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FIRST ORDER NONHOMOGENEOUS LINEAR DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we investigate the complex oscillation of the nonhomogeneous linear differential polynomial $g_f = d_1 f' + d_0 f + b$. Here $d_0(z)$, $d_1(z)$, $b(z)$ are meromorphic functions such that at least one of $d_0(z)$ and $d_1(z)$ does not vanish identically with $\rho_p(d_j) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$, and f is a solution of the differential equation $f'' + A(z) f = 0$, where $A(z)$ is a transcendental meromorphic function with finite iterated order $\rho_p(A) = \rho > 0$.

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1. INTRODUCTION AND MAIN RESULT

For the definition of the iterated order of a meromorphic function, we use the same definition as in [3]([2], p. 317), ([4], p. 129). For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all sufficiently large r the functions $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1 ([3,4])**.** Let f be a meromorphic function. Then the iterated $p-$order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r, f)}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.1}$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. For $p = 1$, this notation is called order and for $p = 2$ hyper-order (see [7]).

Definition 2 ([3]). The finiteness degree of the order of a meromorphic function f is defined by

$$i(f)=\begin{cases} 0, & \text{if } f \text{ is rational,}\\ \min\left\{j\in\mathbb{N}:\rho_j(f)<+\infty\right\}, & \text{if } f \text{ is transcendental such that}\\ & \text{some } j\in\mathbb{N} \text{ with } \rho_j(f)<+\infty \text{ exists,}\\ +\infty, & \text{for } f \text{ with } \rho_j(f)=+\infty \text{ for all } j\in\mathbb{N}.\end{cases}$$

Definition 3 ([3,5]). Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log r}\quad (p\geq 1 \text{ is an integer}), \tag{1.2}$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z:|z|<r\}$. For $p=1$, this notation is called the exponent of convergence of the sequence of distinct zeros and for $p=2$ the hyper-exponent of convergence of the sequence of distinct zeros (see [6]).

Definition 4 ([5]). Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{N}\left(r,\frac{1}{f-z}\right)}{\log r}\quad (p\geq 1 \text{ is an integer}). \tag{1.3}$$

For $p=1$, this notation is called the exponent of convergence of the sequence of distinct fixed points and for $p=2$ the hyper-exponent of convergence of the sequence of distinct fixed points (see [6]). Thus $\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G}\subset\mathbb{C}$. If $\mathbf{G}=\mathbb{C}$, we simply use $\mathcal{L}$ instead of $\mathcal{L}(\mathbb{C})$. A special case of such differential subfields is

$$\mathcal{L}_{p+1,\rho}=\left\{g \text{ meromorphic: } \rho_{p+1}(g)<\rho\right\}, \tag{1.4}$$

where ρ is a positive constant.

Consider the linear differential equation

$$f''+A(z)f=0, \tag{1.5}$$

where $A(z)$ is a transcendental meromorphic function with finite iterated order $\rho_p(A)=\rho>0$. Since the beginning of the last four decades, a substantial number of research articles have been written to describe the fixed points of general transcendental meromorphic functions (see [8]). However, there are few studies on the fixed points of solutions of general differential equations. In [5] Laine and Rieppo have investigated the fixed points and iterated order of equation (1.5) and have obtained the following result:

Theorem 1 ([5]). *Let $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, and let f be a transcendental meromorphic solution of equation (1.5). Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

Then $\rho_{p+1}(f) = \rho_p(A) = \rho$. Moreover, let

$$P[f] = P\left(f, f', \dots, f^{(m)}\right) = \sum_{j=0}^{m} p_j f^{(j)} \tag{1.6}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients p_j does not vanish identically. Then for the fixed points of $P[f]$, we have $\overline{\tau}_{p+1}(P[f]) = \rho$, provided that neither $P[f]$ nor $P[f]-z$ vanishes identically.

Remark 1 ([5], p. 904). In Theorem 1, in order to study $P[f]$, the authors consider $m \leq 1$. Indeed, if $m \geq 2$, we obtain, by repeated differentiation of (1.5), that $f^{(k)} = q_{k,0}f + q_{k,1}f'$, $q_{k,0}, q_{k,1} \in \mathcal{L}_{p+1,\rho}$ for $k = 2, \dots, m$. Substitution into (1.6) yields the required reduction.

The main purpose of this paper is to investigate the fixed points of the nonhomogeneous linear differential polynomial $g_f = d_1 f' + d_0 f + b$, where $d_0(z)$, $d_1(z)$, $b(z)$ are meromorphic functions generated by solutions of equation (1.5). Instead of looking at the zeros of $g_f - z$, we proceed to a slight generalization by considering zeros of $g_f - \varphi(z)$, where φ is a meromorphic function of finite iterated $p-$order, while the solution of respective differential equation is of infinite iterated $p-$order.

Let us denote by

$$\alpha_0 = d_0' - d_1 A, \quad \alpha_1 = d_0 + d_1', \tag{1.7}$$

$$h = d_1\alpha_0 - d_0\alpha_1, \tag{1.8}$$

where A, d_j $(j = 0, 1)$ are meromorphic functions. We obtain:

Theorem 2. *Let $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$, let $d_0(z)$, $d_1(z)$, $b(z)$ be meromorphic functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho_p\left(d_j\right) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$ such that $h \not\equiv 0$. Let $\varphi(z)$ $(\not\equiv 0)$ be a meromorphic function of finite iterated $p-$ order such that $d_1\left(\varphi' - b'\right) - \alpha_1(\varphi - b) \not\equiv 0$. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

If $f(z) \not\equiv 0$ is a meromorphic solution of (1.5), *then the differential polynomial $g_f = d_1 f' + d_0 f + b$ satisfies $\overline{\lambda}_p(g_f - \varphi) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A) = \rho$.*

Setting $p = 1$ and $\varphi(z) = z$ in Theorem 2, we obtain the following corollary:

Corollary 1. *Let $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$, let $d_0(z), d_1(z), b(z)$ be meromorphic functions such that at least one of $d_0(z), d_1(z)$ does not vanish identically with $\rho(d_j) < \infty$ $(j = 0, 1)$, $\rho(b) < \infty$ such that $h \not\equiv 0$ and $d_1\left(1 - b'\right) - \alpha_1(z - b) \not\equiv 0$. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

If $f(z) \not\equiv 0$ is a meromorphic solution of (1.5), *then the differential polynomial $g_f = d_1 f' + d_0 f + b$ has infinitely many fixed points and satisfies $\overline{\tau}(g_f) = \rho(f) = +\infty$, $\overline{\tau}_2(g_f) = \rho_2(f) = \rho(A) = \rho$.*

2. Some Lemmas

We need the following lemmas in the proofs of our theorem.

Lemma 1 (see Remark 1.3 of [3]). *If f is a meromorphic function with $i(f) = p \geq 1$, then $\rho_p(f) = \rho_p\left(f'\right)$.*

Lemma 2 ([5]). *If f is a meromorphic function with $0 < \rho_p(f) < \rho$ $(p \geq 1)$, then $\rho_{p+1}(f) = 0$.*

Lemma 3 ([1]). *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

Then every meromorphic solution $f(z) \not\equiv 0$ of

$$f^{(k)} + A(z) f = 0, \tag{2.1}$$

satisfies $i(f) = p + 1$, $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$.

Lemma 4 ([1]). *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be finite iterated p-order meromorphic functions. If f is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.2}$$

then $\overline{\lambda}_p(f) = \rho_p(f) = +\infty$ *and* $\overline{\lambda}_{p+1}(f) = \rho_{p+1}(f) = \rho$.

Lemma 5. *Let $A(z)$ be a transcendental meromorphic function with finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Let $d_0(z)$, $d_1(z)$, $b(z)$ be meromorphic functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho_p(d_j) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$ such that $h \not\equiv 0$. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

If $f(z) \not\equiv 0$ is a meromorphic solution of (1.5)*, then the differential polynomial*

$$g_f = d_1 f' + d_0 f + b \tag{2.3}$$

satisfies $i(g_f) = p+1$, $\rho_p(g_f) = \rho_p(f) = +\infty$ and $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A) = \rho$.

Proof. Suppose that f $(\not\equiv 0)$ is a meromorphic solution of equation (1.5). Then by Lemma 3, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$. Differentiating both sides of equation (2.3) and replacing f'' with $f'' = -Af$, we obtain

$$g_f' - b' = (d_0 + d_1')f' + (d_0' - d_1 A)f. \tag{2.4}$$

Then by (1.7), (2.3) and (2.4), we have

$$d_1 f' + d_0 f = g_f - b, \tag{2.5}$$

$$\alpha_1 f' + \alpha_0 f = g_f' - b'. \tag{2.6}$$

Set

$$h = d_1\alpha_0 - \alpha_1 d_0 = d_1\left(d_0' - d_1 A\right) - d_0(d_0 + d_1'). \tag{2.7}$$

By the condition $h \not\equiv 0$ and (2.5) – (2.7), we get

$$f = \frac{d_1\left(g_f' - b'\right) - \alpha_1\left(g_f - b\right)}{h}. \tag{2.8}$$

If $\rho_p(g_f) < +\infty$, then by (2.8) and Lemma 1 we obtain $\rho_p(f) < +\infty$, and this is a contradiction. Hence $\rho_p(g_f) = \rho_p(f) = +\infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. By (2.3), Lemma 1 and Lemma 2, we get $\rho_{p+1}(g_f) \le \rho_{p+1}(f)$ and by (2.8) we have $\rho_{p+1}(f) \le \rho_{p+1}(g_f)$. This yield $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. □

Remark 2. In Lemma 5, if we don't have the condition $h \not\equiv 0$, then the differential polynomial can be of finite iterated p-order. For example, if $d_0' - d_1 A \equiv 0$ and $d_1' + d_0 \equiv 0$, then $h \equiv 0$ and $g_f' - b' \equiv 0$. It follows that $\rho_p(g_f) = \rho_p\left(g_f'\right) = \rho_p\left(b'\right) = \rho_p(b) < +\infty$. Hence, the condition $h \not\equiv 0$ is necessary in Theorem 2.

3. Proof of Theorem 2

Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of (1.5). Then by Lemma 3, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$. By Lemma 5, it is clear that $g_f \not\equiv 0$ and $g_f \not\equiv \varphi$. Set $w(z) = d_1 f' + d_0 f + b - \varphi$, since $\rho_p(\varphi) < +\infty$, then by Lemma 5 we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f) = +\infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A) = \rho$. In order to prove $\overline{\lambda}_p(g_f - \varphi) = +\infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_p(A) = \rho$, we need to prove only $\overline{\lambda}_p(w) = +\infty$ and $\overline{\lambda}_{p+1}(w) = \rho_p(A) = \rho$. Substituting $g_f = w + \varphi$ into (2.8)

$$f = \frac{d_1 w' - \alpha_1 w}{h} + \psi, \tag{3.1}$$

where

$$\psi = \frac{d_1\left(\varphi' - b'\right) - \alpha_1(\varphi - b)}{h}. \tag{3.2}$$

Substituting (3.1) into equation (1.5), we obtain

$$\frac{d_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -\left(\psi'' + A(z)\psi\right) = W, \tag{3.3}$$

where ϕ_j $(j = 0,1,2)$ are meromorphic functions with $\rho_p(\phi_j) < \infty$ $(j = 0,1,2)$. By $\rho_p(\psi) < +\infty$ and the condition $\psi \not\equiv 0$, it follows by Lemma 3 that $W \not\equiv 0$. By Lemma 4, we obtain $\overline{\lambda}_p(w) = \rho_p(w) = +\infty$ and $\overline{\lambda}_{p+1}(w) = \rho_{p+1}(w) = \rho$, i.e., $\overline{\lambda}_p(g_f - \varphi) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A) = \rho$.

Remark 3. From the proof of Theorem 2, we see that the condition $d_1\left(\varphi' - b'\right) - \alpha_1(\varphi - b) \not\equiv 0$ is necessary because if $d_1\left(\varphi' - b'\right) - \alpha_1(\varphi - b) \equiv 0$, then $\psi \equiv 0$ and $W \equiv 0$.

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