Miskolc Mathematical Notes
Vol. 19 (2018), No. 2, pp. 1203–1215
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2018.2136

ON THE CESÀRO SUMMABILITY FOR FUNCTIONS OF TWO VARIABLES

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Received 06 November, 2016

Abstract. For a continuous function $f(T,S)$ on $\mathbb{R}^2_+ = [0,\infty)\times[0,\infty)$, we define its integral on $\mathbb{R}^2_+$ by

$$F(T,S) = \int_0^T \int_0^S f(t,s)dtds,$$

and its (C,α,β) mean by

$$\sigma_{\alpha,\beta}(T,S) = \int_0^T \int_0^S \left(1-\frac{t}{T}\right)^{\alpha}\left(1-\frac{s}{S}\right)^{\beta} f(t,s)dtds,$$

where $\alpha > -1$, and $\beta > -1$. We say that $\int_0^\infty \int_0^\infty f(t,s)dtds$ is (C,α,β) integrable to L if $\lim_{T,S\to\infty}\sigma_{\alpha,\beta}(T,S) = L$ exists.

We prove that if $\lim_{T,S\to\infty}\sigma_{\alpha,\beta}(T,S) = L$ exists for some $\alpha > -1$ and $\beta > -1$, then $\lim_{T,S\to\infty}\sigma_{\alpha+h,\beta+k}(T,S) = L$ exists for all $h > 0$ and $k > 0$.

Next, we prove that if $\int_0^\infty \int_0^\infty f(t,s)dtds$ is $(C,1,1)$ integrable to L and

$$T\int_0^S f(T,s)ds = O(1)$$

and

$$S\int_0^T f(t,S)ds = O(1)$$

then $\lim_{T,S\to\infty} F(T,S) = L$ exists.

2010 *Mathematics Subject Classification:* 40A10; 40C10; 40D05; 40G05

Keywords: (C,α,β) integrability, improper double integral, convergence in Pringsheim's sense, Cesàro integrability $(C,1,1)$, $(C,1,0)$ and $(C,0,1)$, Tauberian conditions and theorems

1. INTRODUCTION

Let $f(t)$ be a continuous function on $[0,\infty)$. The improper integral $\int_0^\infty f(t)dt$ is said to be (C,α) integrable to L for some $\alpha > -1$ if the limit

$$\lim_{T\to\infty}\int_0^T \left(1-\frac{t}{T}\right)^{\alpha} f(t)dt = L$$

exists. For all $\alpha,\beta \in \mathbb{R}$ with $-1 < \alpha < \beta$, the (C,α) integrability implies the (C,β) integrability. This implication is a classical result in the summability theory [1, p. 106]. The converse of this implication may be true by adding some suitable condition on the (C,β) integrability of the improper integral $\int_0^\infty f(t)dt$. Any theorem which states that convergence of the improper integral follows from the (C,α) integrability of the improper integral and a Tauberian condition is said to be a Tauberian theorem.

As a special case, Laforgia [7] obtained a sufficient condition under which convergence of the improper integral follows from $(C,1)$ integrability of the improper integral. Móricz and Németh [9] established some one-sided and two-sided bounded Tauberian conditions for real or complex valued functions. Recently, Çanak and Totur ([2, 3]) have proved the generalized Littlewood theorem and Hardy-Littlewood type Tauberian theorems for the $(C,1)$ integrability of a continuous function on $[0,\infty)$ by using the concept of the general control modulo analogous to the one defined by Dik [6]. Çanak and Totur [4] have also given alternative proofs of some classical type Tauberian theorems for the $(C,1)$ integrability of a continuous function on $[0,\infty)$. Çanak and Totur [5] generalized the results of Laforgia [7] for the (C,α) integrability of functions by weighted mean methods.

For a continuous function $f(T,S)$ on $\mathbb{R}^2_+ = [0,\infty)\times[0,\infty)$, we define its integral on $\mathbb{R}^2_+$ by

$$F(T,S) = \int_0^T \int_0^S f(t,s)dtds, \tag{1.1}$$

and its (C,α,β) mean by

$$\sigma_{\alpha,\beta}(T,S) = \int_0^T \int_0^S \left(1-\frac{t}{T}\right)^\alpha \left(1-\frac{s}{S}\right)^\beta f(t,s)dtds,$$

where $\alpha > -1$ and $\beta > -1$. An improper integral

$$\int_0^\infty \int_0^\infty f(t,s)dtds \tag{1.2}$$

is said to be (C,α,β) integrable to L if

$$\lim_{T,S\to\infty} \sigma_{\alpha,\beta}(T,S) = L \tag{1.3}$$

The $(C,0,0)$ integrability is the convergence of the improper integral (1.2).

However, there are some (C,α,β) integrable functions which fail to converge as improper integrals. Adding some Tauberian condition, one may get the converse.

In this paper, we prove that the (C,α,β) integrability of (1.2) where $\alpha > -1$ and $\beta > -1$ implies the $(C,\alpha+h,\beta+k)$ integrability of (1.2) for all $h > 0$ and $k > 0$. As a corollary to this result, we show that if (1.2) converges to L, then (1.2) is (C,h,k) integrable to L for all $h > 0$ and $k > 0$. But, the converse of this implication might be true under some conditions imposed on the function. Furthermore, we give conditions under which (1.2) follows from the $(C,1,1)$ integrability of (1.2). It will be shown as a corollary of our first result in this paper that convergence of the improper integral (1.2) implies the existence of the limit $\lim_{T,S\to\infty}\sigma_{h,k}(T,S)$ for all $h>0$ and $k>0$.

2. Results

The following theorem shows that (C,α,β) integrability of (1.2), where $\alpha>-1$ and $\beta>-1$, implies $(C,\alpha+h,\beta+k)$ integrability of (1.2), where all $h>0$ and $k>0$.

Theorem 1. *If (1.2) is (C,α,β) integrable to L for some $\alpha>-1$ and $\beta>-1$, then it is $(C,\alpha+h,\beta+k)$ integrable to L for all $h>0$ and $k>0$.*

Proof. Consider

$$\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(T,S)dtds, \tag{2.1}$$

where

$$\varphi(t,s;T,S)=\frac{1}{B(\alpha+1,h)}\frac{1}{B(\beta+1,k)}\frac{1}{T}\left(\frac{t}{T}\right)^{\alpha}\left(1-\frac{t}{T}\right)^{h-1}\frac{1}{S}\left(\frac{s}{S}\right)^{\beta}\left(1-\frac{s}{S}\right)^{k-1} \tag{2.2}$$

where B denotes the Beta function defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\qquad x>0,y>0.$$

Letting $u=\frac{t}{T}$ and $v=\frac{s}{S}$, we have

$$\int_0^T\int_0^S \varphi(t,s;T,S)dtds=1. \tag{2.3}$$

We need to prove that

$$\lim_{T,S\to\infty}\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(T,S)dtds=L. \tag{2.4}$$

Since

$$\lim_{T,S\to\infty}\sigma_{\alpha,\beta}(T,S)=L \tag{2.5}$$

by the hypothesis, there exists a value T_ε for any given $\varepsilon>0$ such that

$$|\sigma_{\alpha,\beta}(T,S)-L|<\varepsilon,\qquad T\ge T_\varepsilon,\ S\ge S_\varepsilon. \tag{2.6}$$

It follows from (2.3) that

$$\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(T,S)dtds-L$$
$$=\int_0^T\int_0^S \varphi(t,s;T,S)[\sigma_{\alpha,\beta}(T,S)-L]dtds. \tag{2.7}$$

To prove (2.4), it suffices to show that

$$\left|\int_0^T \int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(T,S)\,dt\,ds - L\right| < 4\varepsilon, \tag{2.8}$$

provided that T and S are large enough.

We notice that by the hypothesis, the function $\sigma_{\alpha,\beta}(T,S)$ is bounded on $\mathbb{R}^2_+$, that is,

$$|\sigma_{\alpha,\beta}(T,S) - L| < K, \quad 0 \le T,S < \infty,$$

for some constant K. Using (2.3) and (2.6), we obtain, by (2.7),

$$\begin{aligned}
&\left|\int_0^T \int_0^S \varphi(t,s;T,S)[\sigma_{\alpha,\beta}(T,S) - L]\,dt\,ds\right| \\
&\le \int_0^{T_\epsilon} \int_0^{S_\epsilon} \varphi(t,s;T,S)|\sigma_{\alpha,\beta}(T,S) - L|\,dt\,ds \\
&\quad + \int_0^{T_\epsilon} \int_{S_\epsilon}^{S} \varphi(t,s;T,S)|\sigma_{\alpha,\beta}(T,S) - L|\,dt\,ds \\
&\quad + \int_{T_\epsilon}^{T} \int_0^{S_\epsilon} \varphi(t,s;T,S)|\sigma_{\alpha,\beta}(T,S) - L|\,dt\,ds \\
&\quad + \epsilon \int_{T_\epsilon}^{T} \int_{S_\epsilon}^{S} \varphi(t,s;T,S)\,dt\,ds \\
&\le K \int_0^{T_\epsilon} \int_0^{S_\epsilon} \varphi(t,s;T,S)\,dt\,ds + K \int_0^{T_\epsilon} \int_{S_\epsilon}^{S} \varphi(t,s;T,S)\,dt\,ds \\
&\quad + K \int_{T_\epsilon}^{T} \int_0^{S_\epsilon} \varphi(t,s;T,S)\,dt\,ds + \epsilon \int_0^{T} \int_0^{S} \varphi(t,s;T,S)\,dt\,ds \\
&= K \int_0^{T_\epsilon} \int_0^{S_\epsilon} \varphi(t,s;T,S)\,dt\,ds + K \int_0^{T_\epsilon} \int_{S_\epsilon}^{S} \varphi(t,s;T,S)\,dt\,ds \\
&\quad + K \int_{T_\epsilon}^{T} \int_0^{S_\epsilon} \varphi(t,s;T,S)\,dt\,ds + \epsilon
\end{aligned}$$

By the substitution $u = \frac{t}{T}$, $v = \frac{s}{S}$, we have

$$\begin{aligned}
\int_0^{T_\epsilon} \int_0^{S_\epsilon} \varphi(t,s;T,S)\,dt\,ds &= \frac{1}{B(\alpha+1,h)} \int_0^{T_\varepsilon} \frac{1}{T}\left(\frac{t}{T}\right)^{\alpha}\left(1-\frac{t}{T}\right)^{h-1} dt \\
&\quad \times \frac{1}{B(\beta+1,k)} \int_0^{S_\epsilon} \frac{1}{S}\left(\frac{s}{S}\right)^{\beta}\left(1-\frac{s}{S}\right)^{k-1} ds
\end{aligned}$$

$$= \frac{1}{B(\alpha+1,h)} \int_0^{T_\varepsilon/T} u^\alpha (1-u)^{h-1} du \times \frac{1}{B(\beta+1,k)} \int_0^{S_\varepsilon/S} v^\beta (1-v)^{k-1} dv$$

which tends to zero when $T, S \to \infty$ for any fixed T_ε and S_ε. Thus, there exist some $\widehat{T_\varepsilon^1}$ and $\widehat{S_\varepsilon^1}$ such that

$$K \int_0^{T_\epsilon} \int_0^{S_\epsilon} \varphi(t,s;T,S)dtds < \varepsilon, \quad T \geq \widehat{T_\varepsilon^1}, \; S \geq \widehat{S_\varepsilon^1}.$$

By the substitution $u = \frac{t}{T}$, $v = \frac{s}{S}$, we have

$$\begin{aligned}\int_0^{T_\epsilon} \int_{S_\epsilon}^{S} \varphi(t,s;T,S)dtds &= \frac{1}{B(\alpha+1,h)} \int_0^{T_\varepsilon} \frac{1}{T} \left(\frac{t}{T}\right)^\alpha \left(1-\frac{t}{T}\right)^{h-1} dt \\ &\quad \times \frac{1}{B(\beta+1,k)} \int_{S_\epsilon}^{S} \frac{1}{S} \left(\frac{s}{S}\right)^\beta \left(1-\frac{s}{S}\right)^{k-1} ds \\ &= \frac{1}{B(\alpha+1,h)} \int_0^{T_\varepsilon/T} u^\alpha (1-u)^{h-1} du \\ &\quad \times \frac{1}{B(\beta+1,k)} \int_{S_\varepsilon/S}^{1} v^\beta (1-v)^{k-1} dv\end{aligned}$$

which tends to zero when $T, S \to \infty$ for any fixed T_ε and S_ε (Note that $\frac{1}{B(\beta+1,k)} \int_{S_\varepsilon/S}^{1} v^\beta (1-v)^{k-1} dv$ tends to 1 as $S \to \infty$). Thus, there exist some $\widehat{T_\varepsilon^2}$ and $\widehat{S_\varepsilon^2}$ such that

$$K \int_0^{T_\epsilon} \int_{S_\epsilon}^{S} \varphi(t,s;T,S)dtds < \varepsilon, \quad T \geq \widehat{T_\varepsilon^2}, \; S \geq \widehat{S_\varepsilon^2}.$$

Similarly, the integral

$$\int_{T_\epsilon}^{T} \int_0^{S_\epsilon} \varphi(t,s;T,S)dtds$$

tends to to zero when $T, S \to \infty$ for any fixed T_ε and S_ε (Note that $\frac{1}{B(\alpha+1,h)} \int_{T_\varepsilon/T}^{1} u^\alpha (1-u)^{k-1} dv$ tends to 1 as $S \to \infty$). Thus, there exist some $\widehat{T_\varepsilon^3}$ and $\widehat{S_\varepsilon^3}$ such that

$$K \int_{T_\epsilon}^{T} \int_0^{S_\epsilon} \varphi(t,s;T,S)dtds < \varepsilon, \quad T \geq \widehat{T_\varepsilon^3}, \; S \geq \widehat{S_\varepsilon^3}.$$

Hence, we have (2.8) for $T \geq \max\{T_\epsilon, \widehat{T_\varepsilon}^1, \widehat{T_\varepsilon}^2, \widehat{T_\varepsilon}^3\}$, $S \geq \max\{S_\epsilon, \widehat{S_\varepsilon}^1, \widehat{S_\varepsilon}^2, \widehat{S_\varepsilon}^3\}$ and this proves (2.4). We obtain

$$\begin{aligned}
&\int_0^T \int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(t,s)dtds \\
&= \int_0^T \int_0^S \varphi(t,s;T,S)\left(\int_0^t \int_0^s \left(1-\frac{u}{t}\right)^{\alpha}\left(1-\frac{v}{s}\right)^{\beta} f(u,v)dudv\right)dtds \\
&= \int_0^T \int_0^S f(u,v)\left(\int_u^T \int_v^S \varphi(t,s;T,S)\left(1-\frac{u}{t}\right)^{\alpha}\left(1-\frac{v}{s}\right)^{\beta} dtds\right)dudv \\
&= \int_0^T \int_0^S f(u,v)I(u,v;T,S)dudv,
\end{aligned}$$

where

$$I(u,v;T,S) = \int_u^T \int_v^S \varphi(t,s;T,S)\left(1-\frac{u}{t}\right)^{\alpha}\left(1-\frac{v}{s}\right)^{\beta} dtds.$$

Here, we write $I(u,v;T,S)$ as

$$\begin{aligned}
I(u,v;T,S) &= \int_u^T \int_v^S \varphi(t,s;T,S)\left(1-\frac{u}{t}\right)^{\alpha}\left(1-\frac{v}{s}\right)^{\beta} dtds \\
&= \left(\frac{1}{B(\alpha+1,h)}\int_u^T \frac{1}{T}\left(\frac{t}{T}\right)^{\alpha}\left(1-\frac{t}{T}\right)^{h-1}\left(1-\frac{u}{t}\right)^{\alpha} dt\right) \\
&\quad \times \left(\frac{1}{B(\beta+1,k)}\int_v^S \frac{1}{S}\left(\frac{s}{S}\right)^{\beta}\left(1-\frac{s}{S}\right)^{k-1}\left(1-\frac{v}{s}\right)^{\beta} ds\right) \\
&= \left(\frac{1}{B(\alpha+1,h)}\left(\frac{1}{T}\right)^{\alpha+1}\int_u^T \left(1-\frac{t}{T}\right)^{h-1}(t-u)^{\alpha} dt\right) \\
&\quad \times \left(\frac{1}{B(\beta+1,k)}\left(\frac{1}{S}\right)^{\beta+1}\int_v^S \left(1-\frac{s}{S}\right)^{k-1}(s-v)^{\beta} ds\right) \\
&= I_1(u,T)I_2(v,S),
\end{aligned}$$

where

$$I_1(u,T) = \frac{1}{B(\alpha+1,h)}\left(\frac{1}{T}\right)^{\alpha+1}\int_u^T \left(1-\frac{t}{T}\right)^{h-1}(t-u)^{\alpha} dt,$$

and

$$I_2(v,S) = \frac{1}{B(\beta+1,k)}\left(\frac{1}{S}\right)^{\beta+1}\int_v^S \left(1-\frac{s}{S}\right)^{k-1}(s-v)^{\beta} ds.$$

Substituting $t = T-(T-u)x$ in $I_1(u,T)$, we have

$$I_1(u,T) = \frac{1}{B(\alpha+1,h)}\left(1-\frac{u}{T}\right)^{h-1}\left(1-\frac{u}{T}\right)^{\alpha+1}\int_0^1 x^{h-1}(1-x)^{\alpha}dx$$
$$= \left(1-\frac{u}{T}\right)^{\alpha+h},$$

and similarly we have

$$I_2(v,S) = \left(1-\frac{v}{S}\right)^{\beta+k}.$$

These show that

$$\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(t,s)dtds$$
$$= \int_0^T\int_0^S \left(1-\frac{u}{T}\right)^{\alpha+h}\left(1-\frac{v}{S}\right)^{\beta+k} f(u,v)dudv = \sigma_{\alpha+h,\beta+k}(T,S)$$

This completes the proof of Theorem 1. □

3. THE CASE $\alpha=1$, $\beta=0$ OR $\alpha=0$, $\beta=1$

Similar to the $(C,1,1)$ integrability, one can improve the theory of the $(C,1,0)$ or the $(C,0,1)$ integrability. Since this theory is similar to the theory of integrability of functions of one variable, we only present it without detailed proofs.

Definition 1. Let $f(T,S)$ be a continuous function on $\mathbb{R}^2_+$ and $F(T,S)$ be defined as in (1.1). We define $(C,1,0)$ and $(C,0,1)$ means of (1.1) by

$$\sigma_{1,0}(T,S) := \int_0^T\int_0^S \left(1-\frac{t}{T}\right) f(t,s)dtds \tag{3.1}$$

and

$$\sigma_{0,1}(T,S) := \int_0^T\int_0^S \left(1-\frac{s}{S}\right) f(t,s)dtds, \tag{3.2}$$

respectively. We say that (1.2) is $(C,1,0)$ and $(C,0,1)$ integrable on $\mathbb{R}^2_+$ if

$$\lim_{T,S\to\infty}\sigma_{1,0}(T,S) \tag{3.3}$$

and

$$\lim_{T,S\to\infty}\sigma_{0,1}(T,S) \tag{3.4}$$

exist and are finite, respectively.

The $(C,1,0)$ and $(C,0,1)$ summability methods are regular method. Namely, if (1.2) converges to L, then (1.2) is both $(C,1,0)$ and $(C,0,1)$ integrable to L.

Theorem 2. *If (1.2) is $(C,1,0)$ integrable to L and*

$$T\int_0^S f(T,s)ds = O(1) \tag{3.5}$$

then (1.2) converges to L.

Theorem 3. *If (1.2) is $(C,0,1)$ integrable to L*

$$S\int_0^T f(t,S)dt = O(1) \tag{3.6}$$

then (1.2) converges to L.

Since the proofs of Theorem 2 and Theorem 3 can be obtained with similar steps as in Theorem 3.2 in [7], we omit the proofs.

4. THE CASE $\alpha = 1$, $\beta = 1$

Definition 2. Let $f(T,S)$ be a continuous function on $\mathbb{R}^2_+$. We say that (1.2) is $(C,1,1)$ integrable on $\mathbb{R}^2_+$, if

$$\lim_{T,S\to\infty}\int_0^T\int_0^S\left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)f(t,s)dtds \tag{4.1}$$

exists and is finite.

As a result of Theorem 1, we have the following corollary.

Corollary 1. *If (1.2) converges to L, then (1.2) is $(C,1,1)$ integrable to L.*

Proof. Take $\alpha = \beta = 0$ and $h = k = 1$ in Theorem 1. □

That the converse of Corollary 1 is not true in general is shown by the following examples.

Example 1. The integral $\int_0^\infty\int_0^\infty \cos t \cos s\, dtds$ converges to zero, in $(C,1,1)$ sense.

By (4.1), we have

$$\int_0^T\int_0^S\left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)\cos t\cos s\, dtds$$
$$=\int_0^T\left(1-\frac{t}{T}\right)\cos t\, dt\int_0^S\left(1-\frac{s}{S}\right)\cos s\, ds = \frac{(1-\cos T)(1-\cos S)}{TS},$$

which tends to zero as $T,S\to\infty$.

Example 2. The integral $\int_0^\infty\int_0^\infty \sin t\sin s\, dtds$ converges to 1, in $(C,1,1)$ sense.

By (4.1), we have

$$\int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right) \sin t \sin s dt ds$$
$$= \int_0^T \left(1-\frac{t}{T}\right) \sin t dt \int_0^S \left(1-\frac{s}{S}\right) \sin s ds = \frac{(T-\sin T)(S-\sin S)}{TS},$$

which tends to 1 as $T,S \to \infty$.

A convolution theorem for $(C,1,1)$ integrability is given by the following theorem.

Theorem 4. *Let the integrals*

$$\int_0^\infty \int_0^\infty f(t,s) dt ds, \int_0^\infty \int_0^\infty g(t,s) dt ds$$

be convergent in $(C,0,0)$ *sense, to* L_1 *and* L_2*, respectively. Then*

$$h(t,s) = \int_0^t \int_0^s f(t-u,s-v)g(t,s) du dv \tag{4.2}$$

converges in $(C,1,1)$ *sense to* $L_1 L_2$.

Proof. We need to show that

$$\lim_{T,S\to\infty} \int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right) h(t,s) dt ds = L_1 L_2 \tag{4.3}$$

We define F and G by

$$F(t,s) = \int_0^t \int_0^s f(t,s) dt ds \tag{4.4}$$

and

$$G(t,s) = \int_0^t \int_0^s g(t,s) dt ds \tag{4.5}$$

By (4.2), we get

$$\int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right) h(t,s) dt ds$$
$$= \int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)\left(\int_0^t \int_0^s f(t-u,s-v)g(t,s) du dv\right) dt ds$$
$$= \int_0^T \int_0^S g(t,s) du dv \left(\int_u^T \int_v^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right) f(t-u,s-v) dt ds\right)$$

The substitutions $t-u=\omega$ and $s-v=\delta$ and the subsequent integration by parts give

$$\int_0^T \int_0^S g(t,s) du dv \int_u^T \int_v^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right) f(t-u,s-v) dt ds$$

$$= \int_0^T \int_0^S g(u,v)dudv \int_0^{T-u} \int_0^{S-v} \left(1 - \frac{u+\omega}{T}\right)\left(1 - \frac{v+\delta}{S}\right) f(\omega,\delta)d\omega d\delta$$
$$= \frac{1}{TS} \int_0^T \int_0^S G(u,v)F(T-u,S-v)dudv$$

Since $F(T,S) \to L_1$ and $G(T,S) \to L_2$ as $T \to \infty$ and $S \to \infty$, we have that for some T_ϵ and S_ϵ

$$|F(T,S) - L_1| \le \epsilon \tag{4.6}$$

and

$$|G(T,S) - L_2| \le \epsilon \tag{4.7}$$

for $T \ge T_\epsilon$ and $S \ge S_\epsilon$. Then we have

$$\begin{aligned}
&\left| \frac{1}{TS} \int_0^T \int_0^S G(u,v)F(T-u,S-v)dudv - L_1 L_2 \right| \\
&= \left| \frac{1}{TS} \int_0^T \int_0^S (G(u,v) - L_2) F(T-u,S-v)dudv \right. \\
&\quad \left. + \int_0^T \int_0^S L_2 (F(T-u,S-v) - L_1) dudv \right| \\
&\le \frac{1}{TS} \int_0^{T_\epsilon} \int_0^{S_\epsilon} |L_2||F(T-u,S-v) - L_1| dudv \\
&\quad + \frac{1}{TS} \int_0^{T_\epsilon} \int_{S_\epsilon}^S |L_2||F(T-u,S-v) - L_1| dudv \\
&\quad + \frac{1}{TS} \int_{T_\epsilon}^T \int_0^{S_\epsilon} |L_2||F(T-u,S-v) - L_1| dudv \\
&\quad + \frac{1}{TS} \int_{T_\epsilon}^T \int_{S_\epsilon}^S |L_2||F(T-u,S-v) - L_1| dudv
\end{aligned}$$

Since $F(T,S) \to L_1$ and $G(T,S) \to L_2$ as $T \to \infty$ and $S \to \infty$, there exist some constants N_1 and N_2 such that

$$|F(T,S) - L_1| \le N_1 \tag{4.8}$$

and

$$|G(T,S) - L_2| \le N_2 \tag{4.9}$$

for all T and S. If we use (4.6), (4.7), (4.8), and (4.9), and then letting T and S tend to ∞ independently, we have the desired result. □

By the next theorem we give a sufficient condition under which $(C,1,1)$ integrability of (1.2) follows from $(C,0,0)$ integrability of (1.2).

Theorem 5. *If (1.2) is* $(C,1,1)$ *integrable to* L *and*

$$T\int_0^S f(T,s)ds = O(1) \tag{4.10}$$

and

$$S\int_0^T f(t,S)dt = O(1) \tag{4.11}$$

then (1.2) converges to L.

Proof. Let (1.2) be $(C,1,1)$ integrable to L; that is,

$$G(T,S) := \int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right) f(t,s)dtds \to L,\ T,S\to\infty \tag{4.12}$$

We rewrite $G(T,S)$ as

$$G(T,S) = \int_0^T \left(1-\frac{s}{S}\right)\frac{\partial}{\partial t}G_1(T,s)dt, \tag{4.13}$$

where

$$G_1(T,S) := \int_0^T \int_0^S \left(1-\frac{t}{T}\right) f(t,s)dtds. \tag{4.14}$$

It follows from (4.12), (4.13), and (4.14) that $\frac{\partial}{\partial S}G_1(T,S)$ is $(C,0,1)$ integrable to L.

By (4.12), we have

$$G(T,S) = G_1(T,S) - H_1(T,S), \tag{4.15}$$

where

$$H_1(T,S) = \frac{1}{S}\int_0^S s\frac{\partial}{\partial s}G_1(T,s)dt. \tag{4.16}$$

We have to show that $H_1(T,S)\to 0$ as $T,S\to\infty$.

By (4.13), we find

$$\frac{\partial}{\partial S}G(T,S) = \frac{1}{S^2}\int_0^S s\frac{\partial}{\partial s}G_1(T,s)ds = \frac{H_1(T,S)}{T} \tag{4.17}$$

We also have

$$\begin{aligned}\int_{S_1}^{S_2}\frac{\partial}{\partial S}G(T,S)dS &= G(T,S_1)-G(T,S_2)\\ &= \int_{S_1}^{S_2}\frac{H_1(T,S)}{S}dT\\ &= \int_{\log S_1}^{\log S_2} H_1(T,e^v)dv\\ &= \int_{\log S_1}^{\log S_2} R(T,v)dv.\end{aligned}$$

Here, we used the substitution $S = e^v$ and $R(T,v) = H_1(T,e^v)$. We need to show that $\lim_{v\to\infty} R(T,v) = 0$. By the simple calculation, we have

$$\frac{\partial}{\partial v}R(T,v) = e^v \frac{\partial}{\partial v}H_1(T,e^v) = S\frac{\partial}{\partial S}H_1(T,S). \tag{4.18}$$

By (4.16), we get

$$SH_1(T,S) = \int_0^S s\frac{\partial}{\partial s}G_1(T,s)ds. \tag{4.19}$$

Differentiation the both sides of (4.19) with respect to T gives

$$H_1(T,S) + S\frac{\partial}{\partial S}H_1(T,S) = S\frac{\partial}{\partial S}G_1(T,S). \tag{4.20}$$

Taking the $(C,1,0)$ mean of the both sides of (4.11) we have

$$S\frac{\partial}{\partial S}G_1(T,S) = O(1), \tag{4.21}$$

which implies that

$$S\frac{\partial}{\partial S}H_1(T,S) = O(1) \tag{4.22}$$

by (4.18).

We can easily obtain $H_1(T,S) \to 0$ as $T,S \to \infty$ by following the steps of Theorem 3.2 in [7]. It follows from (4.12) and (4.15) that $G_1(T,S) \to L$ as $T,S \to \infty$.

Since $G_1(T,S) \to L$ as $T,S \to \infty$ and the condition (4.10), we have $\lim_{T,S\to\infty} F(T,S) = L$ by Theorem 2.

□

Remark 1. Analogous Tauberian results were proved in [8] for double improper integrals with a different perspective.

5. Conclusion

In this paper, we extended the classical Tauberian theorems given for the (C,α) integrability of the improper integrals of functions of one variable to those of the (C,α,β) integrability improper integrals of functions of two by using the methods employed in Laforgia [7]. The analogous results for the functions of several variables can be obtained by the similar techniques.

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