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WEYL PROJECTIVE OBJECTS $\underset{1}{\mathcal{W}}, \underset{2}{\mathcal{W}}, \underset{3}{\mathcal{W}}$ FOR EQUITORSION GEODESIC MAPPINGS

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Abstract. In this paper, invariants of geodesic mappings of non-symmetric affine connection manifolds are studied. It is obtained new generalizations of the Weyl projective tensor of these manifolds. At the end of this paper, generalized invariants of a geodesic mapping between special three dimensional generalized Riemannian manifolds $\mathbb{GR}_3$ and $\mathbb{G}\overline{\mathbb{R}}_3$ are obtained.

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1. INTRODUCTION AND MOTIVATION

Many research papers and monographs [1–19, 21–27] are focused on development of the affine connection spaces theory, on the theory of mappings between these spaces, on the theory of invariants of these mappings and on the applications of them. N. S. Sinyukov [21], J. Mikeš with his research group [9–11, 22], G. Hall [6, 7], G. P. Pokhariyal [16, 17], U. P. Singh [20] and many other researchers have given significant contributions to the development of the concept of symmetric affine connection spaces.

Consider an N-dimensional manifold $\mathcal{M}_N$ on which a non-symmetric affine connection ∇ is defined. If $\mathfrak{X}(\mathcal{M}_N)$ is Lie algebra of smooth vector fields and $X, Y \in \mathfrak{X}(\mathcal{M}_N)$, then the mapping

$$\nabla : \mathfrak{X}(\mathcal{M}_N) \times \mathfrak{X}(\mathcal{M}_N) \to \mathfrak{X}(\mathcal{M}_N)$$

defines **the non-symmetric connection** on $\mathcal{M}_N$ if $\nabla_X Y \neq \nabla_Y X + [X, Y]$. The following identities are satisfied:

$$\begin{aligned} &\nabla_{Y_1+Y_2} X = \nabla_{Y_1} X + \nabla_{Y_2} X, && \nabla_{fY} X = f \nabla_Y X, \\ &\nabla_Y (X_1 + X_2) = \nabla_Y X_1 + \nabla_Y X_2, && \nabla_Y (f X) = Yf \cdot X + f \nabla_Y X, \end{aligned}$$

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for $X, Y, X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(\mathcal{M}_N), f \in \mathcal{F}(\mathcal{M}_N)$, where $\mathcal{F}(\mathcal{M}_N)$ is an algebra of smooth real functions on $\mathcal{M}_N$.

Definition 1 ([4, 25, 27])**.** An N-dimensional differentiable manifold $\mathcal{M}_N$ endowed with a non-symmetric affine connection $\nabla_Y X$ is **the non-symmetric manifold** $\mathbb{GA}_N$.

Remark 1 ([4,8]). A differentiable manifold endowed with a non-symmetric metric $G(X,Y)$ $\big($the basic tensor $G(X,Y)\big)$ is the generalized Riemannian manifold $\mathbb{GR}_N$. Affine connection coefficients of the manifold $\mathbb{GR}_N$ are Christoffel symbols of the second kind of this manifold.

Let ∇ be a non-symmetric affine connection. With regard to local chart $\big(x^1,\dots,x^N\big)$, in the base of $\big(\partial_i\big)_{i=1}^N = \big(\partial/\partial x^i\big)_{i=1}^N$, we obtain that is

$$\nabla_{\partial_k}\partial_j = \Gamma^{\alpha}_{jk}\partial_\alpha.$$

for coefficients Γ^i_{jk} non-symmetric by indices j and k.

In the whole paper, we shall use the capital Latin letters $X, Y, Z, \dots$ to denote smooth vector fields on a smooth manifold $\mathcal{M}_N$.

Let $\mathbb{GA}_N = \big(\mathcal{M}_N, \nabla\big)$ be a non-symmetric affine connection manifold. Because of the non-symmetry $\nabla_Y X + [X,Y] \neq \nabla_X Y$, the symmetric and anti-symmetric part of the affine connection coefficients $\nabla_Y X$ are respectively defined as:

$$\begin{aligned} \widetilde{\nabla}_Y X &= \frac{1}{2}\big(\nabla_Y X + \nabla_X Y - [X,Y]\big),\\ T(X,Y) &= \frac{1}{2}\big(\nabla_Y X - \nabla_X Y + [X,Y]\big). \end{aligned} \tag{1.1}$$

It is evident

$$\widetilde{\nabla}_Y X - \widetilde{\nabla}_X Y = [X,Y]. \tag{1.2}$$

The anti-symmetric part $T(X,Y)$ of the affine connection ∇ is **the torsion tensor**. The coordinates of the affine connection coefficients $\nabla_Y X$ are Γ^i_{jk}. The coordinates of the symmetric and the anti-symmetric parts of the coefficients Γ^i_{jk} are:

$$\widetilde{S}^i_{jk} = \tfrac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) \quad \text{and} \quad T^i_{jk} = \tfrac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}),$$

An N-dimensional manifold $\mathcal{M}_N$ endowed with the symmetric affine connection $\widetilde{\nabla}_Y X$ is called **the associated (symmetric affine connection) manifold** $\mathbb{A}_N$ of the manifold $\mathbb{GA}_N$. Based on the symmetry of affine connection coefficients $\widetilde{\nabla}_Y X$ by X and Y, it exists only one type of covariant differentiation with respect to the affine connection of the manifold $\mathbb{A}_N$. For this reason, it exists only one curvature tensor of this manifold:

$$R(X;Y,Z) = \widetilde{\nabla}_Z\widetilde{\nabla}_Y X - \widetilde{\nabla}_Y\widetilde{\nabla}_Z X + \widetilde{\nabla}_{[Y,Z]}X. \tag{1.3}$$

A. Einstein [1–3] was the first who applied a non-symmetric affine connection in the research about gravity. He worked on the Unified Field Theory (Non-symmetric Gravitational Theory - NGT). In NGT, the basic tensor $G(X,Y)$ is a non-symmetric tensor of the type $(0,2)$. The symmetric part

$$g(X,Y) = \frac{1}{2}\big(G(X,Y) + G(Y,X)\big)$$

of the tensor $G(X,Y)$ is related to gravitation. The anti-symmetric part

$$F(X,Y) = \frac{1}{2}\big(G(X,Y) - G(Y,X)\big) \qquad \big(= G(X,Y) - g(X,Y)\big)$$

of the basic tensor $G(X,Y)$ is related to electromagnetism. Unlike Riemannian and generalized Riemannian manifolds, where affine connection coefficients are functions of the basic tensors $g(X,Y)$ and $G(X,Y)$, in Einstein's works affine connection coefficients of manifolds at NGT satisfy *the Einstein metricity condition*:

$$ZG(X,Y) - G\big(\nabla_Z X, Y\big) - G\big(X, \nabla_Y Z\big) = 0. \tag{1.4}$$

The coordinate form of this condition is

$$\frac{\partial G_{ij}}{\partial x^k} - \Gamma^{\alpha}_{ik} G_{\alpha j} - \Gamma^{\alpha}_{kj} G_{i\alpha} = 0, \tag{1.5}$$

for affine connection coefficients Γ^i_{jk} of the manifold at NGT.

Based on the Einstein's metricity condition, it exists two types ∇^+ and ∇^- of covariant differentiation (see [5]). For example, for a tensor A of the type $(1,1)$ they are:

$$(\nabla^+_Y A)(X) = (\widetilde{\nabla}_Y A)(X) + T(AX,Y) - A\big(T(X,Y)\big), \tag{1.6}$$

$$(\nabla^-_Y A)(X) = (\widetilde{\nabla}_Y A)(X) - T(AX,Y) + A\big(T(X,Y)\big). \tag{1.7}$$

Following the excellent Eisenhart's book [4] and motivated by different results from the theory of symmetric affine connection manifolds, S. Minčić [12–14] has obtained different significant results in the theory of non-symmetric affine connection manifolds. A geometry of these connections has been studied by F. Graiff [5], M. Prvanović [19] and many others.

From the differences

$$\nabla^+_Z \nabla^+_Y X - \nabla^+_Y \nabla^+_Z X, \quad \nabla^-_Z \nabla^-_Y X - \nabla^-_Y \nabla^-_Z X, \quad \nabla^-_Z \nabla^+_Y X - \nabla^+_Y \nabla^-_Z X,$$

one obtains the curvature tensors:

$$\begin{aligned} \underset{1}{R}(X;Y,Z) = R(X;Y,Z) &+ \big(\widetilde{\nabla}_Z T\big)(X,Y) - \big(\widetilde{\nabla}_Y T\big)(X,Z) \\ &+ T\big(T(X,Y),Z\big) - T\big(T(X,Z),Y\big), \end{aligned} \tag{1.8}$$

$$\begin{aligned} \underset{2}{R}(X;Y,Z) = R(X;Y,Z) &- \big(\widetilde{\nabla}_Z T\big)(X,Y) + \big(\widetilde{\nabla}_Y T\big)(X,Z) \\ &+ T\big(T(X,Y),Z\big) - T\big(T(X,Z),Y\big), \end{aligned} \tag{1.9}$$

$$\underset{3}{R}(X;Y,Z) = R(X;Y,Z) + \big(\widetilde{\nabla}_Z T\big)(X,Y) + \big(\widetilde{\nabla}_Y T\big)(X,Z) \\ - T\big(T(X,Y),Z\big) + T\big(T(X,Z),Y\big) - 2T\big(T(Y,Z),X\big), \tag{1.10}$$

where $R(X;Y,Z)$ is given by the formula (1.3). As one can see, these curvature tensors are introduced by using an algebraic approach to the curvature. Some of the geometric meaning of these curvature tensors has been pointed out in [18].

1.1. *Geodesic mappings between affine connection manifolds*

Geodesic mappings between affine connection manifolds have an important role in applications of differential geometry. Definitions and a lot of properties of mappings between symmetric affine connection manifolds are presented in books [9–11, 21]. Many other authors have continued this research (see [16, 17, 22]).

A diffeomorphism $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is called a geodesic mapping of $\mathbb{GA}_N$ onto $\mathbb{G}\overline{\mathbb{A}}_N$ if any geodesic curve in $\mathbb{GA}_N$ it maps onto a geodesic curve in $\mathbb{G}\overline{\mathbb{A}}_N$. The affine connections ∇ and $\overline{\nabla}$ of the manifolds $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ satisfy the equation

$$\overline{\nabla}_Y X = \nabla_Y X + \psi(X)Y + \psi(Y)X + \xi(X,Y) \tag{1.11}$$

for a 1-form ψ and an anti-symmetric tensor ξ of the type (0,2).

Invariants of geodesic mappings of a non-symmetric affine connection manifold $\mathbb{GA}_N$ and some their properties are obtained in [25–27]. The main goal of this paper is to obtain some other generalizations of invariants of geodesic mappings defined on the manifold $\mathbb{GA}_N$.

Having a geodesic mapping of two general affine connection manifolds, we cannot find a generalization of Weyl tensor as an invariant of geodesic mapping in general case. For this reason, one assumes that $\xi(X,Y) = 0$ holds, that is, $\overline{T}(X,Y) = T(X,Y)$. These mappings are **the equitorsion geodesic mappings**.

2. INVARIANTS OF EQUITORSION GEODESIC MAPPINGS

Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be an equitorsion geodesic mapping of a non-symmetric affine connection manifold $\mathbb{GA}_N$. The basic equation of this mapping is (see [25–27])

$$\overline{\nabla}_Y X = \nabla_Y X + \psi(X)Y + \psi(Y)X, \tag{2.1}$$

for a 1-form ψ. The symmetric parts $\widetilde{\nabla}_Y X$ and $\widetilde{\overline{\nabla}}_Y X$ of the affine connection coefficients $\nabla_Y X$ and $\overline{\nabla}_Y X$ satisfy the equation

$$\widetilde{\overline{\nabla}}_Y X = \widetilde{\nabla}_Y X + \frac{1}{N+1}\Big(\big(\mathrm{Tr}\{U \to \nabla_U X\}\big)Y + \big(\mathrm{Tr}\{U \to \nabla_U Y\}\big)X\Big), \tag{2.2}$$

The curvature tensors $\overline{R}(X;Y,Z)$ and $R(X;Y,Z)$ of the associated manifolds $\overline{\mathbb{A}}_N$ and $\mathbb{A}_N$ satisfy the relation

$$\begin{aligned}\overline{R}(X;Y,Z) &= R(X;Y,Z) + \frac{1}{N+1}\big(Ricc(Y,Z) - Ricc(Z,Y)\big)X \\ &- \frac{1}{N+1}\big(\overline{Ricc}(Y,Z) - \overline{Ricc}(Z,Y)\big)X \\ &+ \frac{1}{N^2-1}\Big(\big(NRicc(X,Z)+Ricc(Z,X)\big)Y \\ &\qquad -\big(NRicc(Y,Z)+Ricc(Z,Y)\big)X\Big) \\ &- \frac{1}{N^2-1}\Big(\big(N\overline{Ricc}(X,Z)+\overline{Ricc}(Z,X)\big)Y \\ &\qquad -\big(N\overline{Ricc}(Y,Z)+\overline{Ricc}(Z,Y)\big)X\Big),\end{aligned} \tag{2.3}$$

for the Ricci-curvature tensors $Ricc(X,Y) = \mathrm{Tr}\{U \to R(X;Y,U)\}$ and $\overline{Ricc}(X,Y) = \mathrm{Tr}\{U \to \overline{R}(X;Y,U)\}$ of the manifolds $\mathbb{A}_N$ and $\overline{\mathbb{A}}_N$.

From the equation (2.3), one obtains that the Weyl projective tensor

$$\begin{aligned}W(X,Y,Z) &= R(X;Y,Z) + \frac{1}{N+1}\big(Ricc(Y,Z) - Ricc(Z,Y)\big)X \\ &+ \frac{1}{N^2-1}\Big(\big(NRicc(X,Z) + Ricc(Z,X)\big)Y \\ &\qquad - \big(NRicc(X,Y) + Ricc(Y,X)\big)Z\Big)\end{aligned} \tag{2.4}$$

is an invariant of the mapping f. The coordinates of the Weyl projective tensor W are

$$W^i_{jmn} = R^i_{jmn} + \frac{1}{N+1}\delta^i_j R_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j}. \tag{2.5}$$

2.1. *Weyl projective tensors* $\underset{1}{\mathcal{W}}, \underset{2}{\mathcal{W}}, \underset{3}{\mathcal{W}}$ *of geodesic mappings*

From the equation (1.8), we obtain the following equation:

$$\begin{aligned}Ricc(X,Y) &= \underset{1}{Ricc}(X,Y) \\ &- \mathrm{Tr}\Big\{U \to (\widetilde{\nabla}_U T)(X,Y)\Big\} + \mathrm{Tr}\Big\{U \to (\widetilde{\nabla}_Y T)(X,U)\Big\} \\ &- \mathrm{Tr}\Big\{U \to T\big(T(X,Y),U\big)\Big\} + \mathrm{Tr}\Big\{U \to T\big(T(X,U),Y\big)\Big\},\end{aligned} \tag{2.6}$$

for Ricci-curvature tensor $Ricc(X,Y) = \mathrm{Tr}\{U \to R(X;Y,U)\}$.

The similar equations can be derived for Ricci-curvature tensors $\underset{2}{Ricc}(X,Y)$ and $\underset{3}{Ricc}(X,Y)$. In the same manner, we obtain the corresponding correlations between the Ricci-curvature tensors $\overline{Ricc}(X,Y)$ and $\underset{\theta}{\overline{Ricc}}(X,Y), \theta = 1,2,3$ of the manifolds $\overline{\mathbb{A}}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$. Because $\overline{T}(X,Y) = T(X,Y)$, it holds

$$\begin{aligned}(\widetilde{\overline{\nabla}}_Z T)(X,Y) = (\widetilde{\nabla}_Z T)(X,Y) - \widetilde{\nabla}_{T(X,Y)}Z + T\big(\widetilde{\nabla}_Z X, Y\big) + T\big(X, \widetilde{\nabla}_Z Y\big) \\ + \widetilde{\overline{\nabla}}_{\overline{T}(X,Y)}Z - \overline{T}\big(\widetilde{\overline{\nabla}}_Z X, Y\big) - \overline{T}\big(X, \widetilde{\overline{\nabla}}_Z Y\big).\end{aligned} \tag{2.7}$$

From the expressions (2.3, 2.6, 2.7) involved in the correlation

$$\begin{aligned}\underset{1}{\overline{R}}(X;Y,Z) = \underset{1}{R}(X;Y,Z) &+ \frac{1}{N+1}\big(Ricc(Y,Z) - Ricc(Z,Y) - \overline{Ricc}(Y,Z) + \overline{Ricc}(Z,Y)\big)X \\ &+ \frac{1}{N^2-1}\Big(\big(NRicc(X,Z) + Ricc(Z,X)\big)Y - \big(NRicc(X,Y) + Ricc(Y,X)\big)Z\Big) \\ &- \frac{1}{N^2-1}\Big(\big(N\overline{Ricc}(X,Z) + \overline{Ricc}(Z,X)\big)Y - \big(N\overline{Ricc}(X,Y) + \overline{Ricc}(Y,X)\big)Z\Big) \\ &+ \big(\widetilde{\overline{\nabla}}_Z T\big)(X,Y) - \big(\widetilde{\nabla}_Z T\big)(X,Y) - \big(\widetilde{\overline{\nabla}}_Y T\big)(X,Z) + \big(\widetilde{\nabla}_Y T\big)(X,Z),\end{aligned}$$

we obtain

$$\underset{1}{\overline{\mathcal{W}}}(X,Y,Z) = \underset{1}{\mathcal{W}}(X,Y,Z)$$

for

$$\begin{aligned}\underset{1}{\mathcal{W}}(X,Y,Z) = \underset{1}{R}(X;Y,Z) &+ \frac{1}{N+1}\big(\underset{1}{Ricc}(Y,Z) - \underset{1}{Ricc}(Z,Y)\big)X \\ &+ \frac{1}{N^2-1}\Big(\big(N\underset{1}{Ricc}(X,Z) + \underset{1}{Ricc}(Z,X)\big)Y \\ &\qquad - \big(N\underset{1}{Ricc}(X,Y) + \underset{1}{Ricc}(Y,X)\big)Z\Big) \\ &- \widetilde{\nabla}_{T(X,Y)}Z + T\big(\widetilde{\nabla}_Z X, Y\big) + T\big(X, \widetilde{\nabla}_Z Y\big) \\ &+ \widetilde{\nabla}_{T(X,Z)}Y - T\big(\widetilde{\nabla}_Y X, Z\big) - T\big(X, \widetilde{\nabla}_Y Z\big) \\ &+ \frac{2}{N+1}\mathrm{Tr}\big\{U \to T(\widetilde{\nabla}_U Y, Z) - T(\widetilde{\nabla}_U Z, Y)\big\}X \\ &+ \frac{1}{N+1}\mathrm{Tr}\Big\{U \to \widetilde{\nabla}_{T(U,Z)Y - T(U,Y)Z}X\Big\} \\ &- \frac{1}{N-1}\mathrm{Tr}\Big\{U \to T\big(\widetilde{\nabla}_Z XY - \widetilde{\nabla}_Y XZ, U\big)\Big\}\end{aligned} \tag{2.8}$$

and the corresponding $\underset{1}{\overline{\mathcal{W}}}(X,Y,Z)$. In the same manner, we obtain

$$\underset{2}{\overline{\mathcal{W}}}(X,Y,Z) = \underset{2}{\mathcal{W}}(X,Y,Z) \quad \text{and} \quad \underset{3}{\overline{\mathcal{W}}}(X,Y,Z) = \underset{3}{\mathcal{W}}(X,Y,Z)$$

for

$$\begin{aligned}
\underset{2}{\mathcal{W}}(X,Y,Z) = \underset{2}{R}(X;Y,Z) &+ \frac{1}{N+1}\big(\underset{2}{Ricc}(Y,Z) - \underset{2}{Ricc}(Z,Y)\big)X \\
&+ \frac{1}{N^2-1}\Big(\big(N\underset{2}{Ricc}(X,Z) + \underset{2}{Ricc}(Z,X)\big)Y \\
&\qquad\qquad - \big(N\underset{2}{Ricc}(X,Y) + \underset{2}{Ricc}(Y,X)\big)Z\Big) \\
&+ \widetilde{\nabla}_{T(X,Y)}Z - T\big(\widetilde{\nabla}_Z X, Y\big) - T\big(X, \widetilde{\nabla}_Z Y\big) \\
&- \widetilde{\nabla}_{T(X,Z)}Y + T\big(\widetilde{\nabla}_Y X, Z\big) + T\big(X, \widetilde{\nabla}_Y Z\big) \\
&- \frac{2}{N+1}\operatorname{Tr}\{U \to T(\widetilde{\nabla}_U Y, Z) - T(\widetilde{\nabla}_U Z, Y)\}X \\
&- \frac{1}{N+1}\operatorname{Tr}\Big\{U \to \widetilde{\nabla}_{T(U,Z)Y - T(U,Y)Z}X\Big\} \\
&+ \frac{1}{N-1}\operatorname{Tr}\Big\{U \to T\big(\widetilde{\nabla}_Z XY - \widetilde{\nabla}_Y XZ, U\big)\Big\},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\underset{3}{\mathcal{W}}(X,Y,Z) = \underset{3}{R}(X;Y,Z) &+ \frac{1}{N+1}\big(\underset{3}{Ricc}(Y,Z) - \underset{3}{Ricc}(Z,Y)\big)X \\
&+ \frac{1}{N^2-1}\Big(\big(N\underset{3}{Ricc}(X,Z) + \underset{3}{Ricc}(Z,X)\big)Y \\
&\qquad\qquad - \big(N\underset{3}{Ricc}(X,Y) + \underset{3}{Ricc}(Y,X)\big)Z\Big) \\
&- \widetilde{\nabla}_{T(X,Y)}Z + T\big(\widetilde{\nabla}_Z X, Y\big) + T\big(X, \widetilde{\nabla}_Z Y\big) \\
&- \widetilde{\nabla}_{T(X,Z)}Y + T\big(\widetilde{\nabla}_Y X, Z\big) + T\big(X, \widetilde{\nabla}_Y Z\big) \\
&+ \frac{2}{N+1}\operatorname{Tr}\{U \to T(\widetilde{\nabla}_U Y, Z) - T(\widetilde{\nabla}_U Z, Y)\}X \\
&+ \frac{1}{N+1}\operatorname{Tr}\Big\{U \to \widetilde{\nabla}_{T(U,Z)Y - T(U,Y)Z}X\Big\} \\
&+ \frac{1}{N-1}\operatorname{Tr}\Big\{U \to T\big(\widetilde{\nabla}_Z XY - \widetilde{\nabla}_Y XZ, U\big)\Big\}.
\end{aligned} \tag{2.10}$$

Theorem 1. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be an equitorsion geodesic mapping. The geometrical objects (2.8, 2.9, 2.10) are invariants of the mapping f.* □

Corollary 1. *The invariants $\underset{1}{\mathcal{W}}(X,Y,Z)$, $\underset{2}{\mathcal{W}}(X,Y,Z)$, $\underset{3}{\mathcal{W}}(X,Y,Z)$ and the Weyl projective tensor $W(X,Y,Z)$ satisfy the equations:*

$$\begin{aligned}
\underset{1}{\mathcal{W}}(X,Y,Z) = W(X,Y,Z) &+ ZT(X,Y) - YT(X,Z) \\
&+ T\big(T(X,Y),Z\big) - T\big(T(X,Z),Y\big),
\end{aligned} \tag{2.11}$$

$$\underset{2}{\mathcal{W}}(X,Y,Z) = W(X,Y,Z) - ZT(X,Y) + YT(X,Z) + T\big(T(X,Y),Z\big) - T\big(T(X,Z),Y\big), \tag{2.12}$$

$$\underset{3}{\mathcal{W}}(X,Y,Z) = W(X,Y,Z) + ZT(X,Y) + YT(X,Z) - T\big(T(X,Y),Z\big) + T\big(T(X,Z),Y\big) - 2T\big(T(Y,Z),X\big). \tag{2.13}$$

Example 1. Let a generalized Riemannian manifold $\mathbb{GR}_3$ be determined by the non-symmetric basic matrix

$$\big[G(X,Y)\big] = \begin{bmatrix} 1 & e^{x^1} & -e^{-x^2} \\ -e^{x^1} & 1 & -e^{x^3} \\ e^{-x^2} & e^{x^3} & 1 \end{bmatrix}$$

and let $f : \mathbb{GR}_3 \to \mathbb{G}\overline{\mathbb{R}}_3$ be an equitorsion geodesic mapping of the manifold $\mathbb{GR}_3$.

The symmetric and anti-symmetric part of the basic matrix $\big(G(X,Y)\big)$ are, respectively:

$$\big[g(X,Y)\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \big[F(X,Y)\big] = \begin{bmatrix} 0 & e^{x^1} & -e^{-x^2} \\ -e^{x^1} & 0 & -e^{x^3} \\ e^{-x^2} & e^{x^3} & 0 \end{bmatrix}.$$

The symmetric part $\big(g(X,Y)\big)$ of the basic tensor $\big(G(X,Y)\big)$ is constant. For this reason, the coordinates of the curvature tensor $R(X;Y,Z)$ of the associated manifold $\mathbb{R}_3$ are $R^i_{jmn} = 0$. Furthermore, the coordinates of the Weyl projective tensor $W(X,Y,Z)$ of the manifold $\mathbb{R}_3$ are $W^i_{jmn} = 0$. The coordinates of the torsion tensor $T(X,Y)$ of the manifold $\mathbb{GR}_3$ are

$$T^1_{23} = \frac{1}{2}e^{-x^2}, T^1_{32} = -\frac{1}{2}e^{-x^2}, T^2_{13} = \frac{1}{2}e^{-x^2},$$
$$T^2_{31} = -\frac{1}{2}e^{-x^2}, T^3_{12} = \frac{1}{2}e^{-x^2}, T^3_{21} = -\frac{1}{2}e^{-x^2}$$

and $T^i_{jk} = 0, i,j,k = 1,2,3$, in all other cases.

The coordinates $\underset{1}{\mathcal{W}}{}^i_{jmn}, \underset{2}{\mathcal{W}}{}^i_{jmn}, \underset{3}{\mathcal{W}}{}^i_{jmn}, i,j,m,n = 1,2,3,$ of the invariants $\underset{1}{\mathcal{W}}(X,Y,Z)$, $\underset{2}{\mathcal{W}}(X,Y,Z)$, $\underset{3}{\mathcal{W}}(X,Y,Z)$ of the mapping f are:

$$\underset{1}{\mathcal{W}}{}^i_{jmn} = W^i_{jmn} + T^i_{jm,n} - T^i_{jn,m} + T^\alpha_{jm}T^i_{\alpha n} - T^\alpha_{jn}T^i_{\alpha m},$$
$$\underset{2}{\mathcal{W}}{}^i_{jmn} = W^i_{jmn} - T^i_{jm,n} + T^i_{jn,m} + T^\alpha_{jm}T^i_{\alpha n} - T^\alpha_{jn}T^i_{\alpha m},$$
$$\underset{3}{\mathcal{W}}{}^i_{jmn} = W^i_{jmn} + T^i_{jm,n} + T^i_{jn,m} - T^\alpha_{jm}T^i_{\alpha n} + T^\alpha_{jn}T^i_{\alpha m} - 2T^\alpha_{mn}T^i_{\alpha j}$$

for partial derivations denoted by commas and the above obtained coordinates T^i_{jk} of the torsion tensor $T(X,Y)$ of the manifold $\mathbb{GR}_3$. After a bit of computation, we obtain coordinates $\underset{1}{\mathcal{W}}{}^i_{jmn}, \underset{2}{\mathcal{W}}{}^i_{jmn}, \underset{3}{\mathcal{W}}{}^i_{jmn}$ of the invariants $\underset{1}{\mathcal{W}}(X,Y,Z)$, $\underset{2}{\mathcal{W}}(X,Y,Z), \underset{3}{\mathcal{W}}(X,Y,Z)$ are:

$$\underset{1}{\mathcal{W}}:\begin{cases}\underset{1}{\mathcal{W}}{}^1_{223}=\underset{1}{\mathcal{W}}{}^2_{123}=\underset{1}{\mathcal{W}}{}^2_{312}=\underset{1}{\mathcal{W}}{}^3_{221}=\frac{1}{2}e^{-x^2}\\ \underset{1}{\mathcal{W}}{}^1_{221}=\underset{1}{\mathcal{W}}{}^1_{331}=\underset{1}{\mathcal{W}}{}^2_{112}=\underset{1}{\mathcal{W}}{}^2_{332}=\underset{1}{\mathcal{W}}{}^3_{131}=\underset{1}{\mathcal{W}}{}^3_{232}=\frac{1}{4}e^{-2x^2}\\ \underset{1}{\mathcal{W}}{}^1_{232}=\underset{1}{\mathcal{W}}{}^2_{132}=\underset{1}{\mathcal{W}}{}^2_{321}=\underset{1}{\mathcal{W}}{}^3_{212}=-\frac{1}{2}e^{-x^2}\\ \underset{1}{\mathcal{W}}{}^1_{212}=\underset{1}{\mathcal{W}}{}^1_{313}=\underset{1}{\mathcal{W}}{}^2_{121}=\underset{1}{\mathcal{W}}{}^2_{323}=\underset{1}{\mathcal{W}}{}^3_{113}=\underset{1}{\mathcal{W}}{}^3_{223}=-\frac{1}{4}e^{-2x^2}\end{cases}$$

$$\underset{2}{\mathcal{W}}:\begin{cases}\underset{2}{\mathcal{W}}{}^1_{232}=\underset{2}{\mathcal{W}}{}^2_{132}=\underset{2}{\mathcal{W}}{}^2_{321}=\underset{2}{\mathcal{W}}{}^3_{212}=\frac{1}{2}e^{-x^2}\\ \underset{2}{\mathcal{W}}{}^1_{221}=\underset{2}{\mathcal{W}}{}^1_{331}=\underset{2}{\mathcal{W}}{}^2_{112}=\underset{2}{\mathcal{W}}{}^2_{332}=\underset{2}{\mathcal{W}}{}^3_{131}=\underset{2}{\mathcal{W}}{}^3_{232}=\frac{1}{4}e^{-2x^2}\\ \underset{2}{\mathcal{W}}{}^1_{223}=\underset{2}{\mathcal{W}}{}^2_{123}=\underset{2}{\mathcal{W}}{}^2_{312}=\underset{2}{\mathcal{W}}{}^3_{221}=-\frac{1}{2}e^{-x^2}\\ \underset{2}{\mathcal{W}}{}^1_{212}=\underset{2}{\mathcal{W}}{}^1_{313}=\underset{2}{\mathcal{W}}{}^2_{121}=\underset{2}{\mathcal{W}}{}^2_{323}=\underset{2}{\mathcal{W}}{}^3_{113}=\underset{2}{\mathcal{W}}{}^3_{223}=-\frac{1}{4}e^{-2x^2}\end{cases}$$

$$\underset{3}{\mathcal{W}}:\begin{cases}\underset{3}{\mathcal{W}}{}^1_{322}=e^{-x^2},\underset{3}{\mathcal{W}}{}^3_{122}=-e^{-x^2},\underset{3}{\mathcal{W}}{}^1_{212}=\underset{3}{\mathcal{W}}{}^2_{121}=\underset{3}{\mathcal{W}}{}^3_{131}=\underset{3}{\mathcal{W}}{}^3_{232}=\frac{3}{4}e^{-2x^2}\\ \underset{3}{\mathcal{W}}{}^2_{312}=\underset{3}{\mathcal{W}}{}^2_{321}=\frac{1}{2}e^{-x^2},\\ \underset{3}{\mathcal{W}}{}^1_{221}=\underset{3}{\mathcal{W}}{}^1_{331}=\underset{3}{\mathcal{W}}{}^2_{112}=\underset{3}{\mathcal{W}}{}^2_{332}=\frac{1}{4}e^{-2x^2}\\ \underset{3}{\mathcal{W}}{}^1_{223}=\underset{3}{\mathcal{W}}{}^2_{232}=\underset{3}{\mathcal{W}}{}^2_{123}=\underset{3}{\mathcal{W}}{}^2_{132}=\underset{3}{\mathcal{W}}{}^3_{212}=\underset{3}{\mathcal{W}}{}^3_{221}=\underset{3}{\mathcal{W}}{}^3_{311}=\underset{3}{\mathcal{W}}{}^3_{322}=-\frac{1}{2}e^{-x^2}\\ \underset{3}{\mathcal{W}}{}^1_{313}=\underset{3}{\mathcal{W}}{}^2_{323}=\underset{3}{\mathcal{W}}{}^3_{113}=\underset{3}{\mathcal{W}}{}^3_{223}=-\frac{1}{4}e^{-2x^2}\end{cases}$$

and $\underset{\theta}{\mathcal{W}}{}^i_{jmn}=0, \theta, i, j, m, n = 1,2,3$, in all other cases.

3. Conclusion

We expanded the concept of geodesic mappings defined on spaces with torsion in this paper. Invariants of equitorsion geodesic mappings are obtained above. We found an another generalization of Weyl projective tensor in this way.

The results, which are obtained in this paper, are presented in the corresponding operator forms. Moreover, some of these results are presented coordinately for different applications in physics.

In further research, we will stay focused on NGT, generalized Riemannian spaces in the sense of Eisenhart, as well as semi-symmetric and quarter-symmetric spaces. We will also study conformal invariants of a generalized Riemannian manifold GR_N.

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