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A MEAN VALUE THEOREM FOR N-SIMPLE FUNCTIONALS IN THE SENSE OF POPOVICIU

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Abstract. We establish some mean value theorems involving n-simple functionals in the sense of Popoviciu. In particular, we obtain the Kowalewski mean value formula.

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1. Preliminaries and auxiliary results

Throughout the paper $n \geq 1$ denotes an integer. Let $[a,b]$ be a real interval and $a \leq x_0 < \cdots < x_n \leq b$.

1.1. *The divided difference*

In most books on numerical analysis, the divided differences of a function $f:\{x_0,\ldots,x_n\} \to \mathbb{R}$ are defined recursively:

$$[x_i;f] := f(x_i), \qquad i = 0,\ldots,n,$$

$$[x_0,\ldots,x_k;f] = \frac{[x_1,\ldots,x_k;f]-[x_0,\ldots,x_{k-1};f]}{x_k - x_0}, \qquad k = 1,\ldots,n,$$

but can be written also in terms of determinants,

$$[x_0,\ldots,x_n;f] = \frac{1}{V(x_0,\ldots,x_n)} \begin{vmatrix} 1 & x_0 & \cdots & x_0^{n-1} & f(x_0) \\ 1 & x_1 & \cdots & x_1^{n-1} & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} & f(x_n) \end{vmatrix},$$

where $V(x_0,\ldots,x_n)$ denotes the Vandermonde determinant

$$\begin{vmatrix} 1 & x_0 & \cdots & x_0^{n} \\ 1 & x_1 & \cdots & x_1^{n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n} \end{vmatrix}.$$

1.2. *Convexity of high order*

The standard definition of n-convex functions is that through divided differences (see, e.g., [6]). A function $f:[a,b]\to\mathbb{R}$ is said to be *n-convex* if

$$[x_0,\dots,x_n;f]>0,$$

for any pairwise distinct points $x_0,\dots,x_n\in[a,b]$.

1.3. *Simple form functionals*

Let $e_i:[a,b]\to\mathbb{R}$, $e_i(x)=x^i$, $i=0,1,\dots,n$. The definition and the theorem rewritten below belong to Popoviciu.

Definition 1 ([7,9]). A linear functional $A:C[a,b]\to\mathbb{R}$ is said to be n-simple in the sense of Popoviciu if:

(i) $A(e_0)=A(e_1)=\cdots=A(e_{n-1})=0$,
(ii) $A(f)\neq 0$ for any n-convex function $f\in C[a,b]$.

In particular, $A(e_n)\neq 0$.

A connection between divided differences and n-simple functionals is supplied by the following celebrated result of Popoviciu.

Theorem 1 ([7,9, Popoviciu]). *A linear functional $A:C[a,b]\to\mathbb{R}$ is n-simple if and only if for any $f\in C[a,b]$, there exist distinct points $\xi_0,\dots,\xi_n\in(a,b)$ such that*

$$A(f)=A(e_n)\,[\xi_0,\dots,\xi_n;f]. \tag{1.1}$$

1.4. *The finite difference operator*

The finite difference with step h ($0<h\le(b-a)/n$) of the function $f:[a,b]\to\mathbb{R}$ at the point $x\in[a,b)$ is defined by

$$\Delta_h^n f(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f\Big(x+(n-k)h\Big).$$

Recall that finite difference can be written as a divided difference on equidistant points:

$$\Delta_h^n f(x)=n!h^n\big[x,x+h,\dots,x+nh;f\big].$$

Let f possess a derivative of order n on the interval (a,b) and x, $x+nh\in[a,b]$.

By using the Lagrange mean value theorem, one can prove, step by step, that there exist $c_k\in(x,x+nh)$, $k=1,2,\dots,n$, such that:

$$\Delta_h^n f(x)=h\,\Delta_h^{n-1}f'(c_1)=\cdots=h^k\,\Delta_h^{n-k}f^{(k)}(c_k)=\cdots=h^n\,\Delta_h^0 f^{(n)}(c_n), \tag{1.2}$$

where $\Delta_h^0 f^{(n)}(c_n)=f^{(n)}(c_n)$

One connection between the divided difference and the n-th derivative of a function is given by a classical result of Cauchy (see, e.g., [11, Theorem 2.10]):

Proposition 1 (Cauchy). *Let $f \in C^n[a,b]$ and $x_0, \dots, x_n$ be pairwise distinct points in $[a,b]$. Then there exists at least one point $\xi \in [a,b]$ such that*

$$[x_0,\dots,x_n;f] = \frac{f^{(n)}(\xi)}{n!}.$$

A much deeper mean value property for divided differences is provided by the following theorem of Popoviciu.

Proposition 2 ([8, Popoviciu (1954)]). *If the function f is continuous on an interval $[a,b]$ containing the distinct points $x_0,\dots,x_n$, then there exists $c \in (a,b)$ and $h > 0$ such that*

$$[x_0,\dots,x_n;f] = \frac{\Delta_h^n f(c)}{n!\,h^n}. \tag{1.3}$$

The following result is a generalization of Cauchy's mean-value theorem:

Theorem 2 ([3, Kowalewski (1932), p. 16] and [10, Raikov (1939)]). *If $f, g\colon [a,b] \to \mathbb{R}$ are continuous and n times differentiable on $(a;b)$, and $g^{(n)}(x) \neq 0$ for any $x \in (a,b)$, then there exists a point $c \in (a,b)$ such that*

$$\frac{[x_0,\dots,x_n;f]}{[x_0,\dots,x_n;g]} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

For the purpose of simplicity, and without loss of generality, we will drop out the condition $g^{(n)}(x) \neq 0$ for any $x \in (a,b)$ and consider the following form of the previous equation,

$$[x_0,\dots,x_n;f]\,g^{(n)}(c) = [x_0,\dots,x_n;g]\,f^{(n)}(c). \tag{1.4}$$

2. Main results

We give a simple elementary self-contained proof of Kowalewski's mean value formula (1.4) (see also [4]). Furthermore, we generalize (1.4) in two directions. Firstly, the divided differences are replaced with an n-simple functional. Secondly, we take a combination of forward differences and derivatives in place of the n-th derivatives in (1.4).

Consider the auxiliary function $H : [a,b] \to \mathbb{R}$,

$$H(x) = \begin{vmatrix} 1 & x_0 & \cdots & x_0^{n-1} & f(x_0) & g(x_0) \\ 1 & x_1 & \cdots & x_1^{n-1} & f(x_1) & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} & f(x_n) & g(x_n) \\ 1 & x & \cdots & x^{n-1} & f(x) & g(x) \end{vmatrix}.$$

It is obvious that $H(x_i) = 0$, $i = 0, 1, \ldots, n$. By using the Generalized Rolle Theorem, we deduce that there exists a point $c \in (x_0, x_n)$ such that $H^{(n)}(c) = 0$, i.e.,

$$0 = \begin{vmatrix} 1 & x_0 & \cdots & x_0^{n-1} & f(x_0) & g(x_0) \\ 1 & x_1 & \cdots & x_1^{n-1} & f(x_1) & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} & f(x_n) & g(x_n) \\ 0 & 0 & \cdots & 0 & f^{(n)}(c) & g^{(n)}(c) \end{vmatrix}.$$

By expanding the previous determinant in terms of the last row, and dividing the result by the Vandermonde determinant $V(x_0, \ldots, x_n)$, the proof is complete. □

The following result is a generalization of Kowalewski's mean value formula (1.4).

Theorem 3. *Let $A : C[a,b] \to \mathbb{R}$ be an n-simple functional. and suppose that f, $g : [a,b] \to \mathbb{R}$ are continuous and possess derivatives of suitable order on (a,b). Then there exist points $c_k \in (a,b)$, $k = 0, \ldots, n$, such that*

$$A(f)\, \Delta_h^{n-k} g^{(k)}(c_k) = A(g)\, \Delta_h^{n-k} f^{(k)}(c_k). \tag{2.1}$$

In the particular cases of $k = n$, and $k = 0$, Eq. (2.1) gives:

$$A(f)\, g^{(n)}(c_n) = A(g)\, f^{(n)}(c_n), \tag{2.2}$$

$$A(f)\, \Delta_h^n g(c_0) = A(g)\, \Delta_h^n f(c_0). \tag{2.3}$$

Proof. We consider the auxiliary function $h : [a,b] \to \mathbb{R}$,

$$h = A(f)\, g - A(g)\, f.$$

It follows that $A(h) = 0$. Since $A(e_n) \neq 0$, using (1.1) we deduce that there exist distinct points $\xi_0, \ldots, \xi_n \in (a,b)$ such that

$$[\xi_0, \ldots, \xi_n; h] = 0.$$

By using Popoviciu's mean value formula (1.3), we deduce that there exists a point $c \in (a,b)$ such that

$$\Delta_h^n h(c) = 0.$$

Next, by using (1.2), we obtain that there exist $c_k \in (a,b)$ such that

$$\Delta_h^{n-k} h^{(k)}(c_k) = 0, \qquad k = 1, 2, \ldots, n,$$

and the proof is complete. □

Remark 1. Since the functional $A(f) := [x_0, \ldots, x_n; f]$ is obviously of simple form, from Theorem 3, Eq. (2.2), we deduce the Kowalevski result (1.4).

A variant of the Kowalevski result (1.4) in the case of non-differentiable functions, i.e.,

$$[x_0, \ldots, x_n; f]\, \Delta_h^n g(c_0) = [x_0, \ldots, x_n; g]\, \Delta_h^n f(c_0),$$

follows from (2.3).

We end the paper with an application of Peano's representation formula for continuous linear functionals. For $t \in [a,b]$, let $\varphi_t : [a,b] \to \mathbb{R}$,

$$\varphi_t(x) := (x-t)_+^{n-1} := \left(\frac{x-t+|x-t|}{2}\right)^{n-1}, \qquad x \in [a,b].$$

If $A : C^n[a,b] \to \mathbb{R}$ is a continuous linear functional with the property that

$$A(e_i) = 0, \qquad i = 0, \ldots, n-1,$$

then the following representation formula (see, e.g., [1, 2, 5]),

$$A(f) = \frac{1}{(n-1)!} \int_0^1 A(\varphi_t)\, f^{(n)}(t)\, \mathrm{d}t, \tag{2.4}$$

holds true.

Remark 2. Let $A : C[a,b] \to \mathbb{R}$ be a continuous n-simple functional and suppose that $f,\ g \in C^n[a,b]$. Then there exist points $c_k \in (a,b)$, $k = 0, \ldots, n$, such that

$$\begin{aligned}&\Delta_h^{n-k} g^{(k)}(c_k) \int_0^1 A\big((\cdot - t)_+^{n-1}\big)\, f^{(n)}(t)\, \mathrm{d}t \\ =&\Delta_h^{n-k} f^{(k)}(c_k) \int_0^1 A\big((\cdot - t)_+^{n-1}\big)\, g^{(n)}(t)\, \mathrm{d}t.\end{aligned}$$

Proof. In (2.1) we replace A by its Peano representation (2.4). □

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