Miskolc Mathematical Notes
Vol. 18 (2017), No. 2, pp. 639–642
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.2256

FINITE DIFFERENCES OF EULER'S ZETA FUNCTION

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Received 28 February, 2017

Abstract. We find accurate approximations for certain finite differences of the Euler zeta function, $\zeta(x)$.

2010 *Mathematics Subject Classification:* 41A60; 11M06; 26D15

Keywords: Riemann zeta function, finite differences

1. Introduction

The Euler zeta function is the precursor of the Riemann zeta function [1, 5]. It is defined for real values greater than 1 by

$$\zeta(x) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots, \qquad x > 1. \tag{1.1}$$

As is well known, this infinite series converges for any real $x > 1$, and we have

$$\zeta(x) > \zeta(x+1) \qquad \text{and} \qquad \lim_{x\to\infty} \zeta(x) = 1.$$

A finite difference is an expression of the form

$$f(x+b) - f(x+a). \tag{1.2}$$

The use of finite differences can be traced back to the end of the 16^{th} century. Newton's interpolation formula [4] is an early occurrence of finite differences. For more on the subject we refer the reader to [3].

For many applications, it is useful to consider three particular forms of (1.2): forward, backward, and central differences. A forward difference is an expression of the form

$$\Delta_h f(x) = f(x+h) - f(x).$$

A backward difference is given by

$$\nabla_h f(x) = f(x) - f(x-h).$$

This work was partially supported by a grant from the Simons Foundation (#245997 to Cristina Ballantine).

Finally, a central difference takes the form

$$\delta_h f(x) = f(x+h/2) - f(x-h/2).$$

Iterating, we obtain the n-th order forward, backward, and central differences:

$$\begin{aligned}
\Delta_h^n f(x) &= \Delta_h[\Delta_h^{n-1} f(x)] \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+(n-i)h), \\
\nabla_h^n f(x) &= \nabla_h[\nabla_h^{n-1} f(x)] \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x-ih), \\
\delta_h^n f(x) &= \delta_h[\delta_h^{n-1} f(x)] \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\left(x+\left(\frac{n}{2}-i\right)h\right).
\end{aligned}$$

We note that

$$\nabla_h^n f(x) = \Delta_h^n f(x-nh) = \delta_h^n f(x-nh/2). \tag{1.3}$$

Dividing the finite difference (1.2) by $b-a$, leads to a difference quotient. Thus, finite differences can be used to approximate derivatives or higher order derivatives. Finite difference methods are fundamental in finding numerical solution to differential equations, especially boundary value problems. Higher order finite differences can also be used to express certain recurrences.

In this note, motivated by these facts, we provide accurate approximations for the n-th order forward, backward, and central finite differences of the classical Euler zeta function. We note that Flajolet and Vepstas [2] also explored finite differences of values of the zeta function at integers. Our main result, which is valid for all real x sufficiently large and is of a different flavor than the results of [2], is given below.

Theorem 1. *Let n be a positive integer. For any real numbers $h > 0$ and $x > 1+nh$,*

$$\frac{(2^h-1)^n}{2^x} < (-1)^n \nabla_h^n \zeta(x) < \frac{(2^h-1)^n}{2^x} + R_{n,h}(x),$$

where

$$R_{n,h}(x) = \frac{1}{2^x} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{2^{ih+1}}{x-ih-1}.$$

Using (1.3), similar inequalities can be derived for the finite differences $\Delta_h^k \zeta(x)$ and $\delta_h^k \zeta(x)$. Clearly,

$$\lim_{x\to\infty} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{2^{ih+1}}{x-ih-1} = 0.$$

Thus, we obtain the following asymptotic formulas for the n-th order forward, backward and central differences of Euler's zeta function.

Corollary 1. *Let n be a positive integer. For any real number $h > 0$,*

a) $2^x \Delta_h^n \zeta(x) \sim \left(\dfrac{1-2^h}{2^h}\right)^n$,

b) $2^x \nabla_h^n \zeta(x) \sim (1-2^h)^n$,

c) $2^x \delta_h^n \zeta(x) \sim \left(\dfrac{1-2^h}{\sqrt{2^h}}\right)^n$,

as $x \to \infty$.

The case $n = h = 1$ of this corollary can be used to derive the following (equivalent) limits:

$$\lim_{x\to\infty} 2^x(\zeta(x) - \zeta(x+1)) = \frac{1}{2},$$
$$\lim_{x\to\infty} 2^x(\zeta(x-1) - \zeta(x)) = 1,$$

and

$$\lim_{x\to\infty} 2^x \left(\zeta\left(x-\frac{1}{2}\right) - \zeta\left(x+\frac{1}{2}\right)\right) = \frac{1}{\sqrt{2}}.$$

2. Proof of Theorem 1

We have

$$\begin{aligned}
(-1)^n 2^x \nabla_h^n \zeta(x) &= 2^x \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \zeta(x-ih) \\
&= 2^x \sum_{k=2}^{\infty} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{1}{k^{x-ih}} \\
&= 2^x \sum_{k=2}^{\infty} \left(\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} k^{ih} \right) \frac{1}{k^x} \\
&= 2^x \sum_{k=2}^{\infty} \frac{(k^h-1)^n}{k^x}
\end{aligned}$$

$$= (2^h - 1)^n + 2^x \sum_{k=3}^{\infty} \frac{(k^h - 1)^n}{k^x}.$$

The sum $\sum_{k=3}^{\infty} \frac{(k^h-1)^n}{k^x}$ is a right Riemann sum for the function $g(t) = \frac{(t^h-1)^n}{t^x}$ on the interval $[2,\infty)$. Since $x > 1 + nh$, the function $g(t)$ is positive and decreasing. Thus,

$$\sum_{k=3}^{\infty} \frac{(k^h - 1)^n}{k^x} < \int_2^{\infty} \frac{(t^h - 1)^n}{t^x} dt,$$

On the other hand,

$$\begin{aligned} \int_2^{\infty} \frac{(t^h - 1)^n}{t^x} dt &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \int_2^{\infty} t^{ih-x} dt \\ &= \frac{1}{2^x} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{2^{ih+1}}{x - ih - 1}. \end{aligned}$$

Then, for $x > 1 + nh$, we deduce the double inequality

$$\begin{aligned} (2^h - 1)^n &< (-1)^n 2^x \nabla_h^n \zeta(x) \\ &< (2^h - 1)^n + \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{2^{ih+1}}{x - ih - 1}. \end{aligned}$$

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