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CONTINUOUS SPECTRUM FOR SOME CLASSES OF $(p,2)$-EQUATIONS WITH LINEAR OR SUBLINEAR GROWTH

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Abstract. We are concerned with two classes of nonlinear eigenvalue problems involving equations driven by the sum of the p-Laplace ($p > 2$) and Laplace operators. The main results of this paper establish the existence of a continuous spectrum consisting in an unbounded interval, which is described by using the principal eigenvalue of the Laplace operator.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an open bounded set with smooth boundary. A central result in elementary functional analysis and in the linear theory of partial differential equations asserts that the spectrum of the Laplace operator $(-\Delta)$ in $H_0^1(\Omega)$ is *discrete*. More precisely, the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{1.1}$$

admits a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$. The proof of this result relies on the Riesz-Fredholm theory for compact self-adjoint operators (see, e.g., H. Brezis [6, Ch. VI]).

The anisotropic linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda V(x) u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{1.2}$$

was studied starting with the pioneering papers of M. Bocher [5] and P. Hess and T. Kato [11]. We also refer to S. Minakshisundaram and A. Pleijel [14] who proved that problem (1.2) admits an unbounded sequence (λ_n) of eigenvalues, provided that

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V is nonnegative, $V \in L^{\infty}(\Omega)$ and $V > 0$ in $\omega \subset \Omega$ with $|\omega| > 0$. The case where the weight function V may change sign (that is, V is indefinite) and may have singular points was studied by A. Szulkin and M. Willem [20] who established sufficient conditions for the existence of an unbounded sequence of eigenvalues.

Fix $p \in (1, \infty)$. The quasilinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{1.3}$$

was studied by several mathematicians (see, e.g., A. Anane [1], J. Garcia Azorero and I. Peral Alonso [9], P. Lindqvist [12], A. Szulkin and M. Willem [20]). For instance, A. Anane [1] and P. Lindqvist [12] proved that the first eigenvalue $\lambda = \lambda_1$ of problem (1.3) is *simple* and *isolated* in any bounded domain Ω. By combining topological and variational arguments, A. Szulkin and M. Willem [20] established the existence of a countable family of eigenvalues for a class of quasilinear eigenvalue problems with indefinite weight.

The analysis developed in these papers can be extended to homogeneous eigenvalue problems of the type

$$\begin{cases} -\mathrm{div}\, A(x, \nabla u) u = \lambda V(x) |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases}$$

where $A(x,\xi) \simeq |\xi|^{p-2}\xi$ fulfills restrictive structural conditions and $V \geq 0$, $V \not\equiv 0$.

In the present paper, we are concerned with the spectral analysis of two classes of $(p,2)$-*equations*, that is, equations driven by the sum of the p-Laplace ($p > 2$) and Laplace operators. These equations describe phenomena arising in mathematical physics. We refer to V. Benci, P. D'Avenia, D. Fortunato and L. Pisani [4] (quantum physics) and L. Cherfils and Y. Ilyasov [7] (plasma physics). Problems involving Laplace operators with different homogeneity have been studied recently by S. Barile and G. Figueiredo [2], D. Motreanu and M. Tanaka [15], N. Papageorgiou and V. Rădulescu [17], N. Papageorgiou, V. Rădulescu and D. Repovš [16], etc.

In comparison with the results described in the first part of this section, the properties established in the present paper deal with a *continuous spectrum* that concentrates at infinity.

2. Main results

Consider the eigenvalue problem

$$\begin{cases} -a\Delta u - b\Delta_p u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \tag{2.1}$$

where a, b are positive real numbers and $p > 2$.

We say that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a solution of problem (2.1) if

$$a \int_\Omega \nabla u \nabla v dx + b \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_\Omega u v dx$$

whenever $v \in W_0^{1,p}(\Omega)$.

In such a case, the corresponding λ is called an *eigenvalue* of problem (2.1). Since a and b are positive real numbers, it follows that any eigenvalue λ is positive, too.

Let λ_1 be the *first eigenvalue* (or the *principal frequency*) of the Laplace operator in $H_0^1(\Omega)$, namely the smallest eigenvalue of problem (1.1). The first result of this paper establishes the striking property that the spectrum of problem (2.1) is *continuous*. This description will be performed in terms of λ_1 and does not take into account any contribution of the p-Laplace operator that arises in problem (2.1). More precisely, we prove the following property.

Theorem 1. *Assume that a, b are positive real numbers and $p > 2$. Then λ is an eigenvalue of problem (2.1) if and only if $\lambda > a\lambda_1$.*

This result shows that the eigenvalues of the nonlinear operator $-a\Delta u - b\Delta_p u$ depend *only* on a and λ_1. The spectrum is continuous even for $b \to 0^+$, which corresponds to the case when this operator is "close" to the Laplace operator (hence, with a *discrete* spectrum).

The right-hand side of problem (2.1) is linear. We establish a related continuity property of the spectrum in the case of a suitable linear or sublinear perturbation. In such a case it is not possible to describe the whole spectrum (as done in Theorem 1) but we can assert two facts:
(i) any $\lambda < a\lambda_1$ cannot be an eigenvalue;
(ii) all λ sufficiently large is an eigenvalue.
We refer to [13] and [18] for related concentration properties of the spectrum.

Consider the nonlinear problem

$$\begin{cases} -a\Delta u - b\Delta_p u = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \tag{2.2}$$

where a, b are positive real numbers and $p > 2$.

We assume that $f : \Omega \to \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and we set $F(x,t) := \int_0^t f(x,s)ds$.

We suppose that the following hypotheses are fulfilled:

(f1) we have $|f(x,t)| \le |t|$ for a.a. $x \in \Omega$, all $t \in \mathbb{R}$;

(f2) there exists $t_0 \in \mathbb{R}$ such that $F(x,t_0) > 0$ for all $x \in \Omega$;

(f3) we have $f(x,t) = o(t)$ as $|t| \to \infty$ uniformly for a.a. $x \in \Omega$.

The following functions satisfy the above assumptions:
(i) $f(x,t) = V(x)\sin(\alpha t)$, $\alpha > 0$, $V \in L^\infty(\Omega)$, $V > 0$, $||V||_{L^\infty} \le 1$;
(ii) $f(x,t) = V(x)\log(1+|t|)$, $V \in L^\infty(\Omega)$, $V > 0$, $||V||_{L^\infty} \le 1$;
(iii) $f(x,t) = V(x)(|t|^r - |t|^q)$, $0 < q < r < 1$, $V \in L^\infty(\Omega)$, $V > 0$, $||V||_{L^\infty} \le 1$.

We say that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a solution of problem (2.1) if

$$a\int_\Omega \nabla u \nabla v dx + b\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega f(x,u) v dx \tag{2.3}$$

whenever $v \in W_0^{1,p}(\Omega)$.

In such a case, the corresponding λ is called an *eigenvalue* of problem (2.2).

Theorem 2. *Assume that a, b are positive real numbers, $p > 2$ and hypotheses (f1)-(f3) are fulfilled.*

Then any $0 < \lambda \le a\lambda_1$ is not an eigenvalue of problem (2.2). Moreover, there exists $\lambda^* > 0$ such that all $\lambda > \lambda^*$ is an eigenvalue of problem (2.2).

We do not have any estimate on the value of λ^*. We consider that this is an interesting subject, which should be considered in accordance with the behavior of the nonlinear term f.

The methods developed in this paper allow to consider several classes of differential operators in the left-hand side of problem (2.1), for instance

$$-a\Delta u - b\,\mathrm{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{(p-2)/2}}\right)$$

or

$$-a\Delta u - b\Delta_p u - b\,\mathrm{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{(p-2)/2}}\right).$$

We refer for more details to S. Barile and G. Figueiredo [2].

The approach used in this paper can be applied to the abstract framework developed by Mingione *et al.* [3, 8] and corresponding to differential operators of the form

$$-a\Delta_p u - b\,\mathrm{div}\,(a(x)|\nabla u|^{q-2}\nabla u) \quad \text{with } 1 < p < q,$$

where $0 \le a(\cdot) \in C^{0,\alpha}(\overline{\Omega})$.

Notation: for all $u \in W_0^{1,p}(\Omega)$ we denote

$$u_\pm(x) := \max\{\pm u(x), 0\}, \quad \text{for } x \in \Omega.$$

By [10, Theorem 7.6] we have $u_\pm \in W_0^{1,p}(\Omega)$ and

$$\nabla u_+ = \begin{cases} \nabla u & \text{on } [u > 0] \\ 0 & \text{on } [u \le 0] \end{cases} \qquad \nabla u_- = \begin{cases} \nabla u & \text{on } [u < 0] \\ 0 & \text{on } [u \ge 0]. \end{cases}$$

3. Proof of Theorem 1

We first argue that any $\lambda \le a\lambda_1$ is not an eigenvalue of problem (2.1). Arguing by contradiction, let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ denote the eigenfunction corresponding to the eigenvalue $\lambda \le a\lambda_1$. Then

$$a\int_\Omega |\nabla u|^2 dx + b\int_\Omega |\nabla u|^p dx = \lambda \int_\Omega u^2 dx \le a\lambda_1 \int_\Omega u^2 dx. \tag{3.1}$$

Since $p > 2$, it follows that $u \in H_0^1(\Omega) \setminus \{0\}$, hence the variational characterization of λ_1 yields

$$\lambda_1 \int_\Omega u^2 dx \le \int_\Omega |\nabla u|^2 dx. \tag{3.2}$$

Combining relations (3.1) and (3.2) we deduce that

$$a\lambda_1 \int_\Omega u^2 dx + ab \int_\Omega |\nabla u|^p dx \le a\lambda_1 \int_\Omega u^2 dx,$$

a contradiction.

It remains to show that any $\lambda > a\lambda_1$ is an eigenvalue of problem (2.1).

The energy functional $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated to problem (2.1) is defined by

$$\mathcal{E}(u) = \frac{a}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{2} \int_\Omega u^2 dx.$$

We have

$$\mathcal{E}(u) \ge \frac{b}{p} ||u||^p_{W_0^{1,p}(\Omega)} - \frac{\lambda - a\lambda_1}{2\lambda_1} ||u||^2_{H_0^1(\Omega)}.$$

Our assumption $p > 2$ implies that

$$\lim_{||u||_{W_0^{1,p}(\Omega)} \to \infty} \mathcal{E}(u) = +\infty,$$

hence $\mathcal{E}$ is coercive.

Consider the minimization problem

$$\inf\{\mathcal{E}(u);\ u \in W_0^{1,p}(\Omega)\} \tag{3.3}$$

and let (u_n) be a minimizing sequence of (3.3). Since $\mathcal{E}$ is coercive, it follows that (u_n) is bounded. Thus, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \subset H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, we can also assume that

$$u_n \to u \quad \text{in } L^2(\Omega).$$

Next, using the weakly lower semicontinuity of $\mathcal{E}$, we deduce that $u \in W_0^{1,p}(\Omega)$ minimizes $\mathcal{E}$. In order to show that u is nontrivial (hence, an eigenvalue of problem (2.1)), we argue by contradiction and assume that $u = 0$. This implies that $\mathcal{E}$ takes only nonnegative values, so it is enough to prove that

$$\inf\{\mathcal{E}(v);\ v \in W_0^{1,p}(\Omega)\} < 0.$$

For this purpose we first choose $w \in C_0^\infty(\Omega)$ such that

$$\lambda_1 < \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega w^2 dx} < \frac{\lambda}{a}. \tag{3.4}$$

This choice is possible due to the hypothesis $\lambda > a\lambda_1$ combined with the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$. We also observe that we have $w \in W_0^{1,p}(\Omega) \setminus \{0\}$. So, for all $t > 0$, we have

$$\mathcal{E}(tw) = \frac{at^2}{2} \int_\Omega |\nabla w|^2 dx + \frac{bt^p}{p} \int_\Omega |\nabla w|^p dx - \frac{\lambda t^2}{2} \int_\Omega w^2 dx$$

$$
\begin{aligned}
&= \frac{bt^p}{p}\int_\Omega |\nabla w|^p dx + \frac{t^2}{2}\left(a\int_\Omega |\nabla w|^2 dx - \lambda\int_\Omega w^2 dx\right)\\
&= A\frac{t^2}{2} + B\frac{bt^p}{p},
\end{aligned}
$$

where

$$A := a\int_\Omega |\nabla w|^2 dx - \lambda\int_\Omega w^2 dx < 0$$

and

$$B := \int_\Omega |\nabla w|^p dx > 0.$$

Moreover, by the choice of w, cf. (3.4), we have $A < 0$.

In order to obtain $\mathcal{E}(tw) < 0$ it is enough to choose

$$0 < t < \left(-\frac{pA}{2B}\right)^{1/(p-2)}.$$

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

We first establish that all positive eigenvalues of problem (2.2) are bigger than $a\lambda_1$. Let us observe that relation (2.3) can be rewritten as

$$
\begin{aligned}
&a\int_\Omega (\nabla u_+ - \nabla u_-)\nabla v dx + b\int_\Omega |\nabla u|^{p-2}(\nabla u_+ - \nabla u_-)\nabla v dx =\\
&\lambda\int_\Omega (f(x,u_+) + f(x,-u_-))v dx
\end{aligned}
\tag{4.1}
$$

whenever $v \in W_0^{1,p}(\Omega)$.

In particular, relation (4.1) shows that $u = e_1$ (namely, the first eigenfunction of the Laplace operator in $H_0^1(\Omega)$) cannot be an eigenvalue of problem (2.2), provided that $\lambda \le a\lambda_1$.

Taking $v = u_+$ in (4.1) we obtain

$$a\int_\Omega |\nabla u_+|^2 dx + b\int_\Omega |\nabla u|^{p-2}|\nabla u_+|^2 dx = \lambda\int_\Omega f(u_+)u_+ dx. \tag{4.2}$$

Taking $v = u_-$ in (4.1) we obtain

$$a\int_\Omega |\nabla u_-|^2 dx + b\int_\Omega |\nabla u|^{p-2}|\nabla u_-|^2 dx = -\lambda\int_\Omega f(u_-)u_- dx. \tag{4.3}$$

Relations (4.2) and (4.3) in combination with hypothesis (f1) yield, respectively,

$$a\lambda_1\int_\Omega u_+^2 dx \le a\int_\Omega |\nabla u_+|^2 dx \le \lambda\int_\Omega f(u_+)u_+ dx \le \lambda\int_\Omega u_+^2 dx$$

and

$$a\lambda_1 \int_\Omega u_-^2 dx \le a \int_\Omega |\nabla u_-|^2 dx \le -\lambda \int_\Omega f(u_+)u_+ dx \le \lambda \int_\Omega u_-^2 dx.$$

Since u is nontrivial, at least one of u_+ or u_- is nontrivial. Thus, the above relations imply that $\lambda \ge a\lambda_1$. Moreover, as we have already observed, $\lambda = a\lambda_1$ cannot be an eigenvalue of problem (2.2), since this would imply that $u = e_1$ is an eigenfunction of problem (2.2), which is impossible. In conclusion, if problem (2.2) admits a solution then $\lambda > a\lambda_1$.

It remains to show that problem (2.2) has a solution for all λ large enough.

The energy functional associated to problem (2.2) is $\mathcal{J} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \frac{a}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{p} \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega F(x,u) dx.$$

Fix $\lambda > a\lambda_1$ (which is a necessary condition for the existence of solutions to problem (2.2)).

Hypothesis (f3) implies that there is a positive constant $C = C(\lambda)$ such that

$$\lambda F(x,u) \le \frac{a\lambda_1}{2} u^2 + C \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}.$$

It follows that

$$\begin{aligned} \mathcal{J}(u) &\ge \frac{a}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{p} \int_\Omega |\nabla u|^p dx - \frac{a\lambda_1}{2} \int_\Omega u^2 dx - C|\Omega| \\ &\ge \frac{b}{p} ||u||^p_{W_0^{1,p}} - C|\Omega|, \end{aligned}$$

hence $\mathcal{J}$ is coercive.

Next, we show that there exists $\lambda^* > 0$ such that

$$\inf\{\mathcal{J}(u);\ u \in W_0^{1,p}(\Omega)\} < 0.$$

For this purpose we use our assumption (f2) and fix $t_0 \in \mathbb{R}$ such that

$$F(x,t_0) > 0 \quad \text{for all } x \in \Omega.$$

Fix arbitrarily a compact set $K \subset \Omega$ and let $w \in W_0^{1,p}(\Omega)$ such that $w = t_0$ in K and $0 \le w \le t_0$ in Ω.

Using hypotheses (f1) it follows that

$$\begin{aligned} \int_\Omega F(x,w) dx &= \int_K F(x,w) dx + \int_{\Omega \setminus K} F(x,w) dx \\ &\ge \int_K F(x,t_0) dx - \frac{1}{2} \int_{\Omega \setminus K} w^2 dx \qquad (4.4) \\ &\ge \int_K F(x,t_0) dx - \frac{t_0^2}{2} |\Omega \setminus K|. \end{aligned}$$

Relation (4.4) shows that increasing eventually the size of K (in order to have $|\Omega \setminus K|$ small enough) we can assume that

$$\int_\Omega F(x,w)dx > 0.$$

We deduce that

$$\mathcal{J}(w) = \frac{a}{2}\int_\Omega |\nabla w|^2 dx + \frac{b}{p}\int_\Omega |\nabla w|^p dx - \lambda \int_\Omega F(x,w)dx < 0,$$

provided that $\lambda > 0$ is large enough. For these values of λ, the energy functional $\mathcal{J}$ has a negative global minimum, hence problem (2.2) admits a solution. This completes the proof. □

The proof of Theorem 2 shows that we can assume the growth imposed by hypothesis (f3) only on one side, say at $+\infty$:

$$f(x,t) = o(t) \quad \text{as } t \to +\infty \text{ uniformly for a.a. } x \in \Omega.$$

In such a case, the final part of the proof of Theorem 2 (the existence of λ^*) follows by considering the auxiliary problem

$$\begin{cases} -a\Delta u - b\Delta_p u = \lambda f(x,u_+) & \text{in } \Omega \\ u = 0 & \text{on } \Omega. \end{cases} \tag{4.5}$$

Let u be a solution of problem (4.5). By taking $v = u_-$ as test function we deduce that $u_- = 0$, hence $u \geq 0$. This implies that any solution of (4.5) is also a solution of problem (2.2).

From now on, we follow the same arguments as those developed in the second part of the proof of Theorem 2 by replacing the energy functional $\mathcal{J}$ with $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{p}\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega F(x,u_+)dx.$$

The main result of this paper can be extended in the framework of differential operators with variable exponent; we refer to Rădulescu and Repovš [19] for related results.

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