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NONOSCILLATION OF EVEN ORDER EULER TYPE HALF-LINEAR DIFFERENCE EQUATIONS

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Abstract. We establish nonoscillation criteria for the even order half-linear difference equation of Euler type

$$\sum_{l=0}^{n}(-1)^{n-l}\beta_{n-l}\Delta^{n-l}\left(k^{(\alpha-lp)}\Phi\left(\Delta^{n-l}x_{k+l}\right)\right)=0, \quad \beta_n:=1,$$

where $\Phi(t):=|t|^{p-1}\operatorname{sgn}t,\ p\in(1,\infty)$, $n\in\mathbb{N}$, $k^{(\beta)}$ denotes the falling factorial power (for $\beta\in\mathbb{R}$) and $\alpha,\beta_0,\beta_1,\dots,\beta_{n-1}$ are real constants. For the two-term equation

$$(-1)^n\Delta^n\left(k^{(\alpha)}\Phi\left(\Delta^n x_k\right)\right)+\beta_0 k^{(\alpha-np)}\Phi(x_{k+n})=0$$

we establish the constant $\gamma_{n,p,\alpha}$ such that the two-term equation is nonoscillatory if $\beta_0>-\gamma_{n,p,\alpha}$. The criteria are derived using the variational technique and they are further extended via the theory of regularly varying sequences.

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1. INTRODUCTION

We consider the $2n$-th order half-linear difference equation

$$\sum_{l=0}^{n}(-1)^{n-l}\Delta^{n-l}\left(r_k^{[n-l]}\Phi\left(\Delta^{n-l}x_{k+l}\right)\right)=0, \tag{1.1}$$

where $n\in\mathbb{N}$, $\Phi(t):=|t|^{p-1}\operatorname{sgn}t$ is the odd power function, the real number p is such that $p>1$, $\left\{r_k^{[j]}\right\}_{k=1}^{\infty}$ is real-valued sequence for every $j\in\{0,1,\dots,n\}$ and $r_k^{[n]}\neq 0$ for $k\in\mathbb{N}$. The phrase "half-linear" reflects the fact that the solution space

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is homogenous, but not additive. Further we consider the energy functional

$$\mathcal{F}_n(\{y_k\};N,\infty) := \sum_{k=N}^{\infty}\left[\sum_{l=0}^{n} r_k^{[n-l]}\left|\Delta^{n-l}y_{k+l}\right|^p\right]$$

associated with equation (1.1), where $N \in \mathbb{N}$ and a sequence $\{y_k\}$ is from the set $\mathcal{D}_n(N)$ (definition of this set will be recalled later). We focus on a special cases of (1.1), namely on Euler type equation (1.2) and its extension (3.27). Consider the Euler type half-linear difference equation

$$\sum_{l=0}^{n}(-1)^{n-l}\beta_{n-l}\Delta^{n-l}\left(k^{(\alpha-lp)}\Phi\left(\Delta^{n-l}x_{k+l}\right)\right) = 0, \quad \beta_n := 1, \tag{1.2}$$

where $\beta_0, \beta_1, \dots, \beta_{n-1}$ are real numbers and $\alpha \in \mathbb{R}\setminus\{p-1, 2p-1, \dots, np-1\}$. For $k \in \mathbb{N}$ and $\beta \in \mathbb{R}$ the symbol $k^{(\beta)}$ denotes so-called *falling factorial power* (see [14, Definition 2.3]), which can be expressed as

$$k^{(\beta)} = \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}$$

for $k \in \mathbb{N}\setminus\{\beta - i \mid i \in \mathbb{N}\}$ and $\beta \in \mathbb{R}$, where Γ denotes the Gamma function defined for $t \in \mathbb{R}\setminus\{0,-1,-2,\dots\}$. Recall that for $t \in (0,\infty)$ we have

$$\Gamma(t) := \int_0^{\infty} \mathrm{e}^{-s}s^{t-1}\,\mathrm{d}s.$$

Furthermore, recall that for sequences $\{a_k\}$ and $\{b_k\}$ of non-zero real numbers, we write $a_k \sim b_k$ as $k \to \infty$ and say that the sequences $\{a_k\}$ and $\{b_k\}$ are *asymptotically equivalent*, if $\lim_{k\to\infty} a_k/b_k = 1$. Now, from Stirling's formula

$$\lim_{t\to\infty}\frac{\Gamma(t+1)}{\left(\frac{t}{\mathrm{e}}\right)^t\sqrt{2\pi t}} = 1$$

we get the known relation $\frac{\Gamma(k+\beta)}{\Gamma(k)} \sim k^{\beta}$ as $k \to \infty$ (for $\beta \in \mathbb{R}$), hence,

$$k^{\beta} \sim k^{(\beta)} \quad \text{as} \quad k \to \infty. \tag{1.3}$$

In this article, we focus on getting conditions (for the coeficients $\beta_0, \beta_1, \dots, \beta_{n-1}$) which guarantee the nonoscillation of equation (1.2). We use the variational technique which is at disposal for general equation (1.1) by the results of article [3]. The main result of [3] is formulated (in a slightly different form) in Theorem 3 in this paper.

Our motivation comes mainly from the results for the continuous version of equation (1.2), i.e., for the differential equation

$$\sum_{l=0}^{n}(-1)^{n-l}\beta_{n-l}\left(t^{\alpha-lp}\Phi\left(x^{(n-l)}\right)\right)^{(n-l)} = 0, \quad \beta_n := 1, \tag{1.4}$$

where $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, np-1\}$. Denote

$$\gamma_{n,p,\alpha} := \prod_{i=1}^{n} \left(\frac{|ip-1-\alpha|}{p} \right)^p .$$

Consider also the special cases

$$\gamma_{p,\alpha} := \gamma_{1,p,\alpha} = \left(\frac{|p-1-\alpha|}{p} \right)^p \quad \text{and} \quad \gamma_{n,2,\alpha} = \frac{1}{4^n} \prod_{i=1}^{n} (2i-1-\alpha)^2 .$$

The following results are known criteria for special cases of equation (1.4). In [7, Theorem 1.4.4] it is shown that the second order equation

$$-\left(t^{\alpha} \Phi\left(x'\right)\right)' + \frac{\gamma}{t^{p-\alpha}} \Phi(x) = 0 \tag{1.5}$$

is nonoscillatory if and only if $\gamma + \gamma_{1,p,\alpha} \geq 0$ (for $\alpha = 0$ see the older result in [9]). For equation (1.5) the number $-\gamma_{1,p,\alpha}$ is the *critical constant*, i.e., the constant which is the "borderline" (as for the parameter γ) between oscillation and nonoscillation of equation (1.5). For the two-term $2n$-th order equation

$$(-1)^n \left(t^{\alpha} \Phi\left(x^{(n)}\right)\right)^{(n)} + \gamma t^{\alpha - np} \Phi(x) = 0 \tag{1.6}$$

we have so far only the following implication. Equation (1.6) is nonoscillatory if $\gamma + \gamma_{n,p,\alpha} > 0$ (for general α see [4, Theorem 3.2], for $\alpha = 0$ see the older result in [7, Theorem 9.4.5]).

If $p = 2$ then $\Phi(t) = |t| \operatorname{sgn} t = t$. Therefore, equation (1.4) with $p = 2$ is the linear differential equation. As a special case we get the two-term linear equation

$$(-1)^n \left(t^{\alpha} x^{(n)}\right)^{(n)} + \gamma t^{\alpha - np} x = 0. \tag{1.7}$$

In [13, page 132] (for $\alpha = 0$ see [11, pages 97-98]) it is shown that equation (1.7) is nonoscillatory if and only if $\gamma + \gamma_{n,2,\alpha} \geq 0$, i.e., the number $-\gamma_{n,2,\alpha}$ is the critical constant for equation (1.7).

The variational technique is at disposal (see [7, Theorem 9.4.4]) also for the continuous version of equation (1.1), i.e., for the differential equation

$$\sum_{l=0}^{n} (-1)^{n-l} \left(r_{n-l}(t) \Phi\left(x^{(n-l)}\right)\right)^{(n-l)} = 0. \tag{1.8}$$

In [4], we use the variational principle together with the Wirtinger inequality, which enables us to show positivity of the energy functional associated with equation (1.8). In the discrete case (in this article), to show positivity of the energy functional associated with equation (1.2) we use inequalities obtained by using Lemma 5. This approach is different from the one in the continuous case, because we do not use any discrete Wirtinger type inequality.

In the linear discrete case it is known the following nonoscillation criterion (see [8, Theorem 9]). The two-term linear difference equation

$$(-1)^n \Delta^n \left(k^{(\alpha)} \Delta^n x_k\right) + \frac{\gamma}{k^{(2n-\alpha)}} x_{k+n} = 0 \tag{1.9}$$

is nonoscillatory if $\gamma_{n,2,\alpha} + \gamma > 0$. In (1.9) if we take $k^{(\alpha-2n)}$ instead of $1 \big/ k^{(2n-\alpha)}$ then the proof from [8] of this criterion still works (by using (1.3)) with the same result. Note that the constant $-\gamma_{n,2,\alpha}$ is optimal (critical) for a slightly different type of Euler linear difference equation, namely for equation (4.5) with $-\gamma$ instead of $-\gamma_{n,2,\alpha}$ (see [10, Corollary 4.2]).

This paper is organized as follows. In the second section we rewrite equation (1.1) into a difference system and then we define the concept of generalized zero for the difference system and for equation (1.1) respectively. Further we define the concept of nonoscillation of equation (1.1) and we give two variational lemmas. The second section also contains two nonoscillation criteria, which plays important role in our later proofs. The end of the section is devoted to recalling basic concepts from the theory of regularly varying sequences. Section 3 presents two new nonoscillation criteria for equation (1.2) and is supplemented by remarks on a generalization via the concept of regularly varying sequence.

2. Preliminaries

In order to define the concept of nonoscillation for general half-linear equation (1.1), we need to define the concept of generalized zero for this equation. Further, in order to define the concept of generalized zero for equation (1.1), we transform equation (1.1) into a Hamiltonian type difference system.

Similar observations as in the previous paragraph hold also for the continuous case, i.e. for equation (1.8) (instead of the concept of generalized zero we get the concept of zero point of multiplicity n from the transformation of (1.8) into a Hamiltonian type differential system; see [5]).

The following paragraphs (which lead as to the definition of generalized zero) are modeled according to the article [3]. Let $\{x_k\}$ be a solution of equation (1.1). Set

$$\begin{aligned} u_k^{[i]} &= \Delta^{i-1} x_{k+n-i}, \\ v_k^{[n]} &= r_k^{[n]} \Phi\left(\Delta^n x_k\right), \\ v_k^{[n-j]} &= -\Delta v_k^{[n-j+1]} + r_k^{[n-j]} \Phi\left(\Delta^{n-j} x_{k+j}\right) \end{aligned} \tag{2.1}$$

for $k \in \mathbb{N}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n-1$. Denote the column vectors

$$u_k = \left(u_k^{[i]}\right)_{i=1}^n \quad \text{and} \quad v_k = \left(v_k^{[i]}\right)_{i=1}^n$$

for $k \in \mathbb{N}$, and note that the number $v_k^{[i]}$ can be expressed as

$$v_k^{[i]} = \sum_{l=0}^{n-i} (-1)^{n-i-l} \Delta^{n-i-l} \left(r_k^{[n-l]} \Phi \left(\Delta^{n-l} x_{k+l} \right) \right)$$

for $k \in \mathbb{N}$ and $i = 1, 2, \dots, n$. Then the sequence $\{(u_k, v_k)\}$ is a solution of the Hamiltonian type difference system

$$\Delta u_k = A u_{k+1} + B_k \Phi^{-1}(v_k), \quad \Delta v_k = C_k \Phi(u_{k+1}) - A^T v_k, \tag{2.2}$$

where $\{B_k\}$ and $\{C_k\}$ are square matrix sequences of order n such that

$$B_k = \operatorname{diag}\left\{0, 0, \dots, 0, 1 \Big/ \Phi^{-1}\left(r_k^{[n]}\right)\right\} \quad \text{and} \quad C_k = \operatorname{diag}\left\{r_k^{[0]}, r_k^{[1]}, \dots, r_k^{[n-1]}\right\};$$

and the matrix

$$A = \left(a_{ij}\right)_{i,j=1}^{n} \quad \text{with} \quad a_{ij} = \begin{cases} 1 & \text{for } j = i+1, \\ 0 & \text{elsewhere.} \end{cases}$$

For vector $a = (a_i)_{i=1}^n$, denote $\Phi(a) := (\Phi(a_i))_{i=1}^n$ and $\Phi^{-1}(a) := \left(\Phi^{-1}(a_i)\right)_{i=1}^n$, where $\Phi^{-1}(t) := |t|^{q-2} t$ is the inverse function of $\Phi(t)$. The constant q is the conjugate number of p, i.e., $q := \frac{p}{p-1}$.

Now, we consider the general matrix difference system

$$\Delta u_k = A_k u_{k+1} + B_k \Phi^{-1}(v_k), \quad \Delta v_k = C_k \Phi(u_{k+1}) - A_k^T v_k, \tag{2.3}$$

where $\{B_k\}$ and $\{C_k\}$ are symmetric matrix sequences and $\{I - A_k\}$ is an invertible matrix sequence (symbol I denotes the identity matrix). Let $m \in \mathbb{N}$, then we say that an interval $(m, m+1]$ contains the generalized zero of a solution $\{(u_k, v_k)\}$ of system (2.3) if

$$u_m \neq 0, \quad u_{m+1} \in \operatorname{Im}(I - A_m)^{-1} B_m \quad \text{and} \quad u_m^T B_m^{\dagger} (I - A_m) u_{m+1} \leq 0,$$

where $B_m^{\dagger}$ denotes the Moore-Penrose pseudoinverse of matrix B_m and Im denotes the image.

In order to define generalized zero for equation (1.1) we proceed as follows. Let $m \in \mathbb{N}$ and $m \geq n$. We say that a nontrivial solution $\{x_k\}$ of equation (1.1) has a generalized zero in the interval $(m, m+1]$ if the solution $\{(u_k^*, v_k^*)\}_{k=n}^{\infty}$ of corresponding system (2.2) has a generalized zero in $(m, m+1]$, where $(u_k^*, v_k^*) := (u_{k-n+1}, v_{k-n+1})$ for every $k \in \mathbb{N}$ such that $k \geq n$; and the sequences $\{u_k\}$ and $\{v_k\}$ are given by relations (2.1). The shift ensures that our definition will be the same as the one in article [3] (the same shift is used for the linear case in [1, Remark 5 (ii)]), where it is considered equation (1.1) with $x_{k+1-n+l}$ instead of x_{k+l} (such equation is equivalent to our equation (1.1)).

Now we rewrite this procedure explicitly in terms of equation (1.1). For a real vector $d = (d_i)_{i=1}^n$, we have $d \in \mathrm{Im}(I - A_m)^{-1} B_m$ if and only if there exists the vector $c = (c_i)_{i=1}^n$ such that

$d = (I - A)^{-1} B_m c$ or equivalently (by a direct computation) $d_i = c_n \Big/ \Phi^{-1}\left(r_m^{[n]}\right)$

for $i = 1, 2, \dots, n$, i.e., the vector d has equal components. Let $\{x_k\}$ be a nontrivial solution of (1.1). From the definition of $\{u_k^*\}$ and from the definition of $\{u_k\}$ we have

$$u_{m+1}^* = u_{(m+1)-n+1} = \left(\Delta^{i-1} x_{(m+1-n+1)+n-i}\right)_{i=1}^n = \left(\Delta^{i-1} x_{m-(i-2)}\right)_{i=1}^n.$$

If

$$u_{m+1}^* \in \mathrm{Im}\,(I - A_m)^{-1}\, B_m,$$

then u_{m+1}^* has equal components, i.e., we have

$$\left(x_{m+1}, \Delta x_m, \Delta^2 x_{m-1}, \dots, \Delta^{n-1} x_{m-(n-2)}\right)^T = (x_{m+1}, x_{m+1}, x_{m+1}, \dots, x_{m+1})^T.$$

Hence, $x_{m-(n-2)} = x_{m-(n-3)} = \dots = x_m = 0$. Next, for an arbitrary $i \in \{1, 2, \dots, n\}$ we have

$$\Delta^{i-1} x_{m-(i-1)} = \sum_{l=1}^{i} (-1)^{i-l} \binom{i-1}{l-1} x_{m-(i-l)},$$

which is the i-th component of u_m^*. For $i \in \{1, 2, \dots, n-1\}$ we have $x_{m-(i-1)} = x_{m-(i-2)} = \dots = x_m = 0$, hence,

$$u_m^* = \left(0, 0, \dots, 0, (-1)^{n-1} x_{m-(n-1)}\right)^T.$$

Therefore the relation $u_m^* \neq 0$ is equivalent with the relation $x_{m-(n-1)} \neq 0$. Finally, it can be shown that

$$B_m^\dagger (I - A) = \mathrm{diag}\left\{0, 0, \dots, 0, \Phi^{-1}\left(r_m^{[n]}\right)\right\}.$$

Hence,

$$u_m^{*T} B_m^\dagger (I - A) u_{m+1}^* = (-1)^{n-1} \Phi^{-1}\left(r_m^{[n]}\right) x_{m-(n-1)} x_{m+1}.$$

Definition 1. Let $m \in \mathbb{N}$ and $m \geq n$. We say that a nontrivial solution $\{x_k\}$ of equation (1.1) has a *generalized zero* in the interval $(m, m+1]$ if $x_{m-(n-2)} = x_{m-(n-3)} = \dots = x_m = 0$ (for $n > 1$),

$$x_{m-(n-1)} \neq 0 \quad \text{and} \quad (-1)^{n-1} r_m^{[n]} x_{m-(n-1)} x_{m+1} \leq 0. \tag{2.4}$$

Note that in relation (2.4) the constant $r_m^{[n]}$ appears, but only the sign of $r_m^{[n]}$ is important. This definition agrees with Hartman's one in [12] and it also matches the definition in [1] for the linear case.

Definition 2. We say that equation (1.1) is *nonoscillatory* (at infinity) if there exists $N \in \mathbb{N}$ such that $N \geq n$ and no nontrivial solution of equation (1.1) has two or more generalized zeros in (N, ∞). Otherwise, equation (1.1) is called *oscillatory*.

Note that Definition 1 and Definition 2 agree with the definitions in [15] for the second order equation (equation (1.1) with $n = 1$).

Before we formulate variational lemmas, we make another note on the linear case (equation (1.1) with $p = 2$). System (2.3) with $p = 2$ reduces to the general linear Hamiltonian system. For linear Hamiltonian systems we have the Reid type roundabout theorem (see [1]) which guarantees equivalence between nonoscillation of equation (1.1) with $p = 2$, positivity of the energy functional associated with equation (1.1) with $p = 2$ and solvability of the so-called Riccati matrix equation associated with equation (1.1) with $p = 2$.

Similar remarks as in the previous paragraph hold also in the continuous case, i.e., for the equation

$$\sum_{l=0}^{n}(-1)^{n-l}\left(r_{n-l}(t)x^{(n-l)}\right)^{(n-l)} = 0$$

(see [18]).

Nonoscillation of an equation is equivalent to positivity of its energy functional also for equations (1.1) and (1.8) if $n = 1$, i.e., for the second order half-linear equations (the proof for (1.1) with $n = 1$ is given in [15]).

Next, we formulate the variational lemma for the second order equation

$$-\Delta\left(r_k^{[1]}\Phi(\Delta x_k)\right) + r_k^{[0]}\Phi(x_{k+1}) = 0, \tag{2.5}$$

which is a special case of equation (1.1). Denote

$$\begin{aligned}\mathcal{D}_n(N) := \{\{y_k\}_{k=1}^{\infty} \mid y_k = 0 \text{ for } k \leq N+n-1,\\ \exists m \in \mathbb{N} \text{ such that } m > N+n-1 \text{ and } y_k = 0 \text{ for } k \geq m\}\end{aligned}$$

for $N \in \mathbb{N}$. Note that $\mathcal{D}_n(N) = \mathcal{D}_1(N+n-1)$ for $N \in \mathbb{N}$, and $\mathcal{D}_n(N_2) \subseteq \mathcal{D}_n(N_1)$ for $N_1, N_2 \in \mathbb{N}$ such that $N_1 \leq N_2$.

Lemma 1 ([15])**.** *Equation (2.5) is nonoscillatory if and only if there exists $N \in \mathbb{N}$ such that*

$$\mathcal{F}_1(\{y_k\}; N, \infty) = \sum_{k=N}^{\infty}\left[r_k^{[1]}|\Delta y_k|^p + r_k^{[0]}|y_{k+1}|^p\right]$$

is positive for every nontrivial sequence $\{y_k\} \in \mathcal{D}_1(N)$.

For second order equation (2.5) we have the following two nonoscillation criteria, which will be applied to equation (1.2) with $n = 1$.

Theorem 1 (O. Došlý, P. Řehák [6]). *Suppose that* $\sum^{\infty} r_k^{[0]}$ *is convergent,* $r_k^{[1]} > 0$ *for large* k*,* $\sum^{\infty}\left(r_k^{[1]}\right)^{1-q} = \infty$ *and*

$$\lim_{k\to\infty} \frac{\left(r_k^{[1]}\right)^{1-q}}{\sum^{k-1}\left(r_j^{[1]}\right)^{1-q}} = 0. \tag{2.6}$$

Denote

$$\mathcal{A}_k := \left(\sum^{k-1}\left(r_j^{[1]}\right)^{1-q}\right)^{p-1}\left(\sum_{j=k}^{\infty} r_j^{[0]}\right).$$

If

$$\liminf_{k\to\infty} \mathcal{A}_k > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \quad \text{and} \quad \limsup_{k\to\infty} \mathcal{A}_k < \frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

then equation (2.5) *is nonoscillatory.*

Note that if $r_k^{[1]} > 0$ for large k, $\sum^{\infty}\left(r_k^{[1]}\right)^{1-q} = \infty$ and $\sum^{\infty} r_k^{[0]} = -\infty$, then equation (2.5) is oscillatory (see [15, Theorem 4] or [7, Theorem 8.2.14]).

Further note that if $r_k^{[0]} \le 0$ for large k, then the constant $-\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$ is critical. Indeed, the condition

$$\limsup_{k\to\infty} \mathcal{A}_k < -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

with assumptions $r_k^{[1]} > 0$ for large k, $r_k^{[0]} \le 0$ for large k,

$$\sum^{\infty}\left(r_k^{[1]}\right)^{1-q} = \infty, \quad \sum^{\infty} r_k^{[0]} \text{ is convergent} \quad \text{and} \quad (2.6)$$

implies that equation (2.5) is oscillatory (see [7, Theorem 8.2.15]).

Theorem 2 (O. Došlý, P. Řehák [6]). *Suppose that* $r_k^{[1]} > 0$ *for large* k*,* $\sum^{\infty}\left(r_k^{[1]}\right)^{1-q} < \infty$ *and*

$$\lim_{k\to\infty} \frac{\left(r_k^{[1]}\right)^{1-q}}{\sum_{j=k}^{\infty}\left(r_j^{[1]}\right)^{1-q}} = 0. \tag{2.7}$$

Denote

$$\mathcal{B}_k := \left(\sum_{j=k}^{\infty}\left(r_j^{[1]}\right)^{1-q}\right)^{p-1}\left(\sum^{k-1} r_j^{[0]}\right).$$

If

$$\liminf_{k\to\infty} \mathcal{B}_k > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \quad \text{and} \quad \limsup_{k\to\infty} \mathcal{B}_k < \frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

then equation (2.5) *is nonoscillatory.*

In case of general (even order) equation (1.1), the variational relation presented later in Lemma 2 is obtained from the following theorem.

Theorem 3 (O. Došlý [3])**.** *Let $N_0 \in \mathbb{N}$ be such that $N_0 \ge n$ and let $N_1 \in \mathbb{N}$ be such that $N_1 \ge N_0 + n + 1$. If the interval $(N_0, N_1 + 1]$ contains two generalized zeros of a solution $\{x_k\}$ of equation* (1.1)*, then there exists a nontrivial sequence $\{y_k\} \in \mathcal{D}_1(N_0)$ such that $y_k = 0$ for $k \ge N_1 - n + 2$ and*

$$\mathcal{F}_n(\{y_k\}; N_0 - n + 1, N_1 - n + 1) = \sum_{k=N_0-n+1}^{N_1-n+1} \left[\sum_{l=0}^{n} r_k^{[n-l]} \left|\Delta^{n-l} y_{k+l}\right|^p\right] \le 0.$$

Theorem 3 can be reformulated as follows. If we set $N = N_0 - n + 1$ and $X = N_1 - n + 1$, then the condition $N_0 \ge n$ means that $N \in \mathbb{N}$ and the condition $N_1 \ge N_0 + n + 1$ is reduced to $X \ge N + n + 1$. If we rewrite Theorem 3 in terms of such N and X then we can easily obtain the following variational lemma (in Definition 2 we take $N + n - 1$ (instead of N), which is obviously greater than or equal to n).

Lemma 2. *Equation* (1.1) *is nonoscillatory if there exists $N \in \mathbb{N}$ such that*

$$\mathcal{F}_n(\{y_k\}; N, \infty) = \sum_{k=N}^{\infty} \left[\sum_{l=0}^{n} r_k^{[n-l]} \left|\Delta^{n-l} y_{k+l}\right|^p\right]$$

is positive for any nontrivial sequence $\{y_k\} \in \mathcal{D}_n(N)$.

Next we recall the definition of regularly varying sequences and some of their selected properties (see [2, 17]).

Definition 3. Let $\vartheta \in \mathbb{R}$. A positive sequence $\{a_k\}$ is said to be *regularly varying (at infinity) of index ϑ*, if

$$\lim_{k\to\infty} \frac{a_{[\lambda k]}}{a_k} = \lambda^{\vartheta}$$

for every positive real λ, where $[t]$ denotes the integer part of t. The set of all regularly varying sequences of index ϑ is denoted by $\mathcal{RV}(\vartheta)$. Further $\mathcal{RV} := \bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}(\vartheta)$ and $\mathcal{SV} := \mathcal{RV}(0)$. Sequences from the set $\mathcal{SV}$ are called *slowly varying*.

Lemma 3. *The following statements hold.*

(a) *A sequence $\{a_k\}$ belongs to $\mathcal{RV}(\vartheta)$ if and only if there exists $\{L_k\} \in \mathcal{SV}$ such that $a_k = k^{\vartheta} L_k$ for $k \in \mathbb{N}$.*

(b) *If $\{L_k\} \in \mathcal{SV}$ and $\{K_k\}$ is such that $K_k \sim L_k$ as $k \to \infty$, then $\{K_k\} \in \mathcal{SV}$.*

(c) *A sequence* $\{a_k\} \in \mathcal{RV}(\vartheta)$ *if and only if there exists* $\{L_k\} \in \mathcal{SV}$ *such that* $a_k = k^{(\vartheta)} L_k$ *for* $k \in \mathbb{N}$.

(d) *If* $\{a_k\} \in \mathcal{RV}(\vartheta)$*, then* $\{b_k\} \in \mathcal{RV}(\vartheta\beta)$ *for every* $\beta \in \mathbb{R}$*, where* $b_k = a_k^{\beta}$ *for* $k \in \mathbb{N}$.

(e) *Let* $\{a_k\} \in \mathcal{RV}(\vartheta_1)$ *and* $\{b_k\} \in \mathcal{RV}(\vartheta_2)$*. Then* $\{a_k b_k\} \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$.

Now, we give a few examples of the slowly varying sequences. Trivial examples are positive constant sequences. A typical example is the sequence $\{\ln k\}$. Further the sequences $\{K_k\}$, $\{L_k\}$ and $\{M_k\}$ are slowly varying, where

$$K_k = \prod_{i=1}^{n} (\ln_i k)^{\mu_i} \text{ for } k \in \mathbb{N}, \text{ where } \ln_1 k := \ln k, \ln_{i+1} k := \ln(\ln_i k)$$
$$\text{and } \mu_i \in \mathbb{R} \text{ for } i \in \mathbb{N};$$
$$L_k = \exp\left\{ \prod_{i=1}^{n} (\ln_i k)^{\nu_i} \right\} \text{ for } k \in \mathbb{N}, \text{ where } \nu_i \in (0,1) \text{ for } i \in \{1,2,\dots,n\};$$
$$M_k = \frac{\ln \Gamma(k)}{k} \text{ for } k \in \mathbb{N}.$$

The next statement is very important for the extension of our results (Theorem 5 and Theorem 6) from the next section via the theory of regularly varying sequences.

Theorem 4 (Karamata type theorem [2, 17]). *Let* $\{L_k\} \in \mathcal{SV}$*. Then*

$$\sum_{j=1}^{k-1} j^{\vartheta} L_j \sim \frac{k^{\vartheta+1}}{\vartheta+1} L_k \quad \textit{as} \quad k \to \infty$$

for every real ϑ *such that* $\vartheta > -1$*; and*

$$\sum_{j=k}^{\infty} j^{\vartheta} L_j \sim -\frac{k^{\vartheta+1}}{\vartheta+1} L_k \quad \textit{as} \quad k \to \infty$$

for every real ϑ *such that* $\vartheta < -1$.

3. Nonoscillation criteria

Consider equation (1.2) with $n = 1$ and $\beta_0 = \gamma$, i.e., the equation

$$-\Delta\left(k^{(\alpha)}\Phi\left(\Delta x_k\right)\right) + \gamma k^{(\alpha-p)}\Phi\left(x_{k+1}\right) = 0. \tag{3.1}$$

Lemma 4. *Let* $\alpha \neq p-1$ *and* $\gamma + \gamma_{p,\alpha} > 0$*. Then equation* (3.1) *is nonoscillatory.*

Proof. If $\gamma \geq 0$, then nonoscillation of equation (3.1) immediately follows from Lemma 1. Alternatively, we can use the Sturm comparison theorem (see [6, 15]) for equation (2.5). By this theorem, from nonoscillation of the equation $-\Delta\left(k^{(\alpha)}\Phi\left(\Delta x_k\right)\right) = 0$ it follows nonoscillation of equation (3.1) for $\gamma \geq 0$.

Let $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$. We often use relation (1.3) without referring to it. We prove the cases $\alpha < p-1$ and $\alpha > p-1$ separately. In the both cases we have $\left(k^{(\alpha)}\right)^{1-q} \sim k^{\alpha(1-q)}$ as $k \to \infty$, because

$$\lim_{k\to\infty} \frac{\left(k^{(\alpha)}\right)^{1-q}}{k^{\alpha(1-q)}} = \lim_{k\to\infty} \left(\frac{k^{(\alpha)}}{k^{\alpha}}\right)^{1-q} = 1.$$

Furthermore we have $k^{\alpha(1-q)} \sim k^{(\alpha(1-q))}$ for $k \to \infty$. Hence, $\left(k^{(\alpha)}\right)^{1-q} \sim k^{(\alpha(1-q))}$ for $k \to \infty$. It holds that $\alpha(1-q) = -\frac{\alpha}{p-1}$, hence, $\alpha(1-q) > -1$ for $\alpha < p-1$ and $\alpha(1-q) < -1$ for $\alpha > p-1$.

Let $\alpha < p-1$. We verify the assumptions of Theorem 1 for equation (3.1). We have

$$\lim_{k\to\infty} \frac{\left(k^{(\alpha)}\right)^{1-q}}{\sum^{k-1} \left(j^{(\alpha)}\right)^{1-q}} = \lim_{k\to\infty} \frac{k^{(\alpha(1-q))}}{\sum^{k-1} j^{\alpha(1-q)}} = \alpha(1-q) \lim_{k\to\infty} \frac{k^{(\alpha(1-q)-1)}}{k^{\alpha(1-q)}} = 0. \tag{3.2}$$

Indeed, from the discrete l'Hospital rule we get $\sum^{k-1}\left(j^{(\alpha)}\right)^{1-q} \sim \sum^{k-1} j^{\alpha(1-q)}$ as $k \to \infty$, because $\lim_{k\to\infty} \sum^{k-1} j^{\alpha(1-q)} = \infty$ for $\alpha(1-q) > -1$.

Further, from the limit comparison test,

$$\sum^{\infty} \left(k^{(\alpha)}\right)^{1-q} = \infty \quad \text{and} \quad \sum^{\infty} \gamma k^{(\alpha-p)} \quad \text{is convergent.} \tag{3.3}$$

Now we compute $\lim_{k\to\infty} \mathcal{A}_k$. We have

$$\sum_{j=k}^{\infty} \gamma j^{(\alpha-p)} = -\gamma \frac{k^{(\alpha-p+1)}}{\alpha-p+1} \quad \text{and} \quad \sum^{k-1} \left(j^{(\alpha)}\right)^{1-q} \sim \frac{k^{(\alpha(1-q)+1)}}{\alpha(1-q)+1} \text{ as } k \to \infty,$$

where the latter relation is obtained using the discrete l'Hospital rule. Hence,

$$\lim_{k\to\infty} \mathcal{A}_k = \lim_{k\to\infty} \left[\left(\frac{k^{(\alpha(1-q)+1)}}{\alpha(1-q)+1}\right)^{p-1}\right](-\gamma)\frac{k^{(\alpha-p+1)}}{\alpha-p+1} = \frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p}. \tag{3.4}$$

The inequalities

$$\frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p} > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \quad \text{and} \quad \frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p} < \frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

are equivalent (for $\alpha < p-1$) with the inequalities $\gamma > -\gamma_{p,\alpha}$ and $\gamma < (2p-1)\gamma_{p,\alpha}$, respectively, and the inequality $\gamma < (2p-1)\gamma_{p,\alpha}$ holds for an arbitrary $\gamma \in (-\infty, 0)$. Hence, by Theorem 1, equation (3.1) is nonoscillatory for $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$ and $\alpha < p-1$.

Let $\alpha > p-1$. Similarly as in the previous case, we verify the assumptions of Theorem 2 for equation (3.1). By the limit comparison test we get

$$\sum^{\infty}\left(k^{(\alpha)}\right)^{1-q} < \infty,$$

and by the l'Hospital rule we have

$$\lim_{k\to\infty}\frac{\left(k^{(\alpha)}\right)^{1-q}}{\sum_{j=k}^{\infty}\left(j^{(\alpha)}\right)^{1-q}} = \lim_{k\to\infty}\frac{k^{(\alpha(1-q))}}{\sum_{j=k}^{\infty} j^{\alpha(1-q)}} = -\alpha(1-q)\lim_{k\to\infty}\frac{k^{(\alpha(1-q)-1)}}{k^{\alpha(1-q)}} = 0.$$

Now we compute $\lim_{k\to\infty}\mathcal{B}_k$. By the l'Hospital rule, we can verify that

$$\sum_{j=k}^{\infty}\left(j^{(\alpha)}\right)^{1-q} \sim -\frac{k^{(\alpha(1-q)+1)}}{\alpha(1-q)+1} \quad \text{and} \quad \sum^{k-1}\gamma j^{(\alpha-p)} \sim \gamma\frac{k^{(\alpha-p+1)}}{\alpha-p+1}$$

as $k\to\infty$. Hence,

$$\lim_{k\to\infty}\mathcal{B}_k = \lim_{k\to\infty}\left[\left(\frac{-k^{(\alpha(1-q)+1)}}{\alpha(1-q)+1}\right)^{p-1}\right]\gamma\frac{k^{(\alpha-p+1)}}{\alpha-p+1} = \frac{\gamma(p-1)^{p-1}}{(\alpha-p+1)^p}.$$

The inequalities

$$\frac{\gamma(p-1)^{p-1}}{(\alpha-p+1)^p} > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \quad \text{and} \quad \frac{\gamma(p-1)^{p-1}}{(\alpha-p+1)^p} < \frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

are equivalent (for $\alpha > p-1$) with the inequalities $\gamma > -\gamma_{p,\alpha}$ and $\gamma < (2p-1)\gamma_{p,\alpha}$, respectively. Hence, by Theorem 2, equation (3.1) is nonoscillatory for $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$ and $\alpha > p-1$. The proof is completed. □

Note that we can prove also the following oscillation complement of Lemma 4 (see the text after Theorem 1). If $\alpha \neq p-1$ and $\gamma + \gamma_{p,\alpha} < 0$, then equation (3.1) is oscillatory. We will not present the proof of this result in details since we do not need it.

Remark 1. The previous lemma can be generalized in the following sense. Consider the equation

$$-\Delta\left(f_k\Phi\left(\Delta x_k\right)\right) + \gamma g_k\Phi\left(x_{k+1}\right) = 0, \tag{3.5}$$

where $\{f_k\} \in \mathcal{RV}(\alpha)$, $\{g_k\} \in \mathcal{RV}(\alpha-p)$, $\alpha \neq p-1$, $\gamma \in \mathbb{R}$ and the sequences $\{f_k\}$ and $\{g_k\}$ take the forms

$$f_k = k^{\alpha}K_k,\ k\in\mathbb{N} \quad \text{and} \quad g_k = k^{\alpha-p}L_k,\ k\in\mathbb{N} \tag{3.6}$$

where $\{K_k\}, \{L_k\} \in \mathcal{SV}$ (see Lemma 3). We show that if $\gamma + \gamma_{p,\alpha} > 0$ and $K_k \sim L_k$ as $k\to\infty$, then equation (3.5) is nonoscillatory. We use the proof of Lemma 4 with the following modifications. The paragraphs with relations (3.2), (3.3), (3.4) are replaced by the paragraphs with relations (3.7), (3.8), (3.9) below, respectively.

Let $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$ and $\alpha < p-1$. It holds that

$$\lim_{k\to\infty} \frac{k^{\alpha(1-q)} K_k^{1-q}}{\sum^{k-1} j^{\alpha(1-q)} K_j^{1-q}} = (\alpha(1-q)+1) \lim_{k\to\infty} \frac{k^{\alpha(1-q)} K_k^{1-q}}{k^{\alpha(1-q)+1} K_k^{1-q}} = 0. \tag{3.7}$$

Indeed, we have $\left\{K_k^{1-q}\right\} \in \mathcal{SV}$ and $\alpha(1-q) > -1$ for $\alpha < p-1$, hence, by Theorem 4, we get $\sum^{k-1} j^{\alpha(1-q)} K_j^{1-q} \sim (\alpha(1-q)+1)^{-1} k^{\alpha(1-q)+1} K_k^{1-q}$ as $k\to\infty$. Further, by Theorem 4,

$$\sum^{\infty} k^{\alpha(1-q)} K_k^{1-q} = \infty, \quad \sum^{\infty} \gamma k^{\alpha-p} L_k \quad \text{is convergent} \tag{3.8}$$

and $\sum_{j=k}^{\infty} \gamma j^{\alpha-p} L_j \sim -\gamma \frac{k^{\alpha-p+1}}{\alpha-p+1} L_k$ as $k\to\infty$ for $\alpha - p < -1$.
Now we compute $\lim_{k\to\infty} \mathcal{A}_k$. We have

$$\lim_{k\to\infty} \mathcal{A}_k = \lim_{k\to\infty} \left[\left(\frac{k^{\alpha(1-q)+1} K_k^{1-q}}{\alpha(1-q)+1}\right)^{p-1}\right] (-\gamma) \frac{k^{\alpha-p+1}}{\alpha-p+1} L_k = \frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p}. \tag{3.9}$$

Similarly in the case $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$ and $\alpha > p-1$.

Now we formulate the lemma, which help us to estimate the summands from the energy functional associated with equation (1.2).

Lemma 5. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, mp-1\}$ and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-1}$ be arbitrary positive real numbers. Then there exists $N \in \mathbb{N}$ such that*

$$\sum_{k=N}^{\infty} \left[k^{(\alpha-jp)} \left|\Delta^{m-j} y_{k+j}\right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right) k^{(\alpha-(j+1)p)} \left|\Delta^{m-j-1} y_{k+j+1}\right|^p\right] \tag{3.10}$$

is positive for every nontrivial sequence $\{y_k\} \in \mathcal{D}_m(N)$ and for every $j \in \{0, 1, \dots, m-1\}$.

Proof. First, consider the equation

$$-\Delta\left(k^{(\beta)} \Phi(\Delta x_k)\right) + \gamma k^{(\beta-p)} \Phi(x_{k+1}) = 0, \tag{3.11}$$

where $\beta \in \mathbb{R}$, $p \in (1, \infty)$ and $\gamma \in \mathbb{R}$.

By Lemma 4, equation (3.11) is nonoscillatory if $\gamma + \gamma_{p,\beta} > 0$ and $\beta \neq p-1$. Choose an arbitrary $\varepsilon \in (0, \infty)$, $j \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R} \setminus \{p(j+1)-1\}$. Set $\beta = \alpha - jp$ and $\gamma = \varepsilon - \gamma_{p,\alpha-jp}$, then equation (3.11) becomes the equation

$$-\Delta\left(k^{(\alpha-jp)} \Phi(\Delta x_k)\right) + (\varepsilon - \gamma_{p,\alpha-jp}) k^{(\alpha-(j+1)p)} \Phi(x_{k+1}) = 0 \tag{3.12}$$

and we have

$$\gamma + \gamma_{p,\beta} = \varepsilon - \gamma_{p,\alpha-jp} + \gamma_{p,\alpha-jp} = \varepsilon > 0 \quad \text{and} \quad \beta = \alpha - jp \neq p-1.$$

Hence, by Lemma 4, equation (3.12) is nonoscillatory.

Now, let $m \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \ldots, mp-1\}$ and $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1}$ be positive real numbers. Equation (3.11) with $\beta = \alpha - jp$ and $\gamma = \varepsilon_j - \gamma_{p,\alpha-jp}$, i.e., the equation

$$-\Delta\left(k^{(\alpha-jp)}\Phi(\Delta x_k)\right) + (\varepsilon_j - \gamma_{p,\alpha-jp})k^{(\alpha-(j+1)p)}\Phi(x_{k+1}) = 0$$

is nonoscillatory for any $j \in \{0, 1, \ldots, m-1\}$. By Lemma 1, for every $j \in \{0, 1, \ldots, m-1\}$ there exists $N_j \in \mathbb{N}$ such that the energy functional

$$\sum_{k=N_j}^{\infty} \left[k^{(\alpha-jp)}|\Delta z_k|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}|z_{k+1}|^p\right]$$

is positive for every nontrivial $\{z_k\} \in \mathcal{D}_1(N_j)$.

Denote $N = \max\{N_0, N_1, \ldots, N_{m-1}\}$, then we have $\mathcal{D}_1(N) \subseteq \mathcal{D}_1(N_j)$ for $j = 0, 1, \ldots, m-1$. Note that for $\{z_k\} \in \mathcal{D}_1(N)$ we have $z_k = 0$ for $k \in \{1, 2, \ldots, N\}$. Hence,

$$\sum_{k=N_j}^{N-1} \left[k^{(\alpha-jp)}|\Delta z_k|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}|z_{k+1}|^p\right] = 0$$

for every $\{z_k\} \in \mathcal{D}_1(N)$ and for every $j \in \{0, 1, \ldots, m-1\}$ such that $N_j \leq N-1$, i.e.,

$$\sum_{k=N}^{\infty} \left[k^{(\alpha-jp)}|\Delta z_k|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}|z_{k+1}|^p\right] > 0$$

for every nontrivial $\{z_k\} \in \mathcal{D}_1(N)$ and for every $j \in \{0, 1, \ldots, m-1\}$.

Choose an arbitrary $j \in \{0, 1, \ldots, m-1\}$ and an arbitrary nontrivial $\{y_k\} \in \mathcal{D}_m(N)$. Note that $y_1 = y_2 = \ldots = y_{N+j} = \ldots = y_{N+m-1} = 0$. Then the sequence $\{z_k\}$, defined by the relation $z_k = \Delta^{m-j-1}y_{k+j}$ for $k \in \mathbb{N}$, is nontrivial and it belongs to the set $\mathcal{D}_1(N)$. Hence,

$$\sum_{k=N}^{\infty} \left[k^{(\alpha-jp)}\left|\Delta^{m-j}y_{k+j}\right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}\left|\Delta^{m-j-1}y_{k+j+1}\right|^p\right] > 0$$

for every nontrivial $\{y_k\} \in \mathcal{D}_m(N)$ and for every $j \in \{0, 1, \ldots, m-1\}$. □

Remark 2. Due to Remark 1, Lemma 5 can be generalized in the following way. Expression (3.10) is replaced by

$$\sum_{k=N}^{\infty} \left[k^{(\alpha-jp)}L_k^{[m-j]}\left|\Delta^{m-j}y_{k+j}\right|^p\right.$$

$$+\left(\varepsilon_j-\gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}L_k^{[m-(j+1)]}\left|\Delta^{m-j-1}y_{k+j+1}\right|^p\Big],$$

where $\left\{L_k^{[i]}\right\}\in\mathcal{SV}$ for $i=0,1,\dots,m$ and $L_k^{[m-j]}\sim L_k^{[m-(j+1)]}$ as $k\to\infty$ for $j=0,1,\dots,m-1$.

In the proof, equation (3.11) is replaced by

$$-\Delta\left(k^{(\beta)}K_k\Phi\left(\Delta x_k\right)\right)+\gamma k^{(\beta-p)}L_k\Phi\left(x_{k+1}\right)=0, \tag{3.13}$$

where $\beta\in\mathbb{R}$, $p\in(1,\infty)$, $\gamma\in\mathbb{R}$ and the sequences $\{K_k\}$ and $\{L_k\}$ are from the set $\mathcal{SV}$ such that $K_k\sim L_k$ as $k\to\infty$. Then it is easy to rewrite the rest of the proof. Note that equation (3.13) is equation (3.5), where the sequences

$$\left\{\frac{k^{(\alpha)}}{k^\alpha}K_k\right\}_{n=1}^{\infty}\quad\text{and}\quad\left\{\frac{k^{(\alpha-p)}}{k^{\alpha-p}}L_k\right\}_{n=1}^{\infty}$$

are slowly varying components of $\{f_k\}$ and $\{g_k\}$ (see Lemma 3), respectively.

Consider equation (1.2) with $\beta_1=\beta_2=\ldots=\beta_{n-1}=0$ and $\beta_0=\gamma$, i.e., the two-term equation

$$(-1)^n\Delta^n\left(k^{(\alpha)}\Phi\left(\Delta^n x_k\right)\right)+\gamma k^{(\alpha-np)}\Phi(x_{k+n})=0. \tag{3.14}$$

Theorem 5. *If*

$$\gamma+\gamma_{n,p,\alpha}>0$$

and $\alpha\in\mathbb{R}\setminus\{p-1,2p-1,\dots,np-1\}$*, then equation* (3.14) *is nonoscillatory.*

Proof. By Lemma 2 it is sufficient to prove that there exists $N\in\mathbb{N}$ such that

$$\sum_{k=N}^{\infty}\left[k^{(\alpha)}\left|\Delta^n y_k\right|^p+\gamma k^{(\alpha-np)}\left|y_{k+n}\right|^p\right]>0 \tag{3.15}$$

for every nontrival $\{y_k\}\in\mathcal{D}_n(N)$.

To prove inequality (3.15) we use inequalities obtained via Lemma 5. Therefore, first we determine a set of $n-1$ positive real numbers. Recall that we have

$$\gamma_{p,\alpha}=\left(\frac{|p-1-\alpha|}{p}\right)^p,\quad\gamma_{p,\alpha-lp}=\left(\frac{|(l+1)p-1-\alpha|}{p}\right)^p\ \text{and}$$

$$\gamma_{n,p,\alpha}=\prod_{l=0}^{n-1}\gamma_{p,\alpha-lp}$$

for $l=0,1,\dots,n-1$. Let $\varepsilon\in(0,\infty)$ be such that

$$\varepsilon<\min\left\{\gamma_{n,p,\alpha},\gamma+\gamma_{n,p,\alpha}\right\}. \tag{3.16}$$

Define real numbers $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2n-2}$ by the recurrence relations

$$\varepsilon_{2l-1} = \frac{\varepsilon_{2l+1}}{2\gamma_{p,\alpha-lp}}, \qquad \varepsilon_{2l} = \frac{\varepsilon_{2l+1}\gamma_{p,\alpha-lp}}{2\gamma_{l+1,p,\alpha} - \varepsilon_{2l+1}},$$

$$\varepsilon_{2(n-1)-1} = \frac{\varepsilon}{2\gamma_{p,\alpha-(n-1)p}}, \qquad \varepsilon_{2(n-1)} = \frac{\varepsilon\gamma_{p,\alpha-(n-1)p}}{2\gamma_{n,p,\alpha} - \varepsilon}$$

for $l = 1, 2, \dots, n-2$. Condition (3.16) guarantees that the inequalities

$$\gamma_{p,\alpha-lp} > \varepsilon_{2l} > 0, \quad \gamma_{l,p,\alpha} > \varepsilon_{2l-1} > 0$$

hold for $l = 1, 2, \dots, n-1$. Indeed, we have

$$\begin{aligned}
\gamma_{n,p,\alpha} > \varepsilon > 0 \quad &\text{implies} \quad \gamma_{p,\alpha-(n-1)p} > \varepsilon_{2(n-1)} > 0 \quad \text{and} \\
& \qquad\qquad \gamma_{n-1,p,\alpha} > \varepsilon_{2(n-1)-1} > 0; \\
\gamma_{n-1,p,\alpha} > \varepsilon_{2n-3} > 0 \quad &\text{implies} \quad \gamma_{p,\alpha-(n-2)p} > \varepsilon_{2(n-2)} > 0 \quad \text{and} \\
& \qquad\qquad \gamma_{n-2,p,\alpha} > \varepsilon_{2(n-2)-1} > 0; \\
&\vdots \\
\gamma_{3,p,\alpha} > \varepsilon_5 > 0 \quad &\text{implies} \quad \gamma_{p,\alpha-2p} > \varepsilon_4 > 0 \quad \text{and} \\
& \qquad\qquad \gamma_{2,p,\alpha} > \varepsilon_3 > 0; \\
\gamma_{2,p,\alpha} > \varepsilon_3 > 0 \quad &\text{implies} \quad \gamma_{p,\alpha-p} > \varepsilon_2 > 0 \quad \text{and} \\
& \qquad\qquad \gamma_{1,p,\alpha} = \gamma_{p,\alpha} > \varepsilon_1 > 0.
\end{aligned}$$

Now we use Lemma 5. We have $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, np-1\}$. Denote $H = \{\varepsilon_1\} \cup \{\varepsilon_{2i} \,|\, i = 1, 2, \dots, n-1\}$. By Lemma 5, for the elements of the set H there exists $N \in \mathbb{N}$ such that for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$ we have

$$\sum_{k=N}^{\infty} \Big[k^{(\alpha)} |\Delta^n y_k|^p + (\varepsilon_1 - \gamma_{p,\alpha}) k^{(\alpha-p)} \big|\Delta^{n-1} y_{k+1}\big|^p \Big] > 0 \tag{3.17}$$

and

$$\sum_{k=N}^{\infty} \Big[k^{(\alpha-jp)} \Big|\Delta^{n-j} y_{k+j}\Big|^p + (\varepsilon_{2j} - \gamma_{p,\alpha-jp}) k^{(\alpha-(j+1)p)} \Big|\Delta^{n-j-1} y_{k+j+1}\Big|^p \Big] > 0 \tag{3.18}$$

for $j = 1, 2, \dots, n-1$.

By a direct computation we can easily verify that for $l = 1, 2, \dots, n-2$ we have

$$(\gamma_{l,p,\alpha} - \varepsilon_{2l-1})(\gamma_{p,\alpha-lp} - \varepsilon_{2l}) = \gamma_{l+1,p,\alpha} - \varepsilon_{2(l+1)-1} \tag{3.19}$$

and

$$(\gamma_{n-1,p,\alpha} - \varepsilon_{2n-3})(\gamma_{p,\alpha-(n-1)p} - \varepsilon_{2n-2}) = \gamma_{n,p,\alpha} - \varepsilon. \tag{3.20}$$

Now we are ready to prove inequality (3.15). Among others, we use the relations

$$\gamma_{l,p,\alpha} > \varepsilon_{2l-1} \quad \text{and} \quad \gamma_{n,p,\alpha} - \varepsilon > -\gamma \tag{3.21}$$

for $l = 1,2,\dots,n-1$. We have

$$
\begin{aligned}
\sum_{k=N}^{\infty} k^{(\alpha)} |\Delta^n y_k|^p &\overset{(3.17)}{>} \left(\gamma_{p,\alpha} - \varepsilon_1\right) \sum_{k=N}^{\infty} k^{(\alpha-p)} \left|\Delta^{n-1} y_{k+1}\right|^p \\
&\overset{(3.18),(3.21)}{>} \left(\gamma_{p,\alpha} - \varepsilon_1\right)\left(\gamma_{p,\alpha-p} - \varepsilon_2\right) \sum_{k=N}^{\infty} k^{(\alpha-2p)} \left|\Delta^{n-2} y_{k+2}\right|^p \\
&\overset{(3.19)}{=} \left(\gamma_{2,p,\alpha} - \varepsilon_3\right) \sum_{k=N}^{\infty} k^{(\alpha-2p)} \left|\Delta^{n-2} y_{k+2}\right|^p \\
&\overset{(3.18),(3.21)}{>} \left(\gamma_{2,p,\alpha} - \varepsilon_3\right)\left(\gamma_{p,\alpha-2p} - \varepsilon_4\right) \sum_{k=N}^{\infty} k^{(\alpha-3p)} \left|\Delta^{n-3} y_{k+3}\right|^p \\
&\overset{(3.19)}{=} \left(\gamma_{3,p,\alpha} - \varepsilon_5\right) \sum_{k=N}^{\infty} k^{(\alpha-3p)} \left|\Delta^{n-3} y_{k+3}\right|^p \\
&\ \ \vdots \\
&\overset{(3.18),(3.21)}{>} \left(\gamma_{n-1,p,\alpha} - \varepsilon_{2n-3}\right)\left(\gamma_{p,\alpha-(n-1)p} - \varepsilon_{2n-2}\right) \sum_{k=N}^{\infty} k^{(\alpha-np)} |y_{k+n}|^p \\
&\overset{(3.20)}{=} \left(\gamma_{n,p,\alpha} - \varepsilon\right) \sum_{k=N}^{\infty} k^{(\alpha-np)} |y_{k+n}|^p \\
&\overset{(3.21)}{>} -\gamma \sum_{k=N}^{\infty} k^{(\alpha-np)} |y_{k+n}|^p
\end{aligned}
$$

for every nontrivial $\{y_k\} \in \mathscr{D}_n(N)$. □

Note that the constant $-\gamma_{n,p,\alpha}$ is optimal in the case $n = 1$ of equation (3.14) (see the paragraph below the proof of Lemma 4).

Consider equation (1.2), i.e., the full-term equation

$$
\begin{aligned}
&(-1)^n \Delta^n \left(k^{(\alpha)} \Phi\left(\Delta^n x_k\right)\right) + (-1)^{n-1} \beta_{n-1} \Delta^{n-1} \left(k^{(\alpha-p)} \Phi\left(\Delta^{n-1} x_{k+1}\right)\right) + \dots \\
&\dots - \beta_1 \Delta \left(k^{(\alpha-(n-1)p)} \Phi\left(\Delta x_{k+n-1}\right)\right) + \beta_0 k^{(\alpha-np)} \Phi\left(x_{k+n}\right) = 0.
\end{aligned}
$$

The technique of the previous proof can be also used to obtain a criterion for equation (1.2). In fact, the criterion for the two-term equation is a special case of the criterion for the full-term equation.

The following notation greatly simplifies the formulation of the next theorem. Denote

$$\gamma(1) := \gamma_{p,\alpha} + \beta_{n-1} \quad \text{and} \quad \gamma(k+1) := \gamma(k)\gamma_{p,\alpha-kp} + \beta_{n-1-k}$$

for $k = 1, 2, \ldots, n-1$. Then

$$\begin{aligned}
\gamma(2) &= \gamma_{p,\alpha}\gamma_{p,\alpha-p} + \beta_{n-1}\gamma_{p,\alpha-p} + \beta_{n-2},\\
\gamma(3) &= \gamma_{p,\alpha}\gamma_{p,\alpha-p}\gamma_{p,\alpha-2p} + \beta_{n-1}\gamma_{p,\alpha-p}\gamma_{p,\alpha-2p} + \beta_{n-2}\gamma_{p,\alpha-2p} + \beta_{n-3},\\
&\vdots\\
\gamma(n) &= \gamma_{n,p,\alpha} + \sum_{k=1}^{n-1}\left[\prod_{l=k}^{n-1}\gamma_{p,\alpha-lp}\right]\beta_{n-k} + \beta_0.
\end{aligned}$$

Theorem 6. *If*

$$\gamma(k) > 0$$

for every $k \in \{1, 2, \ldots, n\}$ and $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \ldots, np-1\}$, then equation (1.2) is nonoscillatory.

Proof. By Lemma 2 it is sufficient to prove that there exists $N \in \mathbb{N}$ such that the energy functional

$$\begin{aligned}
&\widetilde{\mathcal{F}}_n(\{y_k\}, N, \infty)\\
&:= \sum_{k=N}^{\infty}\left[k^{(\alpha)}|\Delta^n y_k|^p + \beta_{n-1}k^{(\alpha-p)}\left|\Delta^{n-1}y_{k+1}\right|^p + \ldots + \beta_0 k^{(\alpha-np)}|y_{k+n}|^p\right]
\end{aligned} \tag{3.22}$$

associated with equation (1.2) is positive for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$.

Similarly as in the proof of Theorem 5 we use Lemma 5 here. Let $\varepsilon \in (0, \infty)$ be such that

$$\varepsilon < \gamma(n) \quad \text{and} \quad \varepsilon < 2^{n-l-1}\gamma(l)\prod_{j=l}^{n-1}\gamma_{p,\alpha-jp} \tag{3.23}$$

for every $l = 1, 2, \ldots, n-1$. Define real numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n-2}$ by the relations

$$\varepsilon_{2l-1} = \frac{\varepsilon}{2^{n-l}\prod_{j=l}^{n-1}\gamma_{p,\alpha-jp}} \quad \text{and} \quad \varepsilon_{2l} = \frac{\varepsilon\gamma_{p,\alpha-lp}}{2^{n-l}\gamma(l)\left[\prod_{j=l}^{n-1}\gamma_{p,\alpha-jp}\right] - \varepsilon}$$

for $l = 1, 2, \ldots, n-1$. Note that if $\beta_1 = \beta_2 = \ldots = \beta_{n-1} = 0$, then the constants ε_{2l-1} and ε_{2l} are the same as in the previous proof for each $l \in \{1, 2, \ldots, n-1\}$. From conditions (3.23) we have the inequalities

$$\gamma_{p,\alpha-lp} > \varepsilon_{2l} > 0 \quad \text{and} \quad \gamma(l) > \varepsilon_{2l-1} > 0$$

for $l = 1, 2, \ldots, n-1$.

Denote $H = \{\varepsilon_1\} \cup \{\varepsilon_{2i} \mid i = 1, 2, \ldots, n-1\}$. By Lemma 5, for the elements of the set H there exists $N \in \mathbb{N}$ such that the relations (3.17) and (3.18) hold for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$ and for every $j = 1, 2, \ldots, n-1$.

By direct computation we can easily verify that for $l = 1, 2, \ldots, n-2$ we have

$$(\gamma(l) - \varepsilon_{2l-1})\left(\gamma_{p,\alpha-lp} - \varepsilon_{2l}\right) = [\gamma(l+1) - \beta_{n-1-l}] - \varepsilon_{2(l+1)-1} \tag{3.24}$$

and

$$(\gamma(n-1) - \varepsilon_{2n-3})\left(\gamma_{p,\alpha-(n-1)p} - \varepsilon_{2n-2}\right) = [\gamma(n) - \beta_0] - \varepsilon. \tag{3.25}$$

Now we prove positivity of functional (3.22). Among others, we use the relations

$$\gamma(l) > \varepsilon_{2l-1} \quad \text{and} \quad \gamma(n) - \varepsilon > 0 \tag{3.26}$$

for $l = 1, 2, \ldots, n-1$. We have

$$\begin{aligned}
&\sum_{k=N}^{\infty} \left[k^{(\alpha)} |\Delta^n y_k|^p + \beta_{n-1} k^{(\alpha-p)} \left|\Delta^{n-1} y_{k+1}\right|^p\right] \\
&\overset{(3.17)}{>} \left[\left(\gamma_{p,\alpha} - \varepsilon_1\right) + \beta_{n-1}\right] \sum_{k=N}^{\infty} k^{(\alpha-p)} \left|\Delta^{n-1} y_{k+1}\right|^p \\
&= (\gamma(1) - \varepsilon_1) \sum_{k=N}^{\infty} k^{(\alpha-p)} \left|\Delta^{n-1} y_{k+1}\right|^p \\
&\overset{(3.18),(3.26)}{>} (\gamma(1) - \varepsilon_1)\left(\gamma_{p,\alpha-p} - \varepsilon_2\right) \sum_{k=N}^{\infty} k^{(\alpha-2p)} \left|\Delta^{n-2} y_{k+2}\right|^p \\
&\overset{(3.24)}{=} (\gamma(2) - \beta_{n-2} - \varepsilon_3) \sum_{k=N}^{\infty} k^{(\alpha-2p)} \left|\Delta^{n-2} y_{k+2}\right|^p.
\end{aligned}$$

for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$. Therefore,

$$\begin{aligned}
&\sum_{k=N}^{\infty} \left[k^{(\alpha)} |\Delta^n y_k|^p + \beta_{n-1} k^{(\alpha-p)} \left|\Delta^{n-1} y_{k+1}\right|^p + \beta_{n-2} k^{(\alpha-2p)} \left|\Delta^{n-2} y_{k+2}\right|^p\right] \\
&> [(\gamma(2) - \beta_{n-2} - \varepsilon_3) + \beta_{n-2}] \sum_{k=N}^{\infty} k^{(\alpha-2p)} \left|\Delta^{n-2} y_{k+2}\right|^p \\
&\overset{(3.18),(3.26)}{>} (\gamma(2) - \varepsilon_3)\left(\gamma_{p,\alpha-2p} - \varepsilon_4\right) \sum_{k=N}^{\infty} k^{(\alpha-3p)} \left|\Delta^{n-3} y_{k+3}\right|^p \\
&\overset{(3.24)}{=} (\gamma(3) - \beta_{n-3} - \varepsilon_5) \sum_{k=N}^{\infty} k^{(\alpha-3p)} \left|\Delta^{n-3} y_{k+3}\right|^p.
\end{aligned}$$

for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$. Continuing similarly step by step, we obtain

$$\sum_{k=N}^{\infty} \Big[k^{(\alpha)} |\Delta^n y_k|^p + \ldots + \beta_2 k^{(\alpha-(n-2)p)} \left|\Delta^2 y_{k+n-2}\right|^p$$

$$
\begin{aligned}
&+\beta_1 k^{(\alpha-(n-1)p)}\left|\Delta y_{k+n-1}\right|^p\Big] \\
&> [(\gamma(n-1)-\beta_1-\varepsilon_{2n-3})+\beta_1]\sum_{k=N}^{\infty} k^{(\alpha-(n-1)p)}\left|\Delta y_{k+n-1}\right|^p \\
&\overset{(3.18),(3.26)}{>} (\gamma(n-1)-\varepsilon_{2n-3})\left(\gamma_{p,\alpha-(n-1)p}-\varepsilon_{2n-2}\right)\sum_{k=N}^{\infty} k^{(\alpha-np)}\left|y_{k+n}\right|^p \\
&\overset{(3.25)}{=} (\gamma(n)-\beta_0-\varepsilon)\sum_{k=N}^{\infty} k^{(\alpha-np)}\left|y_{k+n}\right|^p .
\end{aligned}
$$

for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$. Hence, the functional $\widetilde{\mathcal{F}}_n(\{y_k\}, N, \infty)$ is greater than the expression

$$
[(\gamma(n)-\beta_0-\varepsilon)+\beta_0]\sum_{k=N}^{\infty} k^{(\alpha-np)}\left|y_{k+n}\right|^p ,
$$

which is positive (see (3.26)) for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$, i.e., $\widetilde{\mathcal{F}}_n(\{y_k\}, N, \infty)$ is positive for every nontrivial sequence $\{y_k\} \in \mathcal{D}_n(N)$. □

Remark 3. In view of Remark 1 and Remark 2, Theorem 5 and Theorem 6 can be generalized for the equation

$$
\sum_{l=0}^{n}(-1)^{n-l}\beta_{n-l}\Delta^{n-l}\left(f_k^{[n-l]}\Phi\left(\Delta^{n-l}x_{k+l}\right)\right)=0, \quad \beta_n := 1, \tag{3.27}
$$

where $\beta_0, \beta_1, \ldots, \beta_{n-1}$ are real numbers, $p \in (1,\infty)$, $\alpha \in \mathbb{R}\setminus\{p-1, 2p-1, \ldots, np-1\}$ and for every $l \in \{0,1,\ldots,n\}$ the sequence $\left\{f_k^{[n-l]}\right\} \in \mathcal{RV}(\alpha-lp)$ and it has the form

$$
f_k^{[n-l]} = k^{(\alpha-lp)}L_k^{[n-l]},\ k \in \mathbb{N},
$$

where $\left\{L_k^{[n-l]}\right\} \in \mathcal{SV}$.

Now we can formulate the following nonoscillation criterion. If

$$
L_k^{[0]} \sim L_k^{[1]} \sim \cdots \sim L_k^{[n]} \quad \text{as} \quad k \to \infty
$$

and $\gamma(k) > 0$ for every $k \in \{1,2,\ldots,n\}$, then equation (3.27) is nonoscillatory. In particular, if $L_k^{[0]} \sim L_k^{[n]}$ as $k \to \infty$ and $\gamma + \gamma_{n,p,\alpha} > 0$, then equation (3.27) with $\beta_1 = \beta_2 = \ldots = \beta_{n-1} = 0$ and $\beta_0 = \gamma$, i.e., the two-term equation

$$
(-1)^n\Delta^n\left(f_k^{[n]}\Phi\left(\Delta^n x_k\right)\right)+\gamma f_k^{[0]}\Phi\left(x_{k+n}\right)=0 \tag{3.28}
$$

is nonoscillatory.

As far as we know our method of the proof is new even in the linear case ($p = 2$). Moreover, the above criterion is new in the linear case. Here it is worthy of note that equation (4.5) with $-\gamma$ instead of $\gamma_{n,2,\alpha}$ is a special case of (3.28). Thus, equation (3.28) has the higher order special (linear type) case for which the constant $-\gamma_{n,p,\alpha}$ is critical.

Note that if we wanted to give a proof directly for (3.28), then (according to Remark 2) we would have the following possibilities for replacing inequalities (3.17) and (3.18). We can take arbitrary sequences

$$\left\{f_k^{[1]}\right\}, \left\{f_k^{[2]}\right\}, \dots, \left\{f_k^{[n-1]}\right\}$$

such that for every $l \in \{1, 2, \dots, n-1\}$ we have

$$\left\{f_k^{[n-l]}\right\} \in \mathcal{RV}(\alpha - lp)$$

and $f_k^{[n-l]}$ has a slowly varying component asymptotically equivalent to $\left\{L_k^{[n]}\right\}$. Then, inequality (3.17) is replaced by inequality (3.29); and the inequalities contained in (3.18) (for each $j \in \{1, 2, \dots, n-1\}$ we have one inequality) are replaced by the inequalities contained in (3.30) and by inequality (3.31), where

$$\sum_{k=N}^{\infty} \left[k^{(\alpha)} L_k^{[n]} |\Delta^n y_k|^p + \left(\varepsilon_1 - \gamma_{p,\alpha}\right) f_k^{[n-1]} \left|\Delta^{n-1} y_{k+1}\right|^p\right] > 0, \tag{3.29}$$

$$\sum_{k=N}^{\infty} \left[f_k^{[n-j]} \left|\Delta^{n-j} y_{k+j}\right|^p + \left(\varepsilon_{2j} - \gamma_{p,\alpha-jp}\right) f_k^{[n-(j+1)]} \left|\Delta^{n-j-1} y_{k+j+1}\right|^p\right] > 0 \tag{3.30}$$

for $j = 1, 2, \dots, n-2$, and

$$\sum_{k=N}^{\infty} \left[f_k^{[1]} |\Delta y_{k+n-1}|^p + \left(\varepsilon_{2(n-1)} - \gamma_{p,\alpha-(n-1)p}\right) k^{(\alpha-np)} L_k^{[0]} |y_{k+n}|^p\right] > 0. \tag{3.31}$$

Example 1. Consider the second order equation

$$-\Delta\left(f_k \Phi\left(\Delta x_k\right)\right) + \beta_0 g_k \Phi\left(x_{k+1}\right) = 0, \tag{3.32}$$

where $\{f_k\} \in \mathcal{RV}(-2)$ and $\{g_k\} \in \mathcal{RV}(-7)$ have asymptotically equivalent slowly varying components, i.e., equation (3.27) with $n = 1$, $\alpha = -2$ and $p = 5$. By Remark 3, equation (3.32) is nonoscillatory if $\beta_0 > -8$. Indeed, we have $\gamma_{1,5,-2} = \gamma_{5,-2} = \left(\frac{|5-1-(-2)|}{3}\right)^3 = 8$. For more specific example we can take, in addition, $f_k = \frac{\ln k}{k^2}$ and $g_k = \frac{\ln k}{k^{(7)}} + \frac{1}{k^7 \ln k}$ for $k \in \mathbb{N}$.

Example 2. If $n = 3$ then the assumptions of Theorem 6 read as

$$\gamma_{p,\alpha} + \beta_2 > 0,$$
$$\gamma_{p,\alpha}\gamma_{p,\alpha-p} + \beta_2\gamma_{p,\alpha-p} + \beta_1 > 0,$$

$$\gamma_{p,\alpha}\gamma_{p,\alpha-p}\gamma_{p,\alpha-2p}+\beta_2\gamma_{p,\alpha-p}\gamma_{p,\alpha-2p}+\beta_1\gamma_{p,\alpha-2p}+\beta_0>0$$

and $\alpha\in\mathbb{R}\setminus\{p-1,2p-1,3p-1\}$. Hence, $\beta_2\in(-\gamma_{p,\alpha},\infty)$. Assume that $\beta_2=(\varepsilon_1-1)\gamma_{p,\alpha}$ for some $\varepsilon_1\in(0,1)$. Then from the second assumption we have $\beta_1\in(-\varepsilon_1\gamma_{p,\alpha}\gamma_{p,\alpha-p},\infty)$. Note that if $\beta_2=0$, then $\beta_1\in(-\gamma_{2,p,\alpha},\infty)$. Choose $\beta_1=\varepsilon_1(\varepsilon_2-1)\gamma_{p,\alpha}\gamma_{p,\alpha-p}$ for some $\varepsilon_2\in(0,1)$. Then from the third assumption we have $\beta_0\in(-\varepsilon_1\varepsilon_2\gamma_{3,p,\alpha},\infty)$. Note that if $\beta_1=\beta_2=0$, then $\beta_0\in(-\gamma_{3,p,\alpha},\infty)$.

Consider the sixth order equation

$$\begin{aligned}&-\Delta^3\left(a_k\Phi\left(\Delta^3x_k\right)\right)+8(\varepsilon_1-1)\Delta^2\left(b_k\Phi\left(\Delta^2x_{k+1}\right)\right)\\&\qquad-192\varepsilon_1(\varepsilon_2-1)\Delta\left(c_k\Phi\left(\Delta x_{k+2}\right)\right)+\beta_0d_k\Phi\left(x_{k+3}\right)=0,\end{aligned}\tag{3.33}$$

where $\varepsilon_1,\varepsilon_2\in(0,1)$ and $\{a_k\}\in\mathcal{RV}(-4)$, $\{b_k\}\in\mathcal{RV}(-7)$, $\{c_k\}\in\mathcal{RV}(-10)$ and $\{d_k\}\in\mathcal{RV}(-13)$ have asymptotically equivalent slowly varying components, i.e., equation (3.27) with $n=3$, $\alpha=-2$, $p=3$, $\beta_2=8(\varepsilon_1-1)$ and $\beta_1=192\varepsilon_1(\varepsilon_2-1)$.

Then

$$\begin{aligned}\gamma_{3,-4}=8,\ \gamma_{3,-7}=27,\ \gamma_{3,-4}\gamma_{3,-7}=192,\ \gamma_{3,-10}=64,\\ \gamma_{3,-4}\gamma_{3,-7}\gamma_{3,-10}=\gamma_{3,3,-4}=13824,\end{aligned}$$

and equation (3.33) is nonoscillatory if $\beta_0\in(-13824\varepsilon_1\varepsilon_2,\infty)$.

4. Open problems

(i) In the article [16, Theorem 2.2 and Theorem 3.1], which is considered in a more general setting of dynamic equations on time scales, one can found the following generalization of Theorem 1 and Theorem 2. Assume that the limits

$$\lim_{k\to\infty}\frac{\left(r_k^{[1]}\right)^{1-q}}{\sum_{j=a}^{k-1}\left(r_j^{[1]}\right)^{1-q}}=M\quad\text{and}\quad\lim_{k\to\infty}\frac{\left(r_k^{[1]}\right)^{1-q}}{\sum_{j=k}^{\infty}\left(r_j^{[1]}\right)^{1-q}}=N\tag{4.1}$$

exist (less restrictive compare to conditions (2.6) and (2.7)). Denote

$$\gamma_D(M):=\begin{cases}\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} & \text{if } M=0,\\ \left(\frac{(M+1)^{\frac{p-1}{p}}-1}{M}\right)^{\alpha}\frac{M}{(M+1)^{p-1}-1} & \text{if } M\in(0,\infty),\\ 0 & \text{if } M=\infty\end{cases}\tag{4.2}$$

and

$$\gamma_C(N):=\begin{cases}\left(\frac{p-1}{p}\right)^{p}=q^{-p} & \text{if } N=0,\\ (1-N)\left(\frac{1-(1-N)^{\frac{p-1}{p}}}{N}\right)^{p} & \text{if } N\in(0,1),\\ 0 & \text{if } N=1.\end{cases}\tag{4.3}$$

Let $\left\{\sum_{j=k}^{\infty} r_j^{[0]}\right\}_{k=1}^{\infty}$ exist and be nonpositive and nontrivial for large k, $r_k^{[1]} > 0$ for large k, $\sum_{j=a}^{\infty} \left(r_j^{[1]}\right)^{1-q} = \infty$, and M be given by (4.1). Then equation (2.5) is nonoscillatory provided

$$\liminf_{k\to\infty} \left(\sum_{j=a}^{k-1} \left(r_j^{[1]}\right)^{1-q}\right)^{p-1} \left(\sum_{j=k}^{\infty} r_j^{[0]}\right) > -\gamma_D(M).$$

Let $r_k^{[1]} > 0$ for large k, $\sum_{j=a}^{\infty} \left(r_j^{[1]}\right)^{1-q} < \infty$,

$$\left\{\sum_{j=k}^{\infty} \left[\sum_{i=j+1}^{\infty} \left(r_i^{[1]}\right)^{1-q}\right]^p r_j^{[0]}\right\}_{k=1}^{\infty}$$

exist and be nonpositive for large k, and N given by (4.1). Then equation (2.5) is nonoscillatory provided

$$\liminf_{k\to\infty} \left(\sum_{j=k}^{\infty} \left(r_j^{[1]}\right)^{1-q}\right)^{-1} \left(\sum_{j=k}^{\infty} \left[\sum_{i=j+1}^{\infty} \left(r_i^{[1]}\right)^{1-q}\right]^p r_j^{[0]}\right) > -\gamma_C(N). \tag{4.4}$$

If $M = 0$, then the constant $-\gamma_D(0)$ is the same as the estimate for $\liminf_{k\to\infty} \mathcal{A}_k$ in Theorem 1. If $N = 0$, $r_k^{[1]} = k^{(\alpha)}$, $r_k^{[0]} = \gamma k^{(\alpha-p)}$ for $k \in \mathbb{N}$, $\gamma < 0$ and $\alpha > p-1$, then the left side of inequality (4.4) takes the form

$$\liminf_{k\to\infty} \left(\sum_{j=k}^{\infty} \left(j^{(\alpha)}\right)^{1-q}\right)^{-1} \left(\sum_{j=k}^{\infty} \left[\sum_{i=j+1}^{\infty} \left(i^{(\alpha)}\right)^{1-q}\right]^p \gamma j^{(\alpha-p)}\right) = \frac{\gamma(p-1)^p}{(\alpha-p+1)^p}$$

and it holds that

$$\frac{\gamma(p-1)^p}{(\alpha-p+1)^p} > -\gamma_C(0) \quad \text{if and only if} \quad \gamma > -\gamma_{p,\alpha}.$$

Hence, for $N = 0$ and $\alpha > p-1$ we obtained the same result as in the proof of Lemma 4 in the case $\alpha > p-1$, where Theorem 2 is used. Therefore, we believe that for $M \neq 0$ and $N \neq 0$ our criteria can be extended via the results from [16].

(ii) All our results are obtained under the restriction $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, np-1\}$, which is the condition required in our technique. We can try to deal with the cases $\alpha = ip-1$, $i = 1, 2, \dots, n$ in the further research.

(iii) Note that Theorem 1 and Theorem 2 do not require existence of the limits

$$\lim_{k\to\infty} \mathcal{A}_k \quad \text{and} \quad \lim_{k\to\infty} \mathcal{B}_k,$$

but in this article we work only with the coeficients of (3.1), for which these limits exist. Thus, other extension of our results may involve such sequences $\left\{r_k^{[0]}\right\}$ and $\left\{r_k^{[1]}\right\}$ that the limits $\lim_{k\to\infty}\mathcal{A}_k$ and $\lim_{k\to\infty}\mathcal{B}_k$ do not exist.

(iv) Oscillation behavior of two-term equation (3.14) in the critical case $\gamma = -\gamma_{n,p,\alpha}$ is still unknown. We believe that we will be able to prove a half-linear extension of the following result. In [10] it is proved that the linear two-term equation

$$(-1)^n \Delta^n \left(k^{(\alpha)} \Delta^n x_k\right) - \gamma_{n,2,\alpha}(k+n-2c(\alpha))^{(\alpha-2n)} x_{k+n} = 0 \tag{4.5}$$

with

$$c(\alpha) := n\chi_{(2n-1,\infty)}(\alpha) + \sum_{l=1}^{n-1} l\chi_{(2l-1,2l+1)}(\alpha) \quad \text{and} \quad \alpha \in \mathbb{R}\setminus\{1,3,5,\dots,2n-1\}$$

is nonoscillatory (for $t \in \mathbb{R}$ the symbol $\chi_M(t)$ denotes the indicator function for the set M of real numbers).

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