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SHARP POWER MEAN BOUNDS FOR THE SECOND NEUMAN MEAN

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Abstract. In this paper, we prove that the double inequality $M_\alpha(a,b) < N_{GQ}(a,b) < M_\beta(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 2\log 2/(5\log 2 - 2\log\pi) = 1.1785\cdots$ and $\beta \geq 4/3$,where $N_{GQ}(a,b) = [G(a,b) + Q^2(a,b)/U(a,b)]/2$ is the second Neuman mean, $G(a,b) = \sqrt{ab}$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $U(a,b) = (a-b)/[\sqrt{2}\tan^{-1}((a-b)/\sqrt{2ab})]$ are the geometric, quadratic and Yang mean of a and b, respectively.

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1. INTRODUCTION

For $p \in \mathbb{R}$ and $a,b > 0$ with $a \neq b$, the pth power mean $M_p(a,b)$[14] of a and b is defined by

$$M_p(a,b) = \begin{cases} (\frac{a^p+b^p}{2})^{1/p} & \text{if } p \neq 0 \\ \sqrt{ab} & \text{if } p = 0. \end{cases}$$

It is well known that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many bivariate means are the special cases of the power mean, for example, $M_0(a,b) = G(a,b) = \sqrt{ab}$, $M_1(a,b) = A(a,b) = (a+b)/2$ and $M_2(a,b) = Q(a,b) = \sqrt{(a^2+b^2)/2}$ are respectively the arithmetic, geometric and quadratic means. Many properties for the power mean can be found in the literature[2–5, 11, 22, 24, 26, 31, 36].

The Schwab-Borchardt mean $SB(a,b)$[16, 17] defined by

$$SB(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & if\ a < b \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & if\ a > b, \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.It is well-known that $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean, for example, the first Seiffert mean $P(a,b)$, second Seiffert mean $T(a,b)$, Neuman-Sándor mean $M(a,b)$, logarithmic mean $L(a,b)$ and Yang mean $U(a,b)$[29] are respectively defined by

$$P(a,b) = \frac{a-b}{2\sin^{-1}\left[(a-b)/(a+b)\right]} = SB\left[G(a,b), A(a,b)\right],$$

$$T(a,b) = \frac{a-b}{2\tan^{-1}\left[(a-b)/(a+b)\right]} = SB\left[A(a,b), Q(a,b)\right],$$

$$M(a,b) = \frac{a-b}{2\sinh^{-1}\left[(a-b)/(a+b)\right]} = SB\left[Q(a,b), A(a,b)\right],$$

$$L(a,b) = \frac{a-b}{2\tanh^{-1}\left[(a-b)/(a+b)\right]} = SB\left[A(a,b), G(a,b)\right],$$

and

$$U(a,b) = \frac{a-b}{\sqrt{2}\tan^{-1}\left[(a-b)/\sqrt{2ab}\right]} = SB\left[G(a,b), Q(a,b)\right].$$

In 2014, Neuman [15] found a new bivariate means derived from the Schwab-Borchardt mean

$$N(a,b) = \frac{1}{2}\left[a + \frac{b^2}{SB(a,b)}\right].$$

We call $N(a,b)$ is the second Neuman mean[19]. Let $a > b$, $v = (a-b)/(a+b) \in (0,1)$, then Neuman [15] gave explicit formulas

$$N_{AG}(a,b) = \frac{1}{2}A(a,b)\left[1 + (1-v^2)\frac{\tanh^{-1}(v)}{v}\right], N_{GA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1-v^2} + \frac{\sin^{-1}(v)}{v}\right],$$

$$N_{QA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1+v^2} + \frac{\sinh^{-1}(v)}{v}\right], N_{AQ}(a,b) = \frac{1}{2}A(a,b)\left[1 + (1+v^2)\frac{\tan^{-1}(v)}{v}\right].$$

and proved that the inequalities

$$G(a,b) < N_{AG}(a,b) < N_{GA}(a,b) < A(a,b) < N_{QA}(a,b) < N_{AQ}(a,b) < Q(a,b)$$

for $a, b > 0$ with $a \neq b$.

Very recently, Shen et. al. [21] found a new mean $N_{GQ}(a,b)$ derived from the Schwab- Borchardt mean. Let $a > b$, $u = (a-b)/\sqrt{2ab} \in (0, +\infty)$, then explicit formulas for $N_{GQ}(a,b)$ be in the following:

$$N_{GQ}(a,b) = \frac{1}{2}G(a,b)\left[1 + (1+u^2)\frac{\tan^{-1}(u)}{u}\right].$$

Recently, the bounds involving the power and the Schwab-Borchardt means has been the subject of intensive research. In particular, many remarkable inequalities for the power mean, Schwab-Borchardt mean and their related means can be found in the literature [1, 6–10, 12, 13, 18–21, 23, 25, 27–30, 32–35].

Radó[20] (see also [13, 18, 23]) proved that the double inequalities

$$M_p(a,b) < L(a,b) < M_q(a,b), M_\lambda(a,b) < I(a,b) < M_\mu(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $p \leq 0$, $q \geq 1/3$, $\lambda \leq 2/3$ and $\mu \geq \log 2$, where $I(a,b) = (a^a/b^b)^{1/(a-b)}/e$ is the indentric mean of a and b.

In [7–10, 12, 28], the authors proved that $p_1 = \log 2/\log \pi$, $q_1 = 2/3$, $p_2 = \log 2/(\log \pi - \log 2)$, $q_2 = 5/3$, $p_3 = \log 2/\log[2\log(1+\sqrt{2})]$ and $q_3 = 4/3$ are the best possible parameters such that the double inequalities

$$M_{p_1}(a,b) < P(a,b) < M_{q_1}(a,b),$$
$$M_{p_2}(a,b) < T(a,b) < M_{q_2}(a,b),$$
$$M_{p_3}(a,b) < M(a,b) < M_{q_3}(a,b)$$

hold for all $a,b>0$ with $a \neq b$.

Chu [6] and Yang [30] proved that the double inequalities

$$M_{\lambda_1}(a,b) < X(a,b) < M_{\mu_1}(a,b), M_{\lambda_2}(a,b) < U(a,b) < M_{\mu_2}(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\lambda_1 \leq 1/3$, $\mu_1 \geq \log 2/(1+\log 2)$, $\lambda_2 \leq 2\log 2/(2\log \pi - \log 2)$ and $\mu_2 \geq 4/3$, where $X(a,b) = Ae^{G/P-1}$is the Sándor mean of a and b.

In [21], the authors proved the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)G(a,b) < N_{GQ}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)G(a,b),$$
$$\frac{\alpha_2}{G(a,b)} + \frac{1-\alpha_2}{Q(a,b)} < \frac{1}{N_{GQ}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1-\beta_2}{Q(a,b)},$$
$$\alpha_3 Q(a,b) + (1-\alpha_3)U(a,b) < N_{GQ}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)U(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 0$, $\beta_2 \geq 1/3$, $\alpha_3 \leq 0$ and $\beta_3 \geq (\pi^2-8)/[4(\pi-2)] = 0.4094\cdots$

The main purpose of this paper is to present the best possible parameter α and β such that the double inequalities $M_\alpha(a,b) < N_{GQ}(a,b) < M_\beta(a,b)$ hold for all $a,b>0$ with $a \neq b$.

2. Main result

In order to prove our main result we need a lemma, which we present in this section.

Lemma 1. *Let $p \in \mathbb{R}$, and*

$$f(x) = x^{2p+2} + x^{2p+1} + 5x^{2p} + x^{2p-1} + (2p-3)x^{p+3} - 4x^{p+2} + 4x^p - (2p-3)x^{p-1} - x^3 - 5x^2 - x - 1 \quad (2.1)$$

Then the following statements are true:

(1) *If $p = 4/3$, then $f(x) > 0$ for all $x \in (1, +\infty)$;*

(2) *If* $p = 2\log 2/(5\log 2 - 2\log \pi) = 1.1785\cdots$, *then there exists* $\lambda \in (1, +\infty)$ *such that* $f(x) < 0$ *for* $x \in (1, \lambda)$ *and* $f(x) > 0$ *for* $x \in (\lambda, +\infty)$.

Proof. For part (1), if $p = 4/3$, then (2.1) becomes

$$\begin{aligned} f(x) = &\frac{1}{3}(x^{2/3}-1)^3(3x^{8/3} - x^{7/3} + 9x^2 + 6x^{4/3} + 9x^{2/3} - x^{1/3} + 3) \\ &= \frac{1}{3}(x^{2/3}-1)^3\big[2x^{8/3} + x^{7/3}(x^{1/3}-1) + 9x^2 + 6x^{4/3} \\ &\qquad + 8x^{2/3} + x^{1/3}(x^{1/3}-1) + 3\big] \\ &\qquad > \frac{1}{3}(x^{2/3}-1)^3(2x^{8/3} + 9x^2 + 6x^{4/3} + 8x^{2/3} + 3) \end{aligned} \tag{2.2}$$

for $x \in (1, +\infty)$.

Therefore, part (1) follows from (2.2).

For part (2), let $p = 2\log 2/(5\log 2 - 2\log \pi) = 1.1785\cdots$, $f_1(x) = f'(x)$, $f_2(x) = f_1'(x)$, $f_3(x) = f_2'(x)$, $f_4(x) = x^{5-p} f_3'(x)$. Then elaborated computations lead to

$$\lim_{x\to 1} f(x) = 0, \ \lim_{x\to+\infty} f(x) = +\infty, \tag{2.3}$$

$$\begin{aligned} f_1(x) = &2(p+1)x^{2p+1} + (2p+1)x^{2p} + 10px^{2p-1} + (2p-1)x^{2p-2} \\ &+ (p+3)(2p-3)x^{p+2} - 4(p+2)x^{p+1} + 4px^{p-1} \\ &- (p-1)(2p-3)x^{p-2} - 3x^2 - 10x - 1 \end{aligned}$$

$$\lim_{x\to 1} f_1(x) = 24\big(p - \frac{4}{3}\big) < 0, \ \lim_{x\to+\infty} f_1(x) = +\infty, \tag{2.4}$$

$$\begin{aligned} f_2(x) = &2(p+1)(2p+1)x^{2p} + 2p(2p+1)x^{2p-1} + 10p(2p-1)x^{2p-2} \\ &+ 2(p-1)(2p-1)x^{2p-3} + (p+2)(p+3)(2p-3)x^{p+1} \\ &- 4(p+1)(p+2)x^p + 4p(p-1)x^{p-2} \\ &- (p-1)(p-2)(2p-3)x^{p-3} - 6x - 10 \end{aligned}$$

$$\lim_{x\to 1} f_2(x) = 24(2p+1)\big(p - \frac{4}{3}\big) < 0, \ \lim_{x\to+\infty} f_2(x) = +\infty, \tag{2.5}$$

$$\begin{aligned} f_3(x) = &4p(p+1)(2p+1)x^{2p-1} + 2p(4p^2-1)x^{2p-2} + 20p(p-1)(2p-1)x^{2p-3} \\ &+ 2(p-1)(2p-1)(2p-3)x^{2p-4} + (p+1)(p+2)(p+3)(2p-3)x^p \\ &- 4p(p+1)(p+2)x^{p-1} + 4p(p-1)(p-2)x^{p-3} \\ &- (p-1)(p-2)(p-3)(2p-3)x^{p-4} - 6 \end{aligned}$$

$$\lim_{x\to 1} f_3(x) = 4(22p^3 - 33p^2 + 17p - 12) < 0, \ \lim_{x\to+\infty} f_3(x) = +\infty, \tag{2.6}$$

$$\begin{aligned} f_4(x) &= 4p(p+1)(4p^2-1)x^{p+3} + 4p(p-1)(4p^2-1)x^{p+2} \\ &\quad + 20p(p-1)(2p-1)(2p-3)x^{p+1} + 4(p-1)(p-2)(2p-1)(2p-3)x^p \\ &\quad + p(p+1)(p+2)(p+3)(2p-3)x^4 - 4p(p^2-1)(p+2)x^3 \\ &\quad + 4p(p-1)(p-2)(p-3)x - (p-1)(p-2)(p-3)(p-4)(2p-3) \\ &= a_0x^{p+3} + a_2x^{p+2} + a_4x^{p+1} + a_5x^p + a_1x^4 + a_3x^3 + a_6x + a_7. \end{aligned} \tag{2.7}$$

Note that

$$p+3 > 4 > p+2 > 3 > p+1 > p > 1 > 0, \tag{2.8}$$

$$a_0 > 0, a_1 < 0, a_2 > 0, a_3 < 0, a_4 < 0, a_5 > 0, a_6 > 0, a_7 < 0, \tag{2.9}$$

$$23p^2 - 43p + 12 = -6.7311\cdots < 0, 2p^3 - 37p^2 + 89p - 48 = 8.7726\cdots > 0, \tag{2.10}$$

$$2p^3 + 119p^2 - 125p + 70 = 91.2430\cdots > 0, \tag{2.11}$$

$$a_0 + a_1 = p(p^2-1)(2p^2+25p+22) > 0, \tag{2.12}$$

$$a_2 + a_3 + a_4 = 4p(p-1)(23p^2 - 43p + 12) \tag{2.13}$$

$$a_5 + a_6 + a_7 = (p-1)(2-p)(2p^3 - 37p^2 + 89p - 48), \tag{2.14}$$

$$\sum_{i=0}^{4} a_i = p(p-1)(2p^3 + 119p^2 - 125p + 70), \tag{2.15}$$

It follows from (2.7)-(2.15) that

$$\begin{aligned} f_4(x) &> (a_0+a_1)x^4 + (a_2+a_3+a_4)x^3 + (a_5+a_6+a_7)x \\ &> \sum_{i=0}^{4} a_i x^4 + (a_5+a_6+a_7)x > 0 \end{aligned} \tag{2.16}$$

for $x \in (1, +\infty)$.

From (2.16) we clearly see that $f_3(x)$ is strictly increasing on $(1, +\infty)$. Then (2.6) leads to the conclusion that there exists $\lambda_1 > 1$ such that $f_2(x)$ is strictly decreasing on $(1, \lambda_1]$ and strictly increasing on $[\lambda_1, +\infty)$.

It follows from (2.5) and the piecewise monotonicity of $f_2(x)$, we conclude that there exists $\lambda_2 \in (1, +\infty)$ such that $f_1(x)$ is strictly decreasing on $(1, \lambda_2]$ and strictly increasing on $[\lambda_2, +\infty)$.

From (2.4) and the piecewise monotonicity of $f_1(x)$ that there exists $\lambda_3 \in (1, +\infty)$ such that $f(x)$ is strictly decreasing on $(1, \lambda_3]$ and strictly increasing on $[\lambda_3, +\infty)$.

Therefore, part (2) follows from (2.3) and the piecewise monotonicity of $f(x)$. □

Theorem 1. *The double inequality*

$$M_\alpha(a,b) < N_{GQ}(a,b) < M_\beta(a,b),$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leq 2\log 2/(5\log 2 - 2\log \pi) = 1.1785\cdots$ *and* $\beta \geq 4/3$.

Proof. Since $N_{GQ}(a,b)$ and $M_p(a,b)$ are symmetric and homogenous of degree 1, we assume that $a > b > 0$. Let $x = a/b \in (1,+\infty)$, $p \in \mathbb{R}_+$. Then we have

$$\log\left[N_{GQ}(a,b)\right]-\log\left[M_p(a,b)\right] \\ =\log\left[2\sqrt{x}(x-1)+\sqrt{2}(x^2+1)\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right)\right]-\log\left[4(x-1)\right]-\frac{1}{p}\log\left(\frac{x^p+1}{2}\right). \tag{2.17}$$

Let

$$\begin{aligned} F(x)=&\log\left[2\sqrt{x}(x-1)+\sqrt{2}(x^2+1)\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right)\right] \\ &-\log\left[4(x-1)\right]-\frac{1}{p}\log\left(\frac{x^p+1}{2}\right) \end{aligned} \tag{2.18}$$

Then simple computations lead to

$$\lim_{x\to 1^+} F(x)=0, \tag{2.19}$$

$$\lim_{x\to +\infty} F(x)=\frac{1}{p}\log 2+\log\pi-5\log\sqrt{2}, \tag{2.20}$$

$$F'(x)=\frac{x^{p+1}+2x^p-x^{p-1}-x^2+2x+1}{(x-1)(x^p+1)\left[2\sqrt{x}(x-1)+\sqrt{2}(x^2+1)\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right)\right]}F_1(x), \tag{2.21}$$

where

$$F_1(x)=\frac{2\sqrt{x}(x-1)(x^{p-1}+1)}{x^{p+1}+2x^p-x^{p-1}-x^2+2x+1}-\sqrt{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right),$$

$$\lim_{x\to 1} F_1(x)=0, \tag{2.22}$$

$$\lim_{x\to +\infty} F_1(x)=-\frac{\sqrt{2}}{2}\pi<0, \tag{2.23}$$

$$F_1'(x)=-\frac{2(x-1)}{\sqrt{x}(x^2+1)(x^{p+1}+2x^p-x^{p-1}-x^2+2x+1)^2}f(x), \tag{2.24}$$

where $f(x)$ is defined by (2.1).

We divide the proof into four cases.

Case 1. $p = 2\log 2/(5\log 2-2\log\pi)$ Then it follows from Lemma 1(2) and (2.24) that there exists $\lambda \in (1,+\infty)$ such that $F_1(x)$ is strictly increasing on $(1,\lambda]$ and strictly decreasing on $[\lambda,+\infty)$.

Equations (2.21) and (2.22)-(2.23) together with the piecewise monotonicity of $F_1(x)$ lead to the conclusion that there exists $\lambda_0 \in (1,+\infty)$ such that $F(x)$ is strictly increasing on $(1,\lambda_0]$ and strictly decreasing on $[\lambda_0,+\infty)$.

Note that (2.20) becomes

$$\lim_{x\to +\infty} F(x)=0, \tag{2.25}$$

Therefore,

$$N_{GQ}(a,b) > M_{2\log 2/(5\log 2-2\log \pi)}(a,b)$$

for all $a,b>0$ with $a \neq b$ follows from (2.17)-(2.19) and (2.25) together with the piecewise monotonicity of $F(x)$.

Case 2 . $p=4/3$ Then it follows from Lemma 1(1) and (2.24) that $F_1(x)$ is strictly decreasing on $(1,+\infty)$.

Therefore,

$$N_{GQ}(a,b) < M_{4/3}(a,b)$$

for all $a,b>0$ with $a \neq b$ follows from (2.17)-(2.19) and (2.21)-(2.22) together with the monotonicity of $F_1(x)$.

Case 3 . $p > 2\log 2/(5\log 2-2\log \pi)$ Then (2.20) leads to

$$\lim_{x\to+\infty} F(x) < 0, \tag{2.26}$$

Equations (2.17)-(2.18) together with inequality (2.26) imply that there exists large enough $M_0>1$ such that

$$N_{GQ}(a,b) < M_p(a,b)$$

for all $a,b>0$ with $x\in(M_0,+\infty)$.

Case 4 . $p<4/3$ Let $x>0$, $x\to 0$, then making use the Taylor expansion we get

$$\begin{aligned} &N_{GQ}(1,1+x)-M_p(1,1+x)\\ &\quad=\frac{2x\sqrt{x+1}+\sqrt{2}\left[(x+1)^2+1\right]\tan^{-1}\left(\frac{x}{\sqrt{2(x+1)}}\right)}{4x}-\left[\frac{1+(1+x)^p}{2}\right]^{1/p}\\ &\quad=\frac{4-3p}{24}x^2+o(x^2). \end{aligned} \tag{2.27}$$

Equation (2.27) implies that there exists small enough $\delta_0>0$ such that

$$N_{GQ}(1,1+x) > M_p(1,1+x)$$

for all $a,b>0$ with $x\in(0,\delta_0)$.

Therefore, Theorem 1 follows easily from Cases 1-4 and the monotonicity of the function $p\to M_p(a,b)$. □

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