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FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER BASED ON SUBORDINATE CONDITIONS INVOLVING OF THE JACKSON (p,q)-DERIVATIVE

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Abstract. In the present paper, new subclasses of bi-univalent functions of complex order associated with the (p,q)-derivative are introduced. Furthermore, using the Faber polynomial expansions, we get upper bounds for the coefficients of functions belonging to these classes.

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1. INTRODUCTION, DEFINITIONS AND NOTATIONS

Denote by E the unit disc of the complex plane, $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, A the class of functions analytic in E, satisfying the conditions

$$f(0) = 0 \text{ and } \quad f'(0) = 1.$$

Then each function f in A has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by S the subclass of A consisting of functions the form (1.1) which are also univalent in E.

For f and F analytic in E, we say that f is subordinate to F, written $f \prec F$, if there exists a Schwarz function

$$t(z) = \sum_{n=1}^{\infty} c_n z^n$$

with $|t(z)| < 1$ in E, such that $f(z) = F(t(z))$. For the Schwarz function $t(z)$ we note that $|c_n| < 1$. (e.g. see Duren [11]).

It is well known that every function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z\,(z \in E)$ and $f\left(f^{-1}(w)\right) = w\ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E. Let Σ denote the class of bi-univalent functions in E given by (1.1). For a brief history and interesting examples in the class Σ, see [24] (see also [7], [8], [19], [20]). Furthermore, judging by the remarkable flood of papers on the subject (see, for example, [9], [5], [17], [23], [24], [25]).

The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in Geometric Function Theory. Grunsky [14] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalency of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [22] gave a differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (Schaeffer-Spencer [21]).

Not much is known about the bounds on the general coefficient $|a_n|$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([6], [15], [16]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N}\backslash\{1,2\};\ \ \mathbb{N} = \{1,2,3,...\}$) is still an open problem.

1.1. *The Jackson (p,q)-derivative*

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < p \leq 1$. We recall the definitions of fractional q-calculus operators of complex valued function f. We shall follow the notation and terminology in [13].

Definition 1 (see [10])**.** The (p,q)-derivative of the function f is defined as

$$(D_{p,q}f)(z) = \begin{cases} \frac{f(pz)-f(qz)}{(p-q)z} & \text{for } z \neq 0 \\ \\ f'(0) & \text{for } z = 0 \end{cases} . \tag{1.3}$$

From (1.3), we deduce that

$$(D_{p,q}f)(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q}\, a_n z^{n-1},$$

where the symbol $[n]_{p,q}$ denotes the so-called (p,q) *-bracket* or *twin-basic number*

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}.$$

It happens clearly that $D_{p,q}z^n = [n]_{p,q}\, z^{n-1}$. Note also that for $p = 1$, the Jackson (p,q)-derivative reduces to the Jackson q-derivative given by (see [18])

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0.$$

The twin-basic number is a natural generalization of the q-number, that is

$$\lim_{p\to 1} [n]_{p,q} = [n]_q = \frac{1-q^n}{1-q}, \quad q \neq 1.$$

As with ordinary derivative, the action of the (p,q)-derivative of a function is a linear operator. More precisely, for any constants a and b,

$$D_{p,q}(af(z) + bh(z)) = a\,D_{p,q}\,f(z) + bD_{p,q}h(z).$$

The (p,q)-derivative fulfils the following product rules

$$D_{p,q}(f(z)h(z)) = f(pz)D_{p,q}h(z) + h(qz)D_{p,q}\,f(z),$$
$$D_{p,q}(f(z)h(z)) = h(pz)D_{p,q}\,f(z) + f(qz)D_{p,q}h(z).$$

Further, the (p,q)-derivative fulfils the following product rules

$$D_{p,q}\left(\frac{f(z)}{h(z)}\right) = \frac{h(qz)D_{p,q}\,f(z) - f(qz)D_{p,q}h(z)}{h(pz)h(qz)},$$
$$D_{p,q}\left(\frac{f(z)}{h(z)}\right) = \frac{h(pz)D_{p,q}\,f(z) - f(pz)D_{p,q}h(z)}{h(pz)h(qz)}.$$

From (1.2) and (1.3), we also deduce that

$$\begin{aligned}(D_{p,q}g)(w) &= \tfrac{g(pw)-g(qw)}{(p-q)w} \\ &= 1 - [2]_{p,q}\,a_2 w + [3]_{p,q}\left(2a_2^2 - a_3\right)w^2 \\ &\quad - [4]_{p,q}\left(5a_2^3 - 5a_2a_3 + a_4\right)w^3 + \cdots\end{aligned}$$

where the function g is given by (1.2).

2. Preliminary results

By using the Faber polynomial expansion of functions $f \in A$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [3]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots)\, w^n,$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!\,(n-3)!} a_2^{n-3} a_3$$

$$
\begin{aligned}
&+\frac{(-n)!}{(-2n+3)!\,(n-4)!}a_2^{n-4}a_4\\
&+\frac{(-n)!}{[2(-n+2)]!\,(n-5)!}a_2^{n-5}\left[a_5+(-n+2)\,a_3^2\right]\\
&+\frac{(-n)!}{(-2n+5)!\,(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)\,a_3a_4\right]\\
&+\sum_{j\geq 7}a_2^{n-j}V_j,
\end{aligned}
\tag{2.1}
$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ [4]. In particular, the first three terms of K_{n-1}^{-n} are given below:

$$
\begin{aligned}
\frac{1}{2}K_1^{-2} &= -a_2,\\
\frac{1}{3}K_2^{-3} &= 2a_2^2-a_3,\\
\frac{1}{4}K_3^{-4} &= -\left(5a_2^3-5a_2a_3+a_4\right).
\end{aligned}
\tag{2.2}
$$

In general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^{p} is as, [3],

$$
K_{n-1}^{p} = pa_n + \frac{p(p-1)}{2}E_{n-1}^{2} + \frac{p!}{(p-3)!3!}E_{n-1}^{3} + ... + \frac{p!}{(p-n+1)!\,(n-1)!}E_{n-1}^{n-1},
\tag{2.3}
$$

where $E_{n-1}^{p} = E_{n-1}^{p}(a_2, a_3, ...)$ and by [1],

$$
E_{n-1}^{m}(a_2,...,a_n) = \sum_{n=2}^{\infty}\frac{m!\,(a_2)^{\mu_1}...(a_n)^{\mu_{n-1}}}{\mu_1!...\mu_{n-1}!}, \quad \text{for } m \leq n
$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$
\begin{aligned}
\mu_1+\mu_2+\ ...\ +\mu_{n-1} &= m,\\
\mu_1+2\mu_2+\ ...\ +(n-1)\,\mu_{n-1} &= n-1.
\end{aligned}
$$

Evidently, $E_{n-1}^{n-1}(a_2,...,a_n) = a_2^{n-1}$,(see [2]); while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$
\begin{aligned}
\mu_1+\mu_2+\ ...\ +\mu_n &= m,\\
\mu_1+2\mu_2+\ ...\ +n\mu_n &= n.
\end{aligned}
$$

It is clear that $E_n^n(a_1,a_2,...,a_n) = a_1^n$. The first and the last polynomials are:

$$
E_n^1 = a_n, \qquad E_n^n = a_1^n.
$$

In the following, let ϕ be an analytic function with positive real part in E, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(E)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{2.4}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk E with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, and suppose that

$$u(z) = p_1 z + \sum_{n=2}^{\infty} p_n z^n, \quad v(w) = q_1 w + \sum_{n=2}^{\infty} q_n w^n \quad (|z| < 1). \tag{2.5}$$

It is well known that

$$|p_1| \leq 1, \quad |p_2| \leq 1 - |p_1|^2, \quad |q_1| \leq 1, \quad |q_2| \leq 1 - |q_1|^2. \tag{2.6}$$

Next, the equations (2.4) and (2.5) lead to

$$\begin{aligned}
\phi(u(z)) &= 1 + B_1 u(z) + B_2 (u(z))^2 z^2 + \cdots \\
&= 1 + B_1 p_1 z + \left(B_1 p_2 + B_2 p_1^2\right) z^2 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_k E_n^k (p_1, p_2, ..., p_n) z^n,
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\phi(v(w)) &= 1 + B_1 v(w) + B_2 (v(w))^2 z^2 + \cdots \\
&= 1 + B_1 q_1 w + \left(B_1 q_2 + B_2 q_1^2\right) w^2 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_k E_n^k (q_1, q_2, ..., q_n) w^n.
\end{aligned} \tag{2.8}$$

Definition 2. A function $f \in \Sigma$ is said to be in the class $R_{\Sigma,b}^{p,q}(\phi)$ if the following subordination relationships hold true:

$$\left[1 + \frac{1}{b}\left((D_{p,q} f)(z) - 1\right)\right] \prec \phi(z) \tag{2.9}$$

and

$$\left[1 + \frac{1}{b}\left((D_{p,q} g)(w) - 1\right)\right] \prec \phi(w) \tag{2.10}$$

where $b \in \mathbb{C}\backslash\{0\}$, $0 < q < p \leq 1$; $z, w \in E$ and the function g is given by (1.2).

We note from Definition 2 that

$$\lim_{p\to 1} R_{\Sigma,b}^{p,q}(\phi) = \left\{ f \in \Sigma : \left\{ \begin{array}{c} \lim\limits_{p\to 1}\left[1+\frac{1}{b}\left((D_{p,q}f)(z)-1\right)\right] \prec \phi(z) \\ \\ \lim\limits_{p\to 1}\left[1+\frac{1}{b}\left((D_{p,q}g)(w)-1\right)\right] \prec \phi(w) \end{array} \right. \right\} = R_{\Sigma,b}^{q}(\phi).$$

By suitably specializing the parameters b and p, we state the new subclasses of bi-univalent functions of complex order $R_{\Sigma,b}^{p,q}(\phi)$ as illustrated in the following Definitions.

Definition 3. For $b = 1$, a function $f \in \Sigma$ is said to be in the class $R_{\Sigma}^{p,q}(\phi)$ if it satisfies the following conditions respectively:

$$(D_{p,q}f)(z) \prec \phi(z)$$

and

$$(D_{p,q}g)(w) \prec \phi(w)$$

where $0 < q < p \leq 1; z, w \in E$ and the function g is given by (1.2).

Definition 4. For $b = 1$ and $p \to 1$, a function $f \in \Sigma$ is said to be in the class $R_{\Sigma}^{q}(\phi)$ if it satisfies the following conditions respectively:

$$(D_q f)(z) \prec \phi(z)$$

and

$$(D_q g)(w) \prec \phi(w)$$

where $z, w \in E$ and the function g is given by (1.2).

In this paper, we study the class $R_{\Sigma,b}^{p,q}(\phi)$ of analytic bi-univalent functions defined above by using the (p,q)-derivative operator. Moreover, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $R_{\Sigma,b}^{p,q}(\phi)$ as well as we provide estimates for the initial coefficients of these functions. Several new consequences of the results are also pointed out.

3. Main results

Our first main result is given by Theorem 1 below.

Theorem 1. *For $b \in \mathbb{C}\backslash\{0\}$, let $f \in \mathrm{R}_{\Sigma,b}^{p,q}(\phi)$. If $a_m = 0$; $2 \leq m \leq n-1$, then*

$$|a_n| \leq \frac{B_1 |b|}{[n]_{p,q}}, \quad n \geq 3.$$

Proof. Let f be given by (1.1). We have

$$(D_{p,q}f)(z)-1=\sum_{n=2}^{\infty}[n]_{p,q}\,a_n z^{n-1}, \tag{3.1}$$

and, for its inverse map $g=f^{-1}$, it is seen that

$$(D_{p,q}g)(w)-1=\sum_{n=2}^{\infty}K_{n-1}^{-n}(a_2,a_3,...)\,w^{n-1}=\sum_{n=2}^{\infty}[n]_{p,q}\,b_n w^{n-1}. \tag{3.2}$$

From (2.9) and (2.10) yields

$$1+\frac{1}{b}\big((D_{p,q}f)(z)-1\big)=\varphi(u(z)) \tag{3.3}$$

and

$$1+\frac{1}{b}\big((D_{p,q}g)(w)-1\big)=\varphi(v(w)) \tag{3.4}$$

Comparing the corresponding coefficients of (3.3) and (3.4) yields

$$\frac{1}{b}[n]_{p,q}\,a_n=B_1\,p_{n-1}, \tag{3.5}$$

and

$$\frac{1}{b}[n]_{p,q}\,b_n=B_1 q_{n-1}. \tag{3.6}$$

Note that for $a_m=0;\ 2\le m\le n-1$ we have $b_n=-a_n$ and so

$$\frac{1}{b}[n]_{p,q}\,a_n=B_1\,p_{n-1},$$
$$-\frac{1}{b}[n]_{p,q}\,a_n=B_1 q_{n-1}.$$

Now taking the absolute values of either of the above two equations and from (2.6), we obtain

$$|a_n|=\frac{B_1\,|p_{n-1}|\,|b|}{[n]_{p,q}}=\frac{B_1\,|q_{n-1}|\,|b|}{[n]_{p,q}}\le\frac{B_1\,|b|}{[n]_{p,q}}.$$

□

Theorem 2. *For $b\in\mathbb{C}\backslash\{0\}$, let $f\in \mathrm{R}^{q}_{\Sigma,b}(\phi)$. If $a_m=0$; $2\le m\le n-1$, then*

$$|a_n|\le\frac{B_1\,|b|\,(1-q)}{1-q^n};\quad n\ge 3.$$

Theorem 3. *Let $f\in \mathrm{R}^{p,q}_{\Sigma}(\phi)$. If $a_m=0$; $2\le m\le n-1$, then*

$$|a_n|\le\frac{B_1}{[n]_{p,q}};\quad n\ge 3.$$

Theorem 4. *Let* $f \in \mathrm{R}_{\Sigma}^{q}(\phi)$. *If* $a_m = 0$; $2 \le m \le n-1$, *then*

$$|a_n| \le \frac{B_1(1-q)}{1-q^n}; \quad n \ge 3.$$

4. Coefficient estimates

In this section we obtain coefficient estimates for functions belonging to the classes $R_{\Sigma,b}^{p,q}(\phi)$, $R_{\Sigma,b}^{q}(\phi)$, $R_{\Sigma}^{p,q}(\phi)$ and $R_{\Sigma}^{q}(\phi)$.

Theorem 5. *Let* $f \in \mathrm{R}_{\Sigma,b}^{p,q}(\phi)$ $(b \in \mathbb{C}\backslash\{0\})$. *Then*

$$|a_2| \le \min\left\{K(p,q), \frac{|b|\,B_1\sqrt{B_1}}{\sqrt{\left|(B_1^2 b - B_2)(p^2+q^2) + (B_1^2 b - 2B_2)pq\right| + B_1(p^2+2pq+q^2)}}\right\}$$

and

$$|a_3| \le \min\{L(p,q), M(p,q)\}$$

where

$$K(p,q) = \begin{cases} \sqrt{\frac{B_1|b|}{p^2+pq+q^2}}; & |B_2| \le B_1 \\ \sqrt{\frac{|B_2||b|}{p^2+pq+q^2}}; & |B_2| > B_1 \end{cases},$$

$$L(p,q) = \begin{cases} \frac{B_1|b|}{p^2+pq+q^2}; & |B_2| \le B_1 \\ \frac{|B_2||b|}{p^2+pq+q^2}; & |B_2| > B_1 \end{cases},$$

and

$$M(p,q) = \begin{cases} \frac{B_1|b|}{p^2+pq+q^2}; & B_1 \le \frac{p^2+2pq+q^2}{(p^2+pq+q^2)|b|} \\ \frac{|b|B_1\left[\left|(B_1^2 b - B_2)(p^2+q^2)+(B_1^2 b - 2B_2)pq\right| + B_1^2|b|(p^2+pq+q^2)\right]}{\left[\left|(B_1^2 b - B_2)(p^2+q^2)+(B_1^2 b - 2B_2)pq\right| + B_1(p^2+2pq+q^2)\right](p^2+pq+q^2)}; & B_1 > \frac{p^2+2pq+q^2}{(p^2+pq+q^2)|b|}. \end{cases}$$

Proof. Replacing n by 2 and 3 in (3.5) and (3.6), respectively, we find that

$$\frac{1}{b}[2]_{p,q}\,a_2 = B_1 p_1, \tag{4.1}$$

$$\frac{1}{b}[3]_{p,q}\,a_3 = B_1 p_2 + B_2 p_1^2, \tag{4.2}$$

$$-\frac{1}{b}[2]_{p,q}\,a_2 = B_1 q_1, \tag{4.3}$$

$$\frac{1}{b}[3]_{p,q}\,(2a_2^2 - a_3) = B_1 q_2 + B_2 q_1^2. \tag{4.4}$$

From (4.1) and (4.3), we obtain

$$p_1 = -q_1. \tag{4.5}$$

By adding (4.4) to (4.2), further computations using (4.5) lead to

$$\frac{2}{b}[3]_{p,q}\, a_2^2 = B_1\left(p_2+q_2\right)+2B_2\, p_1^2. \tag{4.6}$$

Making use of (4.1) in the above equality (4.6), we get

$$\left[2bB_1^2\,[3]_{p,q} - 2B_2\,[2]_{p,q}^2\right] a_2^2 = b^2 B_1^3\left(p_2+q_2\right). \tag{4.7}$$

Combining (4.7) and (2.6), we obtain

$$\begin{aligned} 2\left|bB_1^2\,[3]_{p,q} - B_2\,[2]_{p,q}^2\right| |a_2|^2 &\le |b|^2 B_1^3\left(|p_2|+|q_2|\right) \\ &\le 2|b|^2 B_1^3\left(1-|p_1|^2\right) \\ &= 2|b|^2 B_1^3 - 2|b|^2 B_1^3 |p_1|^2. \end{aligned} \tag{4.8}$$

It follows from (4.1) that

$$|a_2| \le \frac{|b|\, B_1\sqrt{B_1}}{\sqrt{\left|B_1^2 b\,[3]_{p,q} - B_2\,[2]_{p,q}^2\right| + B_1\,[2]_{p,q}^2}}. \tag{4.9}$$

Moreover, by (2.6) and (4.6)

$$\begin{aligned} \frac{2}{|b|}[3]_{p,q}\,|a_2|^2 &\le B_1\left(|p_2|+|q_2|\right)+2|B_2||p_1|^2 \\ &\le 2B_1\left(1-|p_1|^2\right)+2|B_2||p_1|^2 \\ &= 2B_1 + 2|p_1|^2\left(|B_2|-B_1\right) \end{aligned}$$

$$\frac{1}{|b|}[3]_{p,q}\,|a_2|^2 \le \begin{cases} B_1; |B_2| \le B_1 \\ |B_2|; |B_2| > B_1 \end{cases}.$$

Clearly, we can see that

$$|a_2| \leq \begin{cases} \sqrt{\frac{B_1|b|}{[3]_{p,q}}}; & |B_2| \leq B_1 \\ \\ \sqrt{\frac{|B_2||b|}{[3]_{p,q}}}; & |B_2| > B_1 \end{cases}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (4.4) from (4.2), we obtain

$$\frac{2}{b}[3]_{p,q}\, a_3 = \frac{2}{b}[3]_{p,q}\, a_2^2 + B_1(p_2 - q_2). \tag{4.10}$$

Clearly, from (4.6), we have that

$$a_3 = \frac{b\left[B_1(p_2+q_2) + 2B_2p_1^2\right]}{2[3]_{p,q}} + \frac{bB_1(p_2-q_2)}{2[3]_{p,q}}$$

$$= \frac{bB_1p_2 + bB_2p_1^2}{[3]_{p,q}}$$

and consequently

$$|a_3| \leq \frac{|b|\,B_1\,|p_2| + |b|\,|B_2|\,|p_1|^2}{[3]_{p,q}}$$

$$\leq \frac{|b|\,B_1\left(1-|p_1|^2\right) + |b|\,|B_2|\,|p_1|^2}{[3]_{p,q}}$$

$$= \frac{|b|\,B_1 + |b|\,|p_1|^2\,(|B_2| - B_1)}{[3]_{p,q}}.$$

Hence, we write

$$|a_3| \leq \begin{cases} \frac{B_1|b|}{[3]_{p,q}}; & |B_2| \leq B_1 \\ \\ \frac{|B_2||b|}{[3]_{p,q}}; & |B_2| > B_1. \end{cases}$$

On the other hand, by using (2.6) and (4.10), we have

$$\frac{2}{|b|}[3]_{p,q}\,|a_3| \leq \frac{2}{|b|}[3]_{p,q}\,|a_2|^2 + B_1(|p_2| + |q_2|)$$

$$\leq \frac{2}{|b|}[3]_{p,q}\,|a_2|^2 + 2B_1\left(1-|p_1|^2\right).$$

Then, with the help of (4.1), we have

$$B_1\,|b|\,[3]_{p,q}\,|a_3| \leq \left[B_1\,|b|\,[3]_{p,q} - [2]_{p,q}^2\right]|a_2|^2 + B_1^2\,|b|^2.$$

Now, from (4.9), we obtain

$$|a_3| \leq \frac{B_1\,|b|}{[3]_{p,q}}\left\{1 + \frac{B_1\left[B_1\,|b|\,[3]_{p,q} - [2]_{p,q}^2\right]}{\left|B_1^2 b\,[3]_{p,q} - B_2\,[2]_{p,q}^2\right| + B_1\,[2]_{p,q}^2}\right\}.$$

□

Theorem 6. *Let* $f \in \mathrm{R}_{\Sigma,b}^{q}(\phi)$ $(b \in \mathbb{C}\backslash\{0\})$. *Then*

$$|a_2| \leq \min\left\{K(q), \frac{|b|\,B_1\sqrt{B_1}}{\sqrt{\left|(B_1^2 b - B_2)(1+q^2) + (B_1^2 b - 2B_2)q\right| + B_1(1+2q+q^2)}}\right\}$$

and

$$|a_3| \leq \min\{L(q), M(q)\}$$

where

$$K(q) = \begin{cases} \sqrt{\frac{B_1|b|}{1+q+q^2}}; & |B_2| \leq B_1 \\ \sqrt{\frac{|B_2||b|}{1+q+q^2}}; & |B_2| > B_1 \end{cases},$$

$$L(q) = \begin{cases} \frac{B_1|b|}{1+q+q^2}; & |B_2| \leq B_1 \\ \frac{|B_2||b|}{1+q+q^2}; & |B_2| > B_1 \end{cases},$$

and

$$M(q) = \begin{cases} \frac{B_1|b|}{1+q+q^2}; & B_1 \leq \frac{1+2q+q^2}{(1+q+q^2)|b|} \\ \frac{|b|B_1\left[\left|(B_1^2 b - B_2)(1+q^2) + (B_1^2 b - 2B_2)q\right| + B_1^2|b|(1+q+q^2)\right]}{\left[\left|(B_1^2 b - B_2)(1+q^2) + (B_1^2 b - 2B_2)q\right| + B_1(1+2q+q^2)\right](1+q+q^2)}; & B_1 > \frac{1+2q+q^2}{(1+q+q^2)|b|} \end{cases}.$$

Theorem 7. *Let* $f \in \mathrm{R}_{\Sigma}^{p,q}(\phi)$. *Then*

$$|a_2| \leq \min\left\{K(p,q), \frac{B_1\sqrt{B_1}}{\sqrt{\left|(B_1^2 - B_2)(p^2+q^2) + (B_1^2 - 2B_2)pq\right| + B_1(p^2+2pq+q^2)}}\right\}$$

and

$$|a_3| \leq \min\{L(p,q), M(p,q)\}$$

where

$$K(p,q)=\begin{cases}\sqrt{\frac{B_1}{p^2+pq+q^2}}; & |B_2|\le B_1\\ \sqrt{\frac{|B_2|}{p^2+pq+q^2}}; & |B_2|>B_1\end{cases},$$

$$L(p,q)=\begin{cases}\frac{B_1}{p^2+pq+q^2}; & |B_2|\le B_1\\ \frac{|B_2|}{p^2+pq+q^2}; & |B_2|>B_1\end{cases},$$

and

$$M(p,q)=\begin{cases}\frac{B_1}{p^2+pq+q^2}; & B_1\le\frac{p^2+2pq+q^2}{p^2+pq+q^2}\\ \frac{B_1\left[\left|(B_1^2-B_2)(p^2+q^2)+(B_1^2-2B_2)pq\right|+B_1^2(p^2+pq+q^2)\right]}{\left[\left|(B_1^2-B_2)(p^2+q^2)+(B_1^2-2B_2)pq\right|+B_1(p^2+2pq+q^2)\right](p^2+pq+q^2)}; & B_1>\frac{p^2+2pq+q^2}{p^2+pq+q^2}\end{cases}.$$

Theorem 8. *Let* $f\in \mathrm{R}_{\Sigma}^{q}(\phi)$. *Then*

$$|a_2|\le\min\left\{K(q),\frac{B_1\sqrt{B_1}}{\sqrt{\left|(B_1^2-B_2)(1+q^2)+(B_1^2-2B_2)q\right|+B_1(1+2q+q^2)}}\right\}$$

and

$$|a_3|\le\min\{L(q),M(q)\}$$

where

$$K(q)=\begin{cases}\sqrt{\frac{B_1}{1+q+q^2}}; & |B_2|\le B_1\\ \sqrt{\frac{|B_2|}{1+q+q^2}}; & |B_2|>B_1\end{cases},$$

$$L(q)=\begin{cases}\frac{B_1}{1+q+q^2}; & |B_2|\le B_1\\ \frac{|B_2|}{1+q+q^2}; & |B_2|>B_1\end{cases},$$

and

$$M(q)=\begin{cases}\frac{B_1}{1+q+q^2}; & B_1\le\frac{1+2q+q^2}{1+q+q^2}\\ \frac{B_1\left[\left|(B_1^2-B_2)(1+q^2)+(B_1^2-2B_2)q\right|+B_1^2(1+q+q^2)\right]}{\left[\left|(B_1^2-B_2)(1+q^2)+(B_1^2-2B_2)q\right|+B_1(1+2q+q^2)\right](1+q+q^2)}; & B_1>\frac{1+2q+q^2}{1+q+q^2}\end{cases}.$$

5. Corollaries and concluding Remarks

Corollary 1. *If we take*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \ldots \quad (0 < \alpha \le 1),$$

in Theorem 1 and Theorem 2, respectively, we have

$$|a_n| \le \frac{2\alpha\,|b|}{[n]_{p,q}}; \quad n \ge 3$$

and

$$|a_n| \le \frac{2\alpha\,|b|\,(1-q)}{1-q^n}; \quad n \ge 3.$$

Remark 1. Let $f \in R_{\Sigma}^{p,q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. Then

$$|a_n| \le \frac{2\alpha}{[n]_{p,q}}; \quad n \ge 3.$$

Remark 2. Let $f \in R_{\Sigma}^{q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. Then

$$|a_n| \le \frac{2\alpha(1-q)}{1-q^n}; \quad n \ge 3.$$

Corollary 2. *If we take*

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \le \beta < 1),$$

in Theorem 1 and Theorem 2, respectively, we have

$$|a_n| \le \frac{2(1-\beta)\,|b|}{[n]_{p,q}}; \quad n \ge 3$$

and

$$|a_n| \le \frac{2(1-\beta)\,|b|\,(1-q)}{1-q^n}; \quad n \ge 3.$$

Remark 3. Let $f \in R_{\Sigma}^{p,q}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. Then

$$|a_n| \le \frac{2(1-\beta)}{[n]_{p,q}}; \quad n \ge 3.$$

Remark 4. Let $f \in R_{\Sigma}^{p,q}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. Then

$$|a_n| \le \frac{2(1-\beta)(1-q)}{1-q^n}; \quad n \ge 3.$$

Taking $\phi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$ in Theorem 5 and Theorem 6, then we have the following results.

Corollary 3. *Let* $f\in \mathrm{R}_{\Sigma,b}^{p,q}(\phi)$ $(b\in\mathbb{C}\backslash\{0\})$. *Then*

$$|a_2|\le \min\left\{\sqrt{\frac{2\alpha|b|}{p^2+pq+q^2}},\frac{2\alpha|b|}{\sqrt{\alpha\left|(2b-1)(p^2+q^2)+2(b-1)pq\right|+p^2+2pq+q^2}}\right\}$$

and

$$|a_3|\le \min\left\{\frac{2\alpha|b|}{p^2+pq+q^2},H(p,q)\right\}$$

where

$$H(p,q)=\begin{cases}\frac{2\alpha|b|}{p^2+pq+q^2}; & 0<\alpha\le\frac{p^2+2pq+q^2}{2(p^2+pq+q^2)|b|}\\ \frac{2\alpha^2|b|\left[\left|(2b-1)(p^2+q^2)+2(b-1)pq\right|+2|b|(p^2+pq+q^2)\right]}{\left[\alpha\left|(2b-1)(p^2+q^2)+2(b-1)pq\right|+p^2+2pq+q^2\right](p^2+pq+q^2)}; & \frac{p^2+2pq+q^2}{2(p^2+pq+q^2)|b|}<\alpha\le 1\end{cases}.$$

Corollary 4. *Let* $f\in \mathrm{R}_{\Sigma,b}^{q}(\phi)$ $(b\in\mathbb{C}\backslash\{0\})$. *Then*

$$|a_2|\le \min\left\{\sqrt{\frac{2\alpha|b|}{1+q+q^2}},\frac{2\alpha|b|}{\sqrt{\alpha\left|(2b-1)(1+q^2)+2(b-1)q\right|+1+2q+q^2}}\right\}$$

and

$$|a_3|\le \min\left\{\frac{2\alpha|b|}{1+q+q^2},H(q)\right\}$$

where

$$H(q)=\begin{cases}\frac{2\alpha|b|}{1+q+q^2}; & 0<\alpha\le\frac{1+2q+q^2}{2(1+q+q^2)|b|}\\ \frac{2\alpha^2|b|\left[\left|(2b-1)(1+q^2)+2(b-1)q\right|+2|b|(1+q+q^2)\right]}{\left[\alpha\left|(2b-1)(1+q^2)+2(b-1)q\right|+(1+2q+q^2)\right](1+q+q^2)}; & \frac{1+2q+q^2}{2(1+q+q^2)|b|}<\alpha\le 1\end{cases}.$$

Corollary 5. *Let* $f\in \mathrm{R}_{\Sigma}^{p,q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. *Then*

$$|a_2|\le\frac{2\alpha}{\sqrt{(\alpha+1)(p^2+q^2)+2pq}}$$

and

$$|a_3|\le \min\left\{\frac{2\alpha}{p^2+pq+q^2},H(p,q)\right\}$$

where

$$H(p,q)=\begin{cases}\frac{2\alpha}{p^2+pq+q^2}; & 0<\alpha\le\frac{p^2+2pq+q^2}{2(p^2+pq+q^2)}\\ \frac{2\alpha^2\left[3(p^2+q^2)+2pq\right]}{\left[(\alpha+1)(p^2+q^2)+2pq\right](p^2+pq+q^2)}; & \frac{p^2+2pq+q^2}{2(p^2+pq+q^2)}<\alpha\le 1\end{cases}.$$

Corollary 6. *Let* $f \in \mathrm{R}_{\Sigma}^{q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\alpha+1)(1+q^2)+2q}}$$

and

$$|a_3| \leq \min\left\{\frac{2\alpha}{1+q+q^2}, H(q)\right\}$$

where

$$H(q) = \begin{cases} \frac{2\alpha}{1+q+q^2}; & 0 < \alpha \leq \frac{1+2q+q^2}{2(1+q+q^2)} \\ \frac{2\alpha^2[3(1+q^2)+2q]}{[(\alpha+1)(1+q^2)+2q](1+q+q^2)}; & \frac{1+2q+q^2}{2(1+q+q^2)} < \alpha \leq 1 \end{cases}.$$

By choosing $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$ in Theorem 5 and Theorem 6, then we have the following results.

Corollary 7. *Let* $f \in \mathrm{R}_{\Sigma,b}^{p,q}(\phi)$ $(b \in \mathbb{C}\backslash\{0\})$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)|b|}{p^2+pq+q^2}}, \frac{2(1-\beta)|b|}{\sqrt{\left|2(1-\beta)b[3]_{p,q}-[2]_{p,q}^2\right|+[2]_{p,q}^2}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)|b|}{p^2+pq+q^2}, R(p,q)\right\}$$

where

$$R(p,q) = \begin{cases} \frac{2(1-\beta)|b|}{p^2+pq+q^2}; & \frac{2(p^2+pq+q^2)|b|-(p^2+2pq+q^2)}{2(p^2+pq+q^2)|b|} \leq \beta < 1 \\ \frac{2(1-\beta)|b|\left[\left|2(1-\beta)b(p^2+pq+q^2)-(p^2+2pq+q^2)\right|+2(1-\beta)|b|(p^2+pq+q^2)\right]}{\left[\left|2(1-\beta)b(p^2+pq+q^2)-(p^2+2pq+q^2)\right|+(p^2+2pq+q^2)\right](p^2+pq+q^2)}; & \\ & 0 \leq \beta < \frac{2(p^2+pq+q^2)|b|-(p^2+2pq+q^2)}{2(p^2+pq+q^2)|b|}. \end{cases}$$

Corollary 8. *Let* $f \in \mathrm{R}_{\Sigma,b}^{q}(\phi)$ $(b \in \mathbb{C}\backslash\{0\})$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)|b|}{1+q+q^2}}, \frac{2(1-\beta)|b|}{\sqrt{|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)|+1+q+q^2}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)|b|}{1+q+q^2}, R(q)\right\}$$

where

$$R(q)=\begin{cases}\frac{2(1-\beta)|b|}{1+q+q^2}; & \frac{2(1+q+q^2)|b|-(1+2q+q^2)}{2(1+q+q^2)|b|}\leq\beta<1\\ \frac{2(1-\beta)|b|\left[\left|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)\right|+2(1-\beta)|b|(1+q+q^2)\right]}{\left[\left|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)\right|+1+2q+q^2\right](1+q+q^2)}; & \\ & 0\leq\beta<\frac{2(1+q+q^2)|b|-(1+2q+q^2)}{2(1+q+q^2)|b|}.\end{cases}$$

Corollary 9. *Let* $f\in \mathrm{R}_{\Sigma}^{p,q}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. *Then*

$$|a_2|\leq\min\left\{\sqrt{\frac{2(1-\beta)}{p^2+pq+q^2}},\frac{2(1-\beta)}{\sqrt{|(1-2\beta)(p^2+q^2)-2\beta pq|+p^2+2pq+q^2}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{2(1-\beta)}{p^2+pq+q^2},R(p,q)\right\}$$

where

$$R(p,q)=\begin{cases}\frac{2(1-\beta)}{p^2+pq+q^2}; & \frac{p^2+q^2}{2(p^2+pq+q^2)}\leq\beta<1\\ \frac{(3-4\beta)(p^2+q^2)+2(1-2\beta)pq}{(p^2+pq+q^2)^2}; & 0\leq\beta<\frac{p^2+q^2}{2(p^2+pq+q^2)}\end{cases}.$$

Corollary 10. *Let* $f\in \mathrm{R}_{\Sigma}^{q}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. *Then*

$$|a_2|\leq\min\left\{\sqrt{\frac{2(1-\beta)}{1+q+q^2}},\frac{2(1-\beta)}{\sqrt{\left|(1-2\beta)(1+q^2)-2\beta q)\right|+1+q+q^2}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{2(1-\beta)}{1+q+q^2},R(q)\right\}$$

where

$$R(q)=\begin{cases}\frac{2(1-\beta)}{1+q+q^2}; & \frac{1+q^2}{2(1+q+q^2)}\leq\beta<1\\ \frac{(3-4\beta)(1+q^2)+2(1-2\beta)q}{(1+q+q^2)^2}; & 0\leq\beta<\frac{1+q^2}{2(1+q+q^2)}\end{cases}.$$

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