} f)(x) - f(x)$, while in the case of Szász operators [36], it turns out that this x gets cancelled leading to the uniform convergence of the operators.

Theorem 3. *Let $f \in Lip_M^*(r)$. Then, for all $x \in \left(0, \frac{[n]_{q_n}}{[n+1]_{q_n}}\right]$ we have*

$$|(L_{n,q_n} f)(x) - f(x)| \le M \left(\frac{\delta^2_{n,q_n}(x)}{x} \right)^{r/2}.$$

Proof. Let $r \in (0,1]$, applying the Hölder's inequality for integration and then for summation with $u = 2/r$ and $v = 2/(2-r)$, and Lemma 1, we have,

$$|(L_{n,q_n} f)(x) - f(x)|$$
$$\le \frac{[n+1]^2_{q_n}}{[n]_{q_n}} \sum_{k=0}^{n} q_n^{-k} p_{n,k,q_n}(x) \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t) |f(t) - f(x)| d_{q_n}(t)$$
$$\le \frac{[n+1]^2_{q_n}}{[n]_{q_n}} \sum_{k=0}^{n} q_n^{-k} p_{n,k,q_n}(x) \left\{ \left(\int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} |f(t)-f(x)|^u p_{n,k,q_n}(q_n t) d_{q_n} t \right)^{\frac{1}{u}} \right.$$
$$\left. \times \left(\int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t) d_{q_n} t \right)^{\frac{1}{v}} \right\}$$
$$\le \left(\sum_{k=0}^{n} \frac{[n+1]^2_{q_n}}{[n]_{q_n}} q_n^{-k} p_{n,k,q_n}(x) \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} |f(t)-f(x)|^u p_{n,k,q_n}(q_n t) d_{q_n} t \right)^{1/u}$$

$$\times\left(\sum_{k=0}^{n}\frac{[n+1]_{q_n}^2}{[n]_{q_n}}q_n^{-k}p_{n,k,q_n}(x)\int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}}p_{n,k,q_n}(q_nt)d_{q_n}t\right)^{1/v}$$

$$\leq\frac{M}{(x)^{r/2}}\left(\sum_{k=0}^{n}\frac{[n+1]_{q_n}^2}{[n]_{q_n}}q_n^{-k}p_{n,k,q_n}(x)\int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}}(t-x)^2p_{n,k,q_n}(q_nt)d_{q_n}t\right)^{1/u}$$

$$=M\left\{\frac{\delta_{n,q_n}^2(x)}{x}\right\}^{r/2}.$$

Hence, the proof is completed. □

4. The construction of operators for the bivariate case

In this section, we introduce the bivariate case of the generalized Durrmeyer type operators (1.2). Let $I_j=[0,\frac{[n_j]_{q_{n_j}}}{[n_j+1]_{q_{n_j}}}]\quad j=1,2$. In what follows, let $(q_{n_j}),\ j=1,2,$ be sequences in $(0,1)$ such that $\lim\limits_{n_j\to\infty}q_{n_j}=1$. For $I=I_1\times I_2$, let $C(I)$ denote the space of all real valued continuous functions on I with the norm $||f||_{C(I)}=\sup_{(x,y)\in I}|f(x,y)|$. For $f\in C(I)$, the bivariate case of the operators given by (1.2) is defined as:

$$(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)= \tag{4.1}$$
$$\frac{[n_1+1]_{q_{n_1}}^2}{[n_1]_{q_{n_1}}}\frac{[n_2+1]_{q_{n_2}}^2}{[n_2]_{q_{n_2}}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}q_{n_1}^{-k_1}q_{n_2}^{-k_2}p_{n_1,k_1,q_{n_1}}(x)\ p_{n_2,k_2,q_{n_2}}(y)$$
$$\times\int_{t=0}^{\frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}}\int_{s=0}^{\frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}}p_{n_1,k_1,q_{n_1}}(q_{n_1}t)\ p_{n_2,k_2,q_{n_2}}(q_{n_2}s)f(t,s)\ d_{q_{n_1}}t\ d_{q_{n_2}}s.$$

Lemma 5. *Let $e_{ij}(x,y)=x^iy^j$, $(i,j)\in\mathbb{N}\cup\{0\}\times\mathbb{N}\cup\{0\}$ with $i+j\leq 2$. For the operators given by (4.1), there hold the following equalities:*

i) $(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{00})(x,y)=1;$

ii) $(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{10})(x,y)=\dfrac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}[n_1+2]_{q_{n_1}}}+\dfrac{q_{n_1}[n_1]_{q_{n_1}}x}{[n_1+2]_{q_{n_1}}};$

iii) $(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{01})(x,y)=\dfrac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}[n_2+2]_{q_{n_2}}}+\dfrac{q_{n_2}[n_2]_{q_{n_2}}y}{[n_2+2]_{q_{n_2}}};$

iv) $(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{20})(x,y)=\dfrac{[n_1]_{q_{n_1}}}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}}\Bigg\{[n_1-1]_{q_{n_1}}q_{n_1}^4x^2$
$+\dfrac{q_{n_1}^3x[n_1]_{q_{n_1}}+q_{n_1}(1+2q_{n_1})x[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}+\dfrac{(1+q_{n_1})[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}^2}\Bigg\};$

v) $(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{02})(x,y) = \dfrac{[n_2]_{q_{n_2}}}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}}\Big\{[n_2-1]_{q_{n_2}}q_{n_2}^4 y^2$
$+\dfrac{q_{n_2}^3 y[n_2]_{q_{n_2}}+q_{n_2}(1+2q_{n_2})y[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}+\dfrac{(1+q_{n_2})[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}^2}\Big\}.$

Next, we state Korovkin type theorem, given by Volkov [37]. With the help of this theorem we study the convergence of the sequence $(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)$ to the function $f(x,y)$.

Theorem 4 ([37]). *Let $J_1, J_2 \subset \mathbb{R}$ be compact intervals of the real line and let $\{E_{m,n}f\}$ be a sequence of linear positive operators applying the space $C(J_1\times J_2)$ into itself. Suppose that the following relations*

i) $E_{m,n}(1;x,y) = 1 + a_{m,n}(x,y)$,
ii) $E_{m,n}(t;x,y) = x + b_{m,n}(x,y)$,
iii) $E_{m,n}(s;x,y) = y + c_{m,n}(x,y)$,
iv) $E_{m,n}(t^2+s^2;x,y) = x^2 + y^2 + d_{m,n}(x,y)$,

hold, for each $(x,y)\in J_1\times J_2$.
If the sequence $\{a_{m,n}(x,y)\},\{b_{m,n}(x,y)\},\{c_{m,n}(x,y)\},\{d_{m,n}(x,y)\}$ converge to zero uniformly on $J_1\times J_2$, then the sequence $\{E_{m,n}f\}$ converges to f, uniformly on $J_1\times J_2$, for each $f\in C(J_1\times J_2)$.

Remark 2. In view of Theorem 4 and Lemma 5, it easily follows that for each $f\in C(I)$,
$$\lim_{n_1,n_2\to\infty}(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y) = f(x,y)$$
uniformly on I.

In the following we give some numerical results which show the rate of convergence of the operator $L_{n_1,n_2,q_{n_1},q_{n_2}}f$ to certain functions using Maple algorithms.

Example 1. Let us consider $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = x^2y^2+2x^2y-3y^2$. The convergence of the operator $L_{n_1,n_2,q_{n_1},q_{n_2}}f$ to the function f is illustrated in Figure 1 and Figure 2, respectively for $n_1 = n_2 = 10, q_{n_1} = q_{n_2} = 0.5$ and $n_1 = n_2 = 100, q_{n_1} = q_{n_2} = 0.9$, respectively. We remark that as the values of n_1 and n_2 increase, the error in the approximation of the function by the operator becomes smaller.

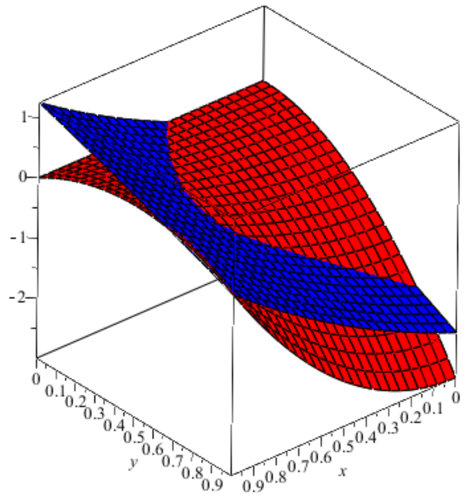

FIGURE 1. The convergence of $(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)$ to $f(x,y)$, for $q_{n_1}=q_{n_2}=0.5$ (red f, blue $L_{n_1,n_2,q_{n_1},q_{n_2}}$)

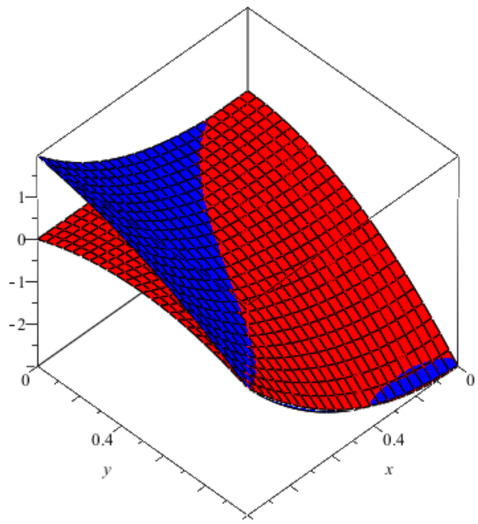

FIGURE 2. The convergence of $(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)$ to $f(x,y)$, for $q_{n_1}=q_{n_2}=0.9$ (red f, blue $L_{n_1,n_2,q_{n_1},q_{n_2}}$)

If $J_1,J_2\subset\mathbb{R}$ are compact intervals and $f\in\mathbb{R}^{J_1\times J_2}$, the modulus of continuity $\omega(\delta_1,\delta_2)$, for any $\delta_1>0,\delta_2>0$ is defined as

$$\omega(\delta_1,\delta_2)=\sup_{(x_1,y_1),(x_2,y_2)\in J_1\times J_2}\left\{|f(x_1,y_1)-f(x_2,y_2)|:|x_1-x_2|\le\delta_1,|y_1-y_2|\le\delta_2\right\}.$$

In what follows, let

$$\delta_{n_1,q_{n_1}}(x) = \Big((L_{n_1,n_2,q_{n_1},q_{n_2}}(t-x)^2)(x,y)\Big)^{1/2} = \Big((L_{n_1,q_{n_1}}(t-x)^2)(x)\Big)^{1/2},$$

and

$$\delta_{n_2,q_{n_2}}(y) = \Big((L_{n_1,n_2,q_{n_1},q_{n_2}}(s-y)^2)(x,y)\Big)^{1/2} = \Big((L_{n_2,q_{n_2}}(s-y)^2)(y)\Big)^{1/2}.$$

Next, we recall the following Shisha-Mond theorem [33].

Theorem 5 ([33])**.** *Let $J_1, J_2 \subset \mathbb{R}$ be compact intervals, $B(J_1\times J_2) = \{f \in \mathbb{R}^{J_1\times J_2} : f$ is bounded on $J_1\times J_2\}$ and let $E : C(J_1\times J_2) \to B(J_1\times J_2)$ be a linear positive operator. For each $f \in C(J_1\times J_2)$, $(x,y) \in J_1\times J_2$ and any $\delta_1 > 0, \delta_2 > 0$, the following inequality*

$$\begin{aligned}&|E(f;x,y) - f(x,y)| \le |f(x,y)||E(1;x,y)-1|\\ &+\Big\{E(1;x,y) + \delta_1^{-1}\sqrt{E(1;x,y)E((t-x)^2;x,y)} + \delta_2^{-1}\sqrt{E(1;x,y)E((s-y)^2;x,y)}\\ &+\delta_1^{-1}\delta_2^{-1}E(1;x,y)\sqrt{E((t-x)^2;x,y)E((s-y)^2;x,y)}\Big\}\,\omega(\delta_1,\delta_2)\end{aligned}$$

holds.

Theorem 6. *Let $f \in C(I)$ and $(x,y) \in I$. Then the operator $L_{n_1,n_2,q_{n_1},q_{n_2}}$ satisfies the following inequality*

$$|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y) - f(x,y)| \le 4\omega(\delta_{n_1,q_{n_1}}(x), \delta_{n_2.q_{n_2}}(y)).$$

Proof. Applying Theorem 5 and Lemma 5, and choosing $\delta_1 = \delta_{n_1,q_{n_1}}(x)$ and $\delta_2 = \delta_{n_2,q_{n_2}}(y)$ we get

$$\begin{aligned}&|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y) - f(x,y)| \le \Big\{1 + \delta_{n_1,q_{n_1}}^{-1}(x)\sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}}(t-x)^2)(x,y)}\\ &+\delta_{n_2,q_{n_2}}^{-1}(y)\sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}}(s-y)^2)(x,y)}\\ &+\delta_{n_1,q_{n_1}}^{-1}(x)\delta_{n_2,q_{n_2}}^{-1}(y)\sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}}(t-x)^2)(x,y)(L_{n_1,n_2,q_{n_1},q_{n_2}}(s-y)^2)(x,y)}\Big\}\\ &\times\omega(\delta_{n_1,q_{n_1}}(x);\delta_{n_2,q_{n_2}}(y))\end{aligned}$$

Hence, we get the required result. □

For $f \in C(I)$ and $\delta > 0$, the first order complete modulus of continuity for the bivariate case is defined as:

$$\bar{\omega}(f;\delta) = \sup\Big\{|f(t,s) - f(x,y)| : (t,s),(x,y) \in I \text{ and } \sqrt{(t-x)^2 + (s-y)^2} \le \delta\Big\}.$$

Further, the partial moduli of continuity with respect to x and y are given by

$$\omega_1(f;\delta)=\sup\left\{|f(x_1,y)-f(x_2,y)| : y\in I_2 \; and \; |x_1-x_2|\le\delta\right\},$$

and

$$\omega_2(f;\delta)=\sup\left\{|f(x,y_1)-f(x,y_2)| : x\in I_1 \; and \; |y_1-y_2|\le\delta\right\}.$$

Theorem 7. *For $f\in C(I)$, there holds the inequality*

$$|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)|\le \; 2(\omega_1(f;\delta_{n_1,q_{n_1}}(x))+\omega_2(f;\delta_{n_2,q_{n_2}}(y))).$$

Proof. Using the definition of partial moduli of continuity, Lemma 5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)|\le(L_{n_1,n_2,q_{n_1},q_{n_2}}|f(t,s)-f(x,y)|)(x,y)\\
&\le(L_{n_1,n_2,q_{n_1},q_{n_2}}|f(t,s)-f(t,y)|)(x,y)+(L_{n_1,n_2,q_{n_1},q_{n_2}}|f(t,y)-f(x,y)|)(x,y)\\
&\le\omega_2(f;\delta_{n_2,q_{n_2}}(y))\left[1+\frac{1}{\delta_{n_2,q_{n_2}}(y)}(L_{n_1,n_2,q_{n_1},q_{n_2}}|s-y|)(x,y)\right]\\
&+\omega_1(f;\delta_{n_1,q_{n_1}}(x))\left[1+\frac{1}{\delta_{n_1,q_{n_1}}(x)}(L_{n_1,n_2,q_{n_1},q_{n_2}}|t-x|)(x,y)\right]\\
&\le\omega_2(f;\delta_{n_2,q_{n_2}}(y))\left[1+\frac{1}{\delta_{n_2,q_{n_2}}(y)}\left((L_{n_1,n_2,q_{n_1},q_{n_2}}(s-y)^2)(x,y)\right)^{1/2}\right]\\
&+\omega_1(f;\delta_{n_1,q_{n_1}}(x))\left[1+\frac{1}{\delta_{n_1,q_{n_1}}(x)}\left((L_{n_1,n_2,q_{n_1},q_{n_2}}(t-x)^2)(x,y)\right)^{1/2}\right],
\end{aligned}$$

from which the required result is immediate. □

4.1. *Degree of approximation*

In this section, let us assume that $\lim_{n_j\to\infty}q_{n_j}=1, j=1,2$. We study the degree of approximation for the bivariate operators (4.1) by means of the Lipschitz class.

For $0<\alpha\le 1$ and $0<\beta\le 1$, we define the Lipschitz class $Lip_M(\alpha,\beta)$ for the bivariate case as follows:

$$|f(t,s)-f(x,y)|\le M|t-x|^{\alpha}|s-y|^{\beta}, \text{ for every } (t,s),(x,y)\in I.$$

Theorem 8. *Let $f\in Lip_M(\alpha,\beta)$. Then, we have*

$$|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)|\le M\delta^{\alpha}_{n_1,q_{n_1}}(x)\delta^{\beta}_{n_2,q_{n_2}}(y).$$

Proof. By our hypothesis, we may write

$$|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)|\le(L_{n_1,n_2,q_{n_1},q_{n_2}}|f(t,s)-f(x,y)|)(x,y)$$

$$\leq M(L_{n_1,n_2,q_{n_1},q_{n_2}}|t-x|^{\alpha}|s-y|^{\beta}(x,y)$$
$$= M(L_{n_1,n_2,q_{n_1},q_{n_2}}|t-x|^{\alpha})(x)(L_{n_1,n_2,q_{n_1},q_{n_2}}|s-y|^{\beta})(y).$$

Now, using the Hölder's inequality with $u_1 = \frac{2}{\alpha}$, $v_1 = \frac{2}{2-\alpha}$ and $u_2 = \frac{2}{\beta}$, $v_2 = \frac{2}{2-\beta}$, and Lemma 5, we have

$$\begin{aligned}
&|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)| \\
&\leq M\{(L_{n_1,n_2,q_{n_1},q_{n_2}}(t-x)^2)(x)\}^{\alpha/2}(L_{n_1,n_2,q_{n_1},q_{n_2}}1)(x) \\
&\times\{(L_{n_1,n_2,q_{n_1},q_{n_2}}(s-y)^2)(y)\}^{\beta/2}(L_{n_1,n_2,q_{n_1},q_{n_2}}1)(y) \\
&= M\delta^{\alpha}_{n_1,q_{n_1}}(x)\delta^{\beta}_{n_2,q_{n_2}}(y).
\end{aligned}$$

Thus, we get the desired result. □

Let $C^1(I)$ be the space of functions $f(x,y)$ whose first order partial derivatives are continuous on I.

Theorem 9. *Let $f \in C^1(I)$. Then, we have*

$$|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)| \leq ||f'_x||_{C(I)}\delta_{n_1,q_{n_1}}(x)+||f'_y||_{C(I)}\delta_{n_2,q_{n_2}}(y).$$

Proof. For $(t,s) \in I$, we have

$$f(t,s)-f(x,y) = \int_x^t f'(u,s)du + \int_y^s f'(x,v)dv.$$

Applying the operator $(L_{n_1,n_2,q_{n_1},q_{n_2}}.)(x,y)$ on both sides of above equality and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)| \leq \left(L_{n_1,n_2,q_{n_1},q_{n_2}}\left|\int_x^t f'_u(u,s)du\right|\right)(x,y) \\
&+\left(L_{n_1,n_2,q_{n_1},q_{n_2}}\left|\int_y^s f'_v(x,v)dv\right|\right)(x,y) \\
&\leq ||f'_x||_{C(I)}(L_{n_1,q_{n_1}}|t-x|)(x)+||f'_y||_{C(I)}(L_{n_2,q_{n_2}}|s-y|)(y) \\
&\leq ||f'_x||_{C(I)}\Big((L_{n_1,q_{n_1}}(t-x)^2)(x)\Big)^{1/2}+||f'_y||_{C(I)}\Big((L_{n_2,q_{n_2}}(s-y)^2)(y)\Big)^{1/2}.
\end{aligned}$$

Hence, we get the required result. □

Let $C^2(I)$ denote the space of functions $f(x,y)$ whose second order partial derivatives are continuous on I, endowed with the norm

$$||f||_{C^2(I)} = ||f||_{C(I)}+\sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C(I)}+\left\|\frac{\partial^i f}{\partial y^i}\right\|_{C(I)}\right).$$

The Peetre's K-functional of the function $f\in C(I)$ is defined by

$$\mathcal{K}(f;\delta)=\inf_{g\in C^2(I)}\{||f-g||_{C(I)}+\delta||g||_{C^2(I)}\},\ \delta>0.$$

Also, from [12, pp.192] it is known that

$$\mathcal{K}(f;\delta)\leq M\left\{\bar{\omega}_2(f;\sqrt{\delta})+min(1,\delta)||f||_{C(I)}\right\}, \tag{4.2}$$

holds for all $\delta>0$. The constant M in the above inequality is independent of δ and f and $\bar{\omega}_2(f;\sqrt{\delta})$ is the second order modulus of continuity for the bivariate case.

Theorem 10. *Let $f\in C(I)$. Then for all $n_1,n_2\in\mathbb{N}$ and each $(x,y)\in I$ there exists a constant $\mathcal{C}>0$ such that*

$$|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)|\leq\mathcal{C}\left\{\bar{\omega}_2\left(f;\frac{1}{2}\sqrt{C_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)}\right)\right.$$
$$\left.+\min\left(1,\frac{C_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)}{4}\right)||f||_{C(I)}\right\}+\omega(f,\sqrt{\psi_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)}),$$

where

$$\psi_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)=\left(\frac{[n_1]_{q_{n_1}}(1+q_{n_1}x[n_1+1]_{q_{n_1}})}{[n_1+1]_{q_{n_1}}[n_1+2]_{q_{n_1}}}-x\right)^2$$
$$+\left(\frac{[n_2]_{q_{n_2}}(1+q_{n_2}y[n_2+1]_{q_{n_2}})}{[n_2+1]_{q_{n_2}}[n_2+2]_{q_{n_2}}}-y\right)^2,$$

and

$$C_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)=\delta^2_{n_1,q_{n_1}}(x)+\delta^2_{n_2,q_{n_2}}(y)+\psi_{n_1,n_2,q_{n_1},q_{n_2}}(x,y).$$

Proof. First, we define the auxiliary operator

$$(L^*_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)=f(x,y)+(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)$$
$$-f\left(\frac{[n_1]_{q_{n_1}}(1+q_{n_1}x[n_1+1]_{q_{n_1}})}{[n_1+1]_{q_{n_1}}[n_1+2]_{q_{n_1}}},\frac{[n_2]_{q_{n_2}}(1+q_{n_2}y[n_2+1]_{q_{n_2}})}{[n_2+1]_{q_{n_2}}[n_2+2]_{q_{n_2}}}\right).$$

Applying Lemma 4 it is obvious that

$(L^*_{n_1,n_2,q_{n_1},q_{n_2}}1)(x,y)=1,\ (L^*_{n_1,n_2,q_{n_1},q_{n_2}}t)(x,y)=x,\ (L^*_{n_1,n_2,q_{n_1},q_{n_2}}s)(x,y)=y,$

and hence

$$(L^*_{n_1,n_2,q_{n_1},q_{n_2}}(t-x))(x,y)=0=(L^*_{n_1,n_2,q_{n_1},q_{n_2}}(s-y))(x,y).$$

Further, for $f\in C(I)$, we have (see also Lemma 3)

$$||L^*_{n_1,n_2,q_{n_1},q_{n_2}}f||_{C(I)}\leq 3||f||_{C(I)}.$$

Let $g \in C^2(I)$ be arbitrary. By Taylor's theorem, we have

$$g(t,s)-g(x,y) = \frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du + \frac{\partial g(x,y)}{\partial y}(s-y)+\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv.$$

Applying the operator $L^*_{n_1,n_2,q_{n_1},q_{n_2}}$ on both sides of the above equation, we get

$$(L^*_{n_1,n_2,q_{n_1},q_{n_2}}g)(x,y)-g(x,y) = \left(L^*_{n_1,n_2,q_{n_1},q_{n_2}}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du\right)\right)(x,y) + \left(L^*_{n_1,n_2,q_{n_1},q_{n_2}}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv\right)\right)(x,y).$$

For simplicity, we set

$$z_1 = \frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}[n_1+2]_{q_{n_1}}}\left(1+q_{n_1}[n_1+1]_{q_{n_1}}x\right)$$

and

$$z_2 = \frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}[n_2+2]_{q_{n_2}}}\left(1+q_{n_2}[n_2+1]_{q_{n_2}}y\right).$$

Then,

$$\begin{aligned}
&|(L^*_{n_1,n_2,q_{n_1},q_{n_2}}g(t,s))(x,y)-g(x,y)| \\
&\le L_{n_1,n_2,q_{n_1},q_{n_2}}\left(\left|\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du\right|\right)(x,y)+\left|\int_x^{z_1}\left|(z_1-u)\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right| \\
&+\left|\int_y^{z_2}\left|(z_2-v)\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|+L_{n_1,n_2,q_{n_1},q_{n_2}}\left(\left|\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv\right|\right)(x,y) \\
&\le \left\{(L_{n_1,n_2,q_{n_1},q_{n_2}}(t-x)^2)(x,y)+(z_1-x)^2\right\}||g||_{C^2(I)} \\
&+\left\{(L_{n_1,n_2,q_{n_1},q_{n_2}}(s-y)^2)(x,y)+(z_2-y)^2\right\}||g||_{C^2(I)} \\
&=\left(\delta^2_{n_1,q_{n_1}}(x)+\delta^2_{n_2,q_{n_2}}(y)+\psi_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)\right)||g||_{C^2(I)} \\
&= C_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)||g||_{C^2(I)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&|(L_{n_1,n_2,q_{n_1},q_{n_2}}f(t,s))(x,y)-f(x,y)| \le |(L^*_{n_1,n_2,q_{n_1},q_{n_2}}\{f-g\})(x,y)| \\
&+|g(x,y)-f(x,y)|+|(L^*_{n_1,n_2,q_{n_1},q_{n_2}}g)(x,y)-g(x,y)|+|f(z_1,z_2)-f(x,y)|
\end{aligned}$$

$$\leq \left(4||f-g||+C_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)||g||_{C^2(I)}\right)+\omega\left(f;\sqrt{\psi_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)}\right).$$

Taking the infimum on the right side over all $g \in C^2(I)$, we get

$$|(L_{n_1,n_2,q_{n_1},q_{n_2}}f(t,s))(x,y)-f(x,y)| \leq 4K_2\left(f,\frac{C_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)}{4}\right)$$
$$+\omega\left(f;\sqrt{\psi_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)}\right).$$

Now, using the relation (4.2), we get the desired result. □

5. Construction of GBS operators of q-Bernstein-Schurer-Durmeyer type

Bögel was the first person who proposed the concepts of B-continuous and B-differentiable functions in his papers [9] and [10]. After that Dobrescu and Matei [17] showed that any B-continuous function on a bounded interval can be uniformly approximated by the boolean sum of bivariate generalization of Bernstein polynomials. For detailed history of the work in this direction we refer the reader to some of the papers in this direction (cf. [3], [6], [7], [19], [34] and [35] etc.).

Next, we recall some definitions which will be used in this section.

A function $f : I \longrightarrow$ R is called a B-continuous (Bögel continuous) at $(x_0,y_0) \in I$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x,y);(x_0,y_0)] = 0,$$

where $\Delta f[(x,y);(x_0,y_0)]$ denotes the mixed difference defined by

$$\Delta f[(x,y);(x_0,y_0)] = f(x,y)-f(x,y_0)-f(x_0,y)+f(x_0,y_0).$$

The space of all B-continuous functions is denoted by $C_b(I)$.

A function $f : I \longrightarrow \mathbb{R}$ is called a B-differentiable (Bögel differentiable) function at $(x_0,y_0) \in I$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(x,y);(x_0,y_0)]}{(x-x_0)(y-y_0)}$$

exists finitely.

The limit is called the B-differential of f at the point (x_0,y_0) and is denoted by $D_B f(x_0,y_0)$ and the space of all B-differentiable functions is denoted by $D_b(I)$ i.e $D_b(I)=\{f : I \to \mathbb{R} | D_B f \text{ exists}\}$. The function $f : I \to \mathbb{R}$ is called B-bounded on I if there exists $M > 0$ such that$|\Delta f\,[(t,s);(x,y)]| \leq M$, for every $(x,y),(t,s) \in I$.

Throughout this paper $B_b\,(I)$, denotes all B-bounded functions on $I \to \mathbb{R}$ equipped with the norm

$$\|f\|_B = \sup_{(x,y),(t,s)\in I} |\Delta f\,[(t,s);(x,y)]|\,.$$

The mixed modulus of smoothness of $f \in C_b(I)$ is a function $\omega_{mixed} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined as

$$\omega_{mixed}(f;\delta_1,\delta_2) := \sup\{|\Delta f[(t,s);(x,y)]| : |x-t| < \delta_1, |y-s| < \delta_2\},$$

for all $(x,y),(t,s) \in I$ and for any $(\delta_1,\delta_2) \in (0,\infty)\times(0,\infty)$.

Let us assume that $\lim_{n_j\to\infty} q_{n_j} = 1$, $j = 1,2$. For $f \in C_b(I)$, the GBS operator $T_{n_1,n_2,q_{n_1},q_{n_2}}$ associated to $L_{n_1,n_2,q_{n_1},q_{n_2}}$ is defined as:

$$\begin{aligned}(T_{n_1,n_2,q_{n_1},q_{n_2}} f)(x,y) &= \frac{[n_1+1]^2_{q_{n_1}}}{[n_1]_{q_{n_1}}} \frac{[n_2+1]^2_{q_{n_2}}}{[n_2]_{q_{n_2}}} \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} q_{n_1}^{-k_1} q_{n_2}^{-k_2} p_{n_1,k_1,q_{n_1}}(x)\, p_{n_2,k_2,q_{n_2}}(y)\times \\ &\int_{t=0}^{\frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}} \int_{s=0}^{\frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}} p_{n_1,k_1,q_{n_1}}(q_{n_1}t)\, p_{n_2,k_2,q_{n_2}}(q_{n_2}s)[f(x,s)+f(t,y)-f(t,s)]\, d_{q_{n_1}}t\, d_{q_{n_2}}s. \end{aligned} \tag{5.1}$$

Example 2. For $n_1 = n_2 = 10$, $q_{n_1} = \dfrac{n_1}{n_1+1}$, $q_{n_2} = 1 - \dfrac{1}{n_2}$, the comparison of convergence of the function $L_{n_1,n_2,q_{n_1},q_{n_2}} f$ (green) and its GBS type function $T_{n_1,n_2,q_{n_1},q_{n_2}} f$ (yellow) to $f(x,y) = x^2y^2 + 2x^2y - 3y^2$ (red) is illustrated in Figure 3. We remark that the operator $T_{n_1,n_2,q_{n_1},q_{n_2}}$ gives a better approximation than the operator $L_{n_1,n_2,q_{n_1},q_{n_2}}$. In the Table 1 we computed the error of approximation for $L_{n_1,n_2,q_{n_1},q_{n_2}}$ and $T_{n_1,n_2,q_{n_1},q_{n_2}}$ at certain points.

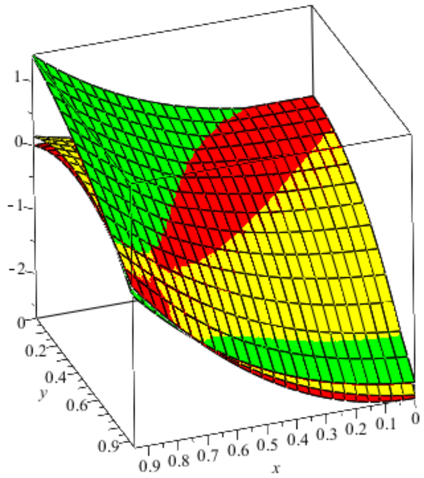

FIGURE 3. The convergence of $L_{n_1,n_2,q_{n_1},q_{n_2}} f$ and $T_{n_1,n_2,q_{n_1},q_{n_2}} f$ to f (red f, green $L_{n_1,n_2,q_{n_1},q_{n_2}}$ and yellow $T_{n_1,n_2,q_{n_1},q_{n_2}}$)

TABLE 1. Error of approximation for $L_{n_1,n_2,q_{n_1},q_{n_2}}$ and $T_{n_1,n_2,q_{n_1},q_{n_2}}$

x	y	$\lvert(L_{n_1,n_2,q_{n_1},q_{n_2}} f)(x,y) - f(x,y)\rvert$	$\lvert(T_{n_1,n_2,q_{n_1},q_{n_2}} f)(x,y) - f(x,y)\rvert$
0.9	0.1	1.0104585870	0.0811695820
0.8	0.1	0.7228289936	0.0026123070
0.7	0.1	0.4853680787	0.0472137716
0.6	0.1	0.2931805555	0.0732162679
0.9	0.2	0.7971775100	0.0721465470
0.8	0.2	0.5423273555	0.0022888760
0.7	0.2	0.3335672866	0.0406177704
0.6	0.2	0.1660020136	0.0614810095
0.4	0.4	0.1556621432	0.0204867702
0.5	0.5	0.1116013761	0.0147209150
0.6	0.6	0.0665209683	0.0077961258
0.2	0.9	0.1925811440	0.1020977850
0.1	0.9	0.1899526260	0.0854533490

Theorem 11. *For every $f \in C_b(I)$, at each point $(x,y) \in I$, we have*

$$|(T_{n_1,n_2,q_{n_1},q_{n_2}} f)(x,y) - f(x,y)| \leq 4\omega_{mixed}(f; \delta_{n_1,q_{n_1}}(x), \delta_{n_2,q_{n_2}}(y)).$$

Proof. By the definition of $\omega_{mixed}(f;\delta_{n_1},\delta_{n_2})$ and using the elementary inequality

$$\omega_{mixed}(f;\lambda_1\delta_{n_1},\lambda_2\delta_{n_2}) \leq (1+\lambda_1)(1+\lambda_2)\,\omega_{mixed}(f;\delta_{n_1},\delta_{n_2}),$$

where $\lambda_1,\lambda_2>0$ and $\delta_{n_1},\delta_{n_2}>0$, we may write

$$\begin{aligned}|\Delta f[(t,s);(x,y)]| &\leq \omega_{mixed}(f;|t-x|,|s-y|)\\ &\leq \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\omega_{mixed}(f;\delta_1,\delta_2),\end{aligned} \tag{5.2}$$

for every $(x,y),(t,s)\in I$ and for any $\delta_1,\delta_2>0$. From the definition of $\Delta f[(t,s);(x,y)]$ we get

$$f(x,s)+f(t,y)-f(t,s)=f(x,y)-\Delta f[(t,s);(x,y)].$$

On applying the linear positive operator $(L_{n_1,n_2,q_{n_1},q_{n_2}}.)(x,y)$ to this equality and taking into account the definition of operator $T_{n_1,n_2,q_{n_1},q_{n_2}}$ given by (5.1) we can write

$$\begin{aligned}&(T_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)\\ &=f(x,y)\,(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{00})(x,y)-(L_{n_1,n_2,q_{n_1},q_{n_2}}\Delta f[(t,s);(x,y)])\,(x,y).\end{aligned}$$

Since $(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{00})(x,y)=1$, considering the inequality (5.2) and applying the Cauchy-Schwarz inequality we obtain,

$$\begin{aligned}&|(T_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)| \leq (L_{n_1,n_2,q_{n_1},q_{n_2}}|\Delta f[(t,s);(x,y)]|)\,(x,y)\\ &\leq \Bigg(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{00})(x,y)+\delta_{n_1}^{-1}\sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}}(t-x)^2)(x,y)}\\ &+\delta_{n_2}^{-1}\sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}}(s-y)^2)(x,y)}+\delta_{n_1}^{-1}\delta_{n_2}^{-1}\sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}}(t-x)^2)(x,y)}\\ &\times\sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}}(s-y)^2)(x,y)}\Bigg)\,\omega_{mixed}(f;\delta_{n_1},\delta_{n_2}),\end{aligned}$$

from which the desired result is immediate, on choosing $\delta_{n_1}=\delta_{n_1,q_{n_1}}(x)$ and $\delta_{n_2}=\delta_{n_2,q_{n_2}}(y)$. □

For $f\in C_b(I)$, the Lipschitz class $Lip_M(\xi,\gamma)$ with $M>0$ and $\xi,\gamma\in(0,1]$ is defined as

$$\begin{aligned}Lip_M(\xi,\gamma)=\Big\{&f\in C_b(I):|\Delta f[(t,s);(x,y)]|\leq M\,|t-x|^{\xi}\,|s-y|^{\gamma},\\ &\text{for}\,(t,s),(x,y)\in I\Big\}.\end{aligned}$$

Our next theorem gives the degree of approximation for the operators $T_{n_1,n_2,q_{n_1},q_{n_2}}$ by means of the Lipschitz class of Bögel continuous functions.

Theorem 12. *Let $f \in Lip_M(\xi,\gamma)$ then we have*

$$\left|(T_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)\right| \le M\delta^{\xi}_{n_1,q_{n_1}}(x)\delta^{\gamma}_{n_2,q_{n_2}}(y),$$

for $M>0$, $\xi,\gamma\in(0,1]$.

Proof. By the definition of the operator $(T_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)$ and Lemma 4, we may write

$$\begin{aligned}(T_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y) &= (L_{n_1,n_2,q_{n_1},q_{n_2}}(f(x,s)+f(t,y)-f(t,s)))(x,y)\\ &= (L_{n_1,n_2,q_{n_1},q_{n_2}}\{f(x,y)-\Delta f[(t,s);(x,y)]\})(x,y)\\ &= f(x,y)(L_{n_1,n_2,q_{n_1},q_{n_2}}e_{00})(x,y)\\ &-(L_{n_1,n_2,q_{n_1},q_{n_2}}(\Delta f[(t,s);(x,y)]))(x,y).\end{aligned}$$

By the hypothesis, we get

$$\begin{aligned}\left|(T_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)\right| &\le (L_{n_1,n_2,q_{n_1},q_{n_2}}|\Delta f[(t,s);(x,y)]|)(x,y)\\ &\le M\left(L_{n_1,n_2,q_{n_1},q_{n_2}}|t-x|^{\xi}|s-y|^{\gamma}\right)(x,y)\\ &= M(L_{n_1,q_{n_1}}|t-x|^{\xi})(x)(L_{n_2,q_{n_2}}|s-y|^{\gamma})(y).\end{aligned}$$

Now, using the Hölder's inequality with $u_1=2/\xi, v_1=2/(2-\xi)$ and $u_2=2/\gamma, v_2=2/(2-\gamma)$, we have

$$\begin{aligned}\left|(L_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)\right| &\le M\left(\left(L_{n_1,q_{n_1}}(t-x)^2\right)(x)\right)^{\xi/2}\left((L_{n_1,q_{n_1}}e_0)(x)\right)^{(2-\xi)/2}\\ &\times(\left(L_{n_2,q_{n_2}}(s-y)^2\right)(y))^{\gamma/2}\left((L_{n_2,q_{n_2}}e_0)(y)\right)^{(2-\gamma)/2}.\end{aligned}$$

Now, applying Lemma 1, we obtain the degree of local approximation for B-continuous functions belonging to $Lip_M(\xi,\gamma)$. □

Theorem 13. *Let the function $f\in D_b(I)$ with $D_Bf\in B(I)$ and $\lim_{n_j\to\infty}q_{n_j}^{n_j}=a_j\in[0,1)$, $j=1,2$. Then, there exists $M>0$ such that for each $(x,y)\in I$, we have*

$$\begin{aligned}&|(T_{n_1,n_2,q_{n_1},q_{n_2}}f)(x,y)-f(x,y)|\\ &\le \frac{M}{[n_1]^{1/2}_{q_{n_1}}[n_2]^{1/2}_{q_{n_2}}}\left(||D_Bf||_\infty+\omega_{mixed}(D_Bf;[n_1]^{-1/2}_{q_{n_1}},[n_2]^{-1/2}_{q_{n_2}})\right).\end{aligned}$$

Proof. We need the following mean value theorem for B-differentiable functions [11, Proposition 17c]

$$\Delta f[(t,s);(x,y)]=(t-x)(s-y)D_Bf(\xi,\eta),\ f\in D_b(I),$$

with ξ between x and t and η between y and s.

It is also evident that

$$D_B f(\xi,\eta) = \Delta D_B f(\xi,\eta) + D_B f(\xi,y) + D_B f(x,\eta) - D_B f(x,y).$$

Since $D_B f \in B(I)$, by above relations, we can write

$$\begin{aligned}
&|(L_{n_1,n_2,q_{n_1},q_{n_2}} \Delta f[(t,s);(x,y)])(x,y)| \\
&= |(L_{n_1,n_2,q_{n_1},q_{n_2}} (t-x)(s-y) D_B f(\xi,\eta))(x,y)| \\
&\leq (L_{n_1,n_2,q_{n_1},q_{n_2}} |t-x||s-y||\Delta D_B f(\xi,\eta)|)(x,y) \\
&+ (L_{n_1,n_2,q_{n_1},q_{n_2}} |t-x||s-y|(|D_B f(\xi,y)| \\
&+ |D_B f(x,\eta)| + |D_B f(x,y)|)(x,y) \\
&\leq (L_{n_1,n_2,q_{n_1},q_{n_2}} (|t-x||s-y| \omega_{mixed}(D_B f; |\xi-x|,|\eta-y|) \\
&+ 3\,||D_B f||_\infty \,(L_{n_1,n_2,q_{n_1},q_{n_2}} |t-x||s-y|)(x,y).
\end{aligned}$$

Using the property (5.2) of ω_{mixed}, and applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&|(T_{n_1,n_2,q_{n_1},q_{n_2}} f)(x,y) - f(x,y)| = |(L_{n_1,n_2,q_{n_1},q_{n_2}} \Delta f[(t,s);(x,y)])(x,y)| \\
&\leq 3||D_B f||_\infty \sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}} (t-x)^2(s-y)^2)(x,y)} \\
&+ \Big(L_{n_1,n_2,q_{n_1},q_{n_2}} |t-x||s-y|)(x,y) + \delta_{n_1}^{-1} (L_{n_1,n_2,q_{n_1},q_{n_2}} (t-x)^2 |s-y|)(x,y) \\
&+ \delta_{n_2}^{-1} (L_{n_1,n_2,q_{n_1},q_{n_2}} |t-x|(s-y)^2)(x,y) \\
&\quad + \delta_{n_1}^{-1}\delta_{n_2}^{-1} (L_{n_1,n_2,q_{n_1},q_{n_2}} (t-x)^2(s-y)^2)(x,y) \Big) \omega_{mixed}(D_B f; \delta_{n_1}, \delta_{n_2}) \\
&\quad \leq 3||D_B f||_\infty \sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}} ((t-x)^2(s-y)^2)(x,y)} \\
&\quad + \Big(\sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}} (t-x)^2(s-y)^2)(x,y)} \\
&\quad + \delta_{n_1}^{-1} \sqrt{(L_{n_1,n_2,;q_{n_1},q_{n_2}} (t-x)^4(s-y)^2)(x,y)} \\
&\quad + \delta_{n_2}^{-1} \sqrt{(L_{n_1,n_2,q_{n_1},q_{n_2}} (t-x)^2(s-y)^4)(x,y)} \qquad (5.3) \\
&\quad + \delta_{n_1}^{-1}\delta_{n_2}^{-1} (L_{n_1,n_2,q_{n_1},q_{n_2}} (t-x)^2(s-y)^2)(x,y) \Big) \omega_{mixed}(D_B f; \delta_{n_1}, \delta_{n_2}).
\end{aligned}$$

We observe that for $(x,y),(t,s) \in I^2$ and $i,j \in \{1,2\}$

$$\begin{aligned}
&(L_{n_1,n_2,q_{n_1},q_{n_2}} (t-x)^{2i}(s-y)^{2j})(x,y) \qquad (5.4) \\
&= (L_{n_1,q_{n_1}} (t-x)^{2i})(x)(L_{n_2,q_{n_2}} (s-y)^{2j})(y).
\end{aligned}$$

If (q_{n_j}), $j=1,2$, are sequences in $(0,1)$ such that $q_{n_j}\to 1$ and $q_{n_j}^{n_j}\to a_j$ $(0\le a_j<1)$, as $n_j\to\infty$, using Remark 1, it follows

$$(L_{n_j,q_{n_j}}(t-x)^2)(x)\le \frac{M_1}{[n_j]_{q_{n_j}}} \tag{5.5}$$

$$(L_{n_j,q_{n_j}}(t-x)^4)(x)\le \frac{M_2}{[n_j]_{q_{n_j}}^2}, \tag{5.6}$$

for some constants $M_1,M_2>0$.
Let $\delta_{n_1}=\frac{1}{[n_1]_{q_{n_1}}^{1/2}}$, and $\delta_{n_2}=\frac{1}{[n_2]_{q_{n_2}}^{1/2}}$.

Then, combining (5.3)-(5.6), we obtain the desired result. □

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