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New sharp inequalities for approximating the factorial function and the digamma function

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NEW SHARP INEQUALITIES FOR APPROXIMATING THE FACTORIAL FUNCTION AND THE DIGAMMA FUNCTION

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Abstract. The aim of this paper is to establish new sharp upper and lower bounds for the gamma and digamma functions, starting from the Stirling's formula.

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1. INTRODUCTION

Stirling's formula

$$n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

is probably the most widely known and used result for approximation of the factorial function. It was discovered by the French mathematician Abraham de Moivre (1667–1754) as

$$n! \approx \text{constant} \cdot n^{n+\frac{1}{2}} e^{-n},$$

while the Scottish mathematician James Stirling (1692–1770) discovered the constant $\sqrt{2\pi}$ (see, e. g., [2, 21, 25, 27] for the proofs and further details)

Although such an approximation is satisfactory for the needs of the probability theory, in pure mathematics, more precise estimates are necessary. The following refined estimate

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}} \quad (1.1)$$

was first established by Robbins [23] and it can also be found, e. g., in [3, 4, 6, 8, 24]. Successively better results about the gamma and polygamma functions were obtained in [5, 9, 12–18, 20, 26]. In fact, (1.1) is the initial form of the more accurate expression

$$n! = \sqrt{2\pi} n^{n+1/2} e^{-n} \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \dots\right),$$

see, e. g., [8, 19, 24].

In this paper, we study the complete monotonicity of the functions

$$f(x) = \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2}\right) \ln x + x - \frac{1}{12x}$$

and

$$g(x) = \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2}\right) \ln x + x - \frac{1}{12x+1}$$

associated with the approximations (1.1). More precisely, we show that $-f$ and g are completely monotonic. As direct consequences, we establish the following double inequalities for $x \geq 1$:

$$\omega \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}} \leq \Gamma(x+1) < \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}},$$

where $\omega = \frac{1}{\sqrt{2\pi}} e^{\frac{11}{12}} = 0.99773\dots$ is the best possible and

$$\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x+1}} < \Gamma(x+1) \leq \eta \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x+1}},$$

where $\eta = \frac{1}{\sqrt{2\pi}} e^{\frac{12}{13}} = 1.004146965\dots$ is the best possible.

With this occasion, we state the following double inequalities for $x \geq 1$:

$$\ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) \leq \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \tau, \quad (1.2)$$

where $\tau = -\gamma + \frac{7}{12} = 0.0061177\dots$ is the best possible and

$$\ln x - \frac{1}{2x} - \frac{1}{12\left(x + \frac{1}{12}\right)^2} - \nu \leq \psi(x) < \ln x - \frac{1}{2x} - \frac{1}{12\left(x + \frac{1}{12}\right)^2}, \quad (1.3)$$

where $\nu = \frac{193}{338} - \gamma = 0.0062097\dots$ is the best possible. Estimates (1.2) and (1.3) refine other known results [1, 7, 10, 11, 22] of the form

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}, \quad x > 1.$$

2. THE RESULTS

The gamma Γ and digamma ψ functions are defined by the equalities

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \psi(x) = \frac{d}{dx} (\ln \Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)},$$

for an arbitrary positive real x . We also have the recurrence relation

$$\psi(x+1) = \psi(x) + \frac{1}{x}$$

valid for all $x > 0$. The gamma function is an extension of the factorial function because, as is well-known, $\Gamma(n+1) = n!$ for $n = 0, 1, 2, 3, \dots$. The derivatives ψ' , ψ'' , \dots , known as polygamma functions, admit the integral representations

$$\psi^{(n)}(x) = (-1)^{n-1} \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}} dt \quad (2.1)$$

for $n = 1, 2, 3, \dots$ (see, e.g., [2] for the proofs and other details). We also use the following integral representation

$$\frac{1}{x^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-xt} dt, \quad n \geq 1. \quad (2.2)$$

Recall that a function f is (strictly) completely monotonic in an interval I if f has derivatives of all orders in I such that $(-1)^n f^{(n)}(x) \geq 0$ (resp., $(-1)^n f^{(n)}(x) > 0$) for all $x \in I$ and $n = 0, 1, 2, 3, \dots$

Completely monotonic functions involving $\ln \Gamma(x)$ are important because they produce bounds for the polygamma functions. The famous Hausdorff–Bernstein–Widder theorem states that f is completely monotonic on $[0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a non-negative measure on $[0, \infty)$ such that the integral converges for all $x > 0$, see [27].

We are now in position to give the following

Theorem 1. *Let the function $f : (0, \infty) \rightarrow \mathbb{R}$ be given by the equality*

$$f(x) = \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2}\right) \ln x + x - \frac{1}{12x}.$$

Then $-f$ is completely monotonic.

Proof. We have

$$f'(x) = \psi(x) + \frac{1}{2x} - \ln x + \frac{1}{12x^2}$$

and

$$f''(x) = \psi'(x) - \frac{1}{2x^2} - \frac{1}{x} - \frac{1}{6x^3}.$$

Now, using (2.1) and (2.2), we have

$$f''(x) = \int_0^\infty \frac{te^{-tx}}{1-e^{-t}} dt - \int_0^\infty \frac{1}{2} te^{-xt} dt - \int_0^\infty e^{-xt} dt - \int_0^\infty \frac{1}{12} t^2 e^{-xt} dt,$$

or

$$f''(x) = \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} \varphi(t) dt,$$

where

$$\varphi(t) = t - \frac{1}{2}t(1 - e^{-t}) - 1 + e^{-t} - \frac{1}{12}t^2(1 - e^{-t}).$$

We have $\varphi'''(t) = -\frac{1}{12}t^2e^{-t} < 0$, for every $t > 0$, so φ'' is strictly decreasing. But $\varphi''(0) = 0$, thus $\varphi'' < 0$ on $(0, \infty)$. Now, φ' is strictly decreasing, with $\varphi'(0) = 0$, so $\varphi' < 0$. Finally, φ is strictly decreasing, with $\varphi(0) = 0$, so $\varphi < 0$ on $(0, \infty)$.

In consequence, $-f''$ is completely monotonic. Furthermore, f' is strictly decreasing, since $f'' < 0$. But we have $\lim_{x \rightarrow \infty} f'(x) = 0$, so $f'(x) > 0$ and consequently, f is strictly increasing. Using the fact that $\lim_{x \rightarrow \infty} f(x) = 0$, we deduce that $f < 0$. Finally, $-f$ is completely monotonic. \square

Corollary 1. *The following assertions hold:*

(i) *For all $x \geq 1$, we have*

$$\omega \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}} \leq \Gamma(x+1) < \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}},$$

where $\omega = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{12}} = 0.99773\dots$ is the best possible.

(ii) *For all $x \geq 1$, we have*

$$\ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) \leq \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \tau,$$

where $\tau = -\gamma + \frac{7}{12} = 0.0061177\dots$ is the best possible.

Proof. (i) The function f is strictly increasing, so for $x \geq 1$, we have

$$f(1) \leq f(x) < \lim_{x \rightarrow \infty} f(x) = 0.$$

As $f(1) = \frac{1}{12} - \ln \sqrt{2\pi}$, we get, by exponentiating

$$\frac{1}{\sqrt{2\pi}} e^{\frac{1}{12}} \leq \frac{\Gamma(x+1)}{\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}}} < 1,$$

or

$$\omega \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}} \leq \Gamma(x+1) < \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}},$$

where $\omega = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{12}} = 0.99773\dots$ is the best possible.

(ii) The function f' is strictly decreasing, so for $x \geq 1$, we have

$$0 = \lim_{x \rightarrow \infty} f'(x) < f'(x) \leq f'(1).$$

As $f'(1) = -\gamma + \frac{7}{12}$, we get

$$\ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) \leq \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \tau,$$

where $\tau = -\gamma + \frac{7}{12} = 0.0061177\dots$ is the best possible. \square

In order to prove the next result, we need the following

Lemma 1. *Let*

$$a_n = (6n - 13)13^{n-1} + 6n + 1 - 12^{n-1}n(n-1), \quad n \geq 4.$$

Then $a_n > 0$ for every $n \geq 4$.

Proof. Since we have $a_4 = 3456$, $a_5 = 70848$, $a_6 = 1074816$, $a_7 = 14566176$, $a_8 = 189616896$, $a_9 = 2486277504$, $a_{10} = 34031238912$, $a_{11} = 495590003424$, $a_{12} = 7660358317440$, it suffices to show that $a_n > 0$, for every $n \geq 13$. We prove moreover that

$$(6n - 13)13^{n-1} > 12^{n-1}n(n-1), \quad n \geq 13.$$

This follows immediately from the inequalities

$$\left(\frac{13}{12}\right)^{n-1} > \frac{n}{5} > \frac{n(n-1)}{6n-13},$$

because the first inequality can be easily proved by induction, with respect to $n \geq 13$, using

$$\frac{13}{12} \frac{n}{5} - \frac{n+1}{5} = \frac{n-12}{60} > 0,$$

and for the second inequality, we have

$$\frac{n}{5} - \frac{n(n-1)}{6n-13} = \frac{n(n-8)}{5(6n-13)} > 0.$$

□

Theorem 2. *Let the function $g: (0, \infty) \rightarrow \mathbb{R}$ be given by the equality*

$$g(x) = \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2}\right) \ln x + x - \frac{1}{12x+1}.$$

Then g is completely monotonic.

Proof. We have

$$g'(x) = \psi(x) + \frac{1}{2x} - \ln x + \frac{1}{12} \frac{1}{\left(x + \frac{1}{12}\right)^2}$$

and

$$g''(x) = \psi'(x) - \frac{1}{2x^2} - \frac{1}{x} - \frac{1}{6} \frac{1}{\left(x + \frac{1}{12}\right)^3}.$$

Now, using (2.1) and (2.2), we have

$$g''(x) = \int_0^\infty \frac{te^{-tx}}{1-e^{-t}} dt - \int_0^\infty \frac{1}{2} te^{-xt} dt - \int_0^\infty e^{-xt} dt - \int_0^\infty \frac{1}{12} t^2 e^{-xt} dt,$$

or

$$g''(x) = \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} v(t) dt,$$

where

$$v(t) = t - \frac{1}{2}t(1 - e^{-t}) - 1 + e^{-t} - \frac{1}{12}t^2e^{-\frac{1}{2}t}(1 - e^{-t}).$$

Now, $v > 0$ on $(0, \infty)$, since straightforward computations lead us to the following power series expansion:

$$v(t) = \sum_{n=4}^{\infty} \frac{a_n}{n!} t^n > 0,$$

where the sequence $(a_n)_{n \geq 4}$ is defined in Lemma 1.

As a consequence, g'' is completely monotonic. Furthermore, g' is strictly increasing because $g'' > 0$. However, we have $\lim_{x \rightarrow \infty} g'(x) = 0$, so $g'(x) < 0$ and, consequently, g is strictly decreasing. Using the fact that $\lim_{x \rightarrow \infty} g(x) = 0$, we deduce that $g > 0$. Finally, g is completely monotonic. \square

Corollary 2. *The following assertions hold:*

(i) *For all $x \geq 1$, we have*

$$\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}} < \Gamma(x+1) \leq \eta\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}},$$

where $\eta = \frac{1}{\sqrt{2\pi}}e^{\frac{12}{13}} = 1.004146965\dots$ is the best possible value.

(ii) *For all $x \geq 1$, we have*

$$\ln x - \frac{1}{2x} - \frac{1}{12(x + \frac{1}{12})^2} - v \leq \psi(x) < \ln x - \frac{1}{2x} - \frac{1}{12(x + \frac{1}{12})^2},$$

where $v = \frac{193}{338} - \gamma = 0.0062097\dots$ is the best possible value.

Proof. (i) The function g is strictly decreasing, so for $x \geq 1$, we have

$$0 = \lim_{x \rightarrow \infty} g(x) < g(x) \leq g(1).$$

As $g(1) = \frac{12}{13} - \ln \sqrt{2\pi} = 0.0041384\dots$, we get, by exponentiating

$$1 < \frac{\Gamma(x+1)}{\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}}} \leq \frac{1}{\sqrt{2\pi}}e^{\frac{12}{13}},$$

or

$$\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}} \leq \Gamma(x+1) < \eta\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}},$$

where $\eta = \frac{1}{\sqrt{2\pi}}e^{\frac{12}{13}} = 1.004146965\dots$ is the best possible.

(ii) The function g' is strictly increasing, so for $x \geq 1$, we have

$$g'(1) \leq g'(x) < \lim_{x \rightarrow \infty} g'(x) = 0.$$

As $g'(1) = -\gamma + \frac{193}{338} = -0.0062097\dots$, we get

$$\ln x - \frac{1}{2x} - \frac{1}{12(x + \frac{1}{12})^2} - v \leq \psi(x) < \ln x - \frac{1}{2x} - \frac{1}{12(x + \frac{1}{12})^2},$$

where $\nu = \frac{193}{338} - \gamma = 0.0062097\dots$ is the best possible. \square

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