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# BALANCING WITH POWERS OF THE LUCAS SEQUENCE OF RECURRENCE $u_n = Au_{n-1} - u_{n-2}$

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*Abstract.* In this paper, we show that there is no solution of the diophantine equation

$$u_1^k + u_2^k + \cdots + u_{n-1}^k = u_{n+1}^l + u_{n+2}^l + \cdots + u_{n+r}^l$$

for special cases of $k$ and $l$ where the elements of sequence $\{u_n\}$ satisfy the relation $u_n = Au_{n-1} - u_{n-2}$ with $u_0 = 0$, $u_1 = 1$ and $A \geq 3$ is a positive integer.

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## 1. INTRODUCTION

Let the sequence $\{u_n\}$ is defined by the recurrence relation

$$u_n = Au_{n-1} - u_{n-2} \tag{1.1}$$

where $A$ is a positive integer with $u_0 = 0$ and $u_1 = 1$. It's Binet form of the sequence $\{u_n\}$ is known as

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.2}$$

where $\alpha = \frac{A+\sqrt{A^2-4}}{2}$ and $\beta = \frac{A-\sqrt{A^2-4}}{2}$. Assume the $\{v_n\}$ be the associate sequence of the sequence $\{u_n\}$. Namely, the elements of the sequence $\{v_n\}$ satisfies the following recurrence,

$$v_n = Av_{n-1} - v_{n-2}$$

with initial conditon $v_0 = 2$ and $v_1 = A$. The Binet formula of the sequnce $\{v_n\}$ is

$$v_n = \alpha^n + \beta^n. \tag{1.3}$$

The case $A = 6$ coincides with the sequence of balancing number (see [2, 4, 7, 8]) whose elements satisfy the equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

Recently, several authors handled some diophantine equations including balancing-type rules. For example, Behera et al. [3] showed that the diophantine equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l \tag{1.4}$$

has no solution in the positive integers $n, r, k, l$ with $n \geq 2$ in the case

$$k \leq l \quad \text{and} \quad (k,l) = (2,1),\ (3,1),\ (3,2). \tag{1.5}$$

Here $F_n$ denotes the $n^{th}$ Fibonacci number. They also conjectured in [3] that only the quadruple $(n,r,k,l) = (4,3,8,2)$ satisfies the equation (1.4). Their conjecture was proved by Alvarado et al. [1]. In the sequel, Irmak [5] replaced the Fibonacci numbers with balancing numbers in (1.4), and conjectured that there is no solution of the equation

$$B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l.$$

In this paper, we generalize the conjecture of Irmak [5]. Our conjecture is following,

**Conjecture 1.** *Assume that $A \geq 3$. For the sequence $\{u_n\}_{n\geq 0}$, there is no solution of the equation*

$$u_1^k + u_2^k + \cdots + u_{n-1}^k = u_{n+1}^l + u_{n+2}^l + \cdots u_{n+r}^l \tag{1.6}$$

*for positive integers $n \geq 2$, $r$, $k$ and $l$.*

## 2. Preliminaries

In this section, we present several lemmas to confirm the conjecture. First lemma presents some formulas including the sums of the elements of the sequence $\{u_n\}$.

**Lemma 1.** *For any positive integer $k$, the following identities hold.*

*(a)* $\sum_{k=1}^{n} u_k = \frac{1}{A-2}\left(u_{n+1} - u_n - 1\right)$

*(b)* $\sum_{k=1}^{n} u_k^2 = \frac{1}{A^2-4}\left(u_{2n+1} - (2n+1)\right)$

*(c)* $\sum_{k=1}^{n} u_k^3 = \frac{1}{A^2-4}\left\{\frac{u_{3n+3}-u_{3n}-u_3}{A^3-3A-2} - \frac{3}{A-2}\left(u_{n+1}-u_n-1\right)\right\}$

*(d)* $\sum_{k=1}^{n} u_{2k-1} = u_n^2$.

*Proof.* We prove the third one. We follow the Binet formula (1.2). Other identities can be proven by similar way.

$$\begin{aligned}
\sum_{k=1}^{n} u_k^3 &= \frac{1}{(\alpha-\beta)^3}\sum_{k=0}^{n-1}\left(\alpha^{k+1}-\beta^{k+1}\right)^3 \\
&= \frac{1}{(\alpha-\beta)^3}\sum_{k=0}^{n-1}\alpha^{3k+3}-\beta^{3k+3}-3\left(\alpha^{k+1}-\beta^{k+1}\right)
\end{aligned}$$

$$= \frac{1}{(\alpha-\beta)^3}\left\{\alpha^3\frac{1-\alpha^{3n}}{1-\alpha^3}-\beta^3\frac{1-\beta^{3n}}{1-\beta^3}-3\left(\alpha\frac{1-\alpha^n}{1-\alpha}-\beta\frac{1-\beta^n}{1-\beta}\right)\right\}$$
$$= \frac{1}{(\alpha-\beta)^2}\left\{\frac{\alpha^{3n+3}-\beta^{3n+3}-\left(\alpha^{3n}-\beta^{3n}\right)-\left(\alpha^3-\beta^3\right)}{(\alpha-\beta)\left(\alpha^3+\beta^3-2\right)}\right.$$
$$\left.-\frac{3}{A-2}\left(u_{n+1}-u_n-1\right)\right\}$$
$$= \frac{1}{A^2-4}\left\{\frac{u_{3n+3}-u_{3n}-u_3}{A^3-3A-2}-\frac{3}{A-2}\left(u_{n+1}-u_n-1\right)\right\}$$

as claimed. □

**Lemma 2.** *For positive integer $n$,*

$$u_n^2-u_{n-1}^2=u_{2n-1}$$

*follows.*

*Proof.* It can proven by the Binet formula of the sequence $\{u_n\}$ or Lemma 1 d. □

**Lemma 3.** *For the positive integer $A\geq 3$ and $n\geq 1$, then the inequalities*

$$(A-2)\,u_n<u_{n+1}-u_n<(A-1)\,u_n$$

*and*

$$\left(A^3-3A-2\right)u_{3n}<u_{3n+3}-u_{3n}-u_3<\left(A^3-3A-1\right)u_{3n}$$

*hold.*

*Proof.* By the Binet formulas of the sequences $\{u_n\}$ and $\{v_n\}$, we obtain the formula

$$u_{rn}=v_r u_{r(n-1)}+(-1)^r\,u_{r(n-2)}.$$

Assume that $r=3$. By the recurrence relation (1.1),

$$\begin{aligned} u_{3(n+1)}-u_{3n} &= \left(A^3-3A\right)u_{3n}-u_{3(n-1)}-u_{3n}\\ &= \left(A^3-3A-1\right)u_{3n}-u_{3(n-1)}\\ &< \left(A^3-3A-1\right)u_{3n}\end{aligned}$$

follows. Since the inequality $-u_{3n}<-u_{3(n-1)}$ holds, then

$$u_{3(n+1)}-u_{3n}=\left(A^3-3A-1\right)u_{3n}-u_{3(n-1)}>\left(A^3-3A-2\right)u_n$$

follows. Second one can be proven by similar way. □

**Lemma 4.** *Suppose that $A\geq 3$. Then for all integers $n\geq 3$, the inequality*

$$\alpha^{n-1}<u_n<\alpha^{n-0.83} \tag{2.1}$$

*hold.*

*Proof.* See Lemma 2.2 in [6]. □

**Lemma 5.** *Suppose that $a > 0$ and $b \geq 0$ are real numbers, and that $t_0$ is a positive integer. Then for all integers $t_0 \leq t \leq t_1$, the inequality*

$$\alpha^{t+\kappa_1} \leq a\alpha^t + b \leq \alpha^{t+\kappa_0}$$

*where $\kappa_i = \log_\alpha\left(1 + \frac{b}{\alpha^{t_i}}\right)$ for $i = 0, 1$.*

*Proof.* It is obvious. □

## 3. Results

In this section, we prove the several theorems which the special cases of the diophantine equation (1.6). These theorems confirm the conjecture.

**Theorem 1.** *If $l \geq k$, then there is no solution of the diophantine equation (1.6) for positive integers $r$ and $n \geq 2$.*

*Proof.* By Lemma 1 (a),

$$\begin{aligned} u_1^k + u_2^k + \cdots + u_{n-1}^k &< (u_1 + u_2 + \cdots + u_{n-1})^k \\ &< \left(\frac{u_n - u_{n-1} - 1}{A-2}\right)^l < u_n^l \end{aligned}$$

which completes the proof. □

**Theorem 2.** *If $k = 2$ and $l = 1$ in (1.6), then there is no solution of the diophantine equation (1.6) for positive integers $r$ and $n \geq 2$.*

*Proof.* The Lemma 1 (a) and (b) yield that

$$\frac{u_{2n-1} - (2n-1)}{A^2 - 4} = \frac{1}{A-2}\left(u_{n+r+1} - u_{n+r} - u_{n+1} + u_n\right)$$

which gives

$$u_{2n-1} - (2n-1) + (A+2)(u_{n+1} - u_n) = (A+2)(u_{n+r+1} - u_{n+r}).$$

Let

$$LS := u_{2n-1} - (2n-1) + (A+2)(u_{n+1} - u_n)$$

and

$$RS := (A+2)(u_{n+r+1} - u_{n+r}).$$

Together with Lemma 3, we have

$$u_{2n-1} - (2n-1) + \left(A^2 - 4\right)u_n < LS < u_{2n-1} + (A+2)(A-1)u_n.$$

So, $LS > \alpha^{2n-2}$ holds. The Lemma 4 gives

$$\begin{aligned} LS &< u_{2n-1} + (A+2)(A-1)u_n \\ &< \alpha^{2n-1.83} + \alpha^{n+1.57} \\ &= \alpha^{n+1.57}\left(\alpha^{n-3.4} + 1\right) < 2\alpha^{n+1.57}\alpha^{n-2.8} < \alpha^{2n-0.5}. \end{aligned} \tag{3.1}$$

Similarly, we have the followings with Lemma 3

$$(A+2)(A-2)u_{n+r} < RS < (A+2)(A-1)u_{n+r}.$$

So, the inequalities

$$\alpha^{n+r+0.67} < RS < \alpha^{n+r+1.57} \tag{3.2}$$

follow by Lemma 4. Combining the inequalities (3.1) and (3.2), we deduce that

$$\max\{2n-2, n+r+0.67\} < \min\{n+r+1.57, 2n-0.5\}$$

which yields that

$$1.17 < n-r < 3.57.$$

So, there are two possibilities which are $n = r+2$ and $n = r+3$.

If $n = r+2$, then the equation (1.6) turns to the equation

$$u_1^2 + u_2^2 + \cdots + u_{r+1}^2 = u_{r+1} + u_{r+2} + \cdots + u_{2r+2}.$$

Above equation yields that

$$\frac{1}{A^2-4}\left(u_{2r+3} - (2r+3)\right) = u_{r+1} + u_{r+2} + \cdots + u_{2r+2}.$$

Obviously, this is not possible. For the case $n = r+3$, we arrive at a contradiction similarly since we obtain the following equation.

$$\frac{1}{A^2-4}\left(u_{2r+5} - (2r+5)\right) = u_{r+1} + u_{r+2} + \cdots + u_{2r+3}.$$

Therefore, we complete the proof of Theorem 2. □

**Theorem 3.** *If $k = 3$ and $l = 1$ in (1.6), then there is no solution of the diophantine equation (1.6) for positive integers $r$ and $n \geq 2$.*

*Proof.* By Lemma 1 (a) and (c), the equation turns to

$$\frac{1}{A-2}\left\{\frac{u_{3n} - u_{3n-3} - u_3}{A^3-3A-2} - \frac{3}{A-2}\left(u_n - u_{n-1} - 1\right)\right\} + \left(u_{n+1} - u_n\right)$$
$$= u_{n+r+1} - u_{n+r}.$$

Let

$$LS := \frac{1}{(A-2)\left(A^3-3A-2\right)}\left(u_{3n} - u_{3n-3} - u_3\right) - \left(\frac{3}{(A-2)^2} + 1\right)\left(u_n - u_{n-1} - 1\right) + \frac{3}{A-2}$$

and

$$RS := u_{n+r+1} - u_{n+r}$$

We have the followings by Lemma 3,

$$\frac{1}{(A-2)}u_{3n-3} - \left(\frac{3}{(A-2)^2} + 1\right)(A-1)u_{n-1} < LS$$

and

$$LS < \frac{\left(A^3-3A-1\right)}{(A-2)\left(A^3-3A-2\right)} u_{3n-3} - \left(\frac{3}{(A-2)^2}+1\right)(A-2)\, u_{n-1}.$$

Since $\alpha^{-1} < \frac{(A^3-3A-1)}{(A-2)(A^3-3A-2)} < \alpha^{0.63}$ and $1 < \left(\frac{3}{(A-2)^2}+1\right)(A-2) < \alpha^{1.45}$ follow, then the inequalities

$$\alpha^{3n-5} - \alpha^{n-0.38} < LS < \alpha^{3n-0.2} - \alpha^{n-1}$$

hold which gives that

$$\alpha^{3n-6.5} < LS < \alpha^{3n-0.2}. \tag{3.3}$$

By Lemma 3, we obtain

$$(A-2)\, u_{n+r} < RS < (A-1)\, u_{n+r}$$

which yields that

$$\alpha^{n+r-1} < RS < \alpha^{n+r-0.17}. \tag{3.4}$$

Together with the inequalities (3.3) and (3.4), we get that

$$\max\{3n-6.5, n+r-1\} < \min\{3n-0.2, n+r-0.17\}.$$

So there are seven possibilities which are $2n-r=j$ where $j \in \{0,1,2,3,4,5,6\}$.

If $r=2n$, the the diophantine equation turns the equation

$$u_1^3+u_2^3+\cdots+u_{n-1}^3 = u_{n+1}+u_{n+2}+\cdots+u_{3n}.$$

By Lemma 1 (c), we get

$$\frac{1}{A^2-4}\left\{\frac{u_{3n}-u_{3n-3}-u_3}{A^3-3A-2} - \frac{3}{A-2}(u_n-u_{n-1}-1)\right\}$$
$$= u_{n+1}+u_{n+2}+\cdots+u_{3n}.$$

Since $\alpha^{-5} < \frac{1}{(A^2-4)(A^3-3A-2)} < \alpha^{-3.26}$ holds, then $\alpha^{3n-6} < \frac{u_{3n}}{(A^2-4)(A^3-3A-2)} < \alpha^{3n-4.09}$ follows. But this is not possible since the inequality $\alpha^{3n-1} < u_{3n} < \alpha^{3n-0.83}$ holds. Since we have similar calculations for the cases $j \in \{1,2,3,4,5,6\}$, we omit the proofs of these cases to cut unnecessary calculations. So, we complete the proof of Theorem 3. □

**Theorem 4.** *If $k=3$ and $l=2$ in (1.6), then there is no solution of the diophantine equation (1.6) for positive integers $r$ and $n \geq 2$.*

*Proof.* By Lemma 1 (b) and (c), we have

$$\frac{u_{3n}-u_{3n-3}-u_3}{A^3-3A-2} - \frac{3}{A-2}(u_n-u_{n-1}-1) + u_{2n+1}$$
$$= u_{2(n+r)+1} - 2r.$$

Let

$$RS := u_{2(n+r)+1} - 2r$$

and

$$LS := \frac{u_{3n} - u_{3n-3} - u_3}{A^3 - 3A - 2} - \frac{3}{A-2}(u_n - u_{n-1} - 1) + u_{2n+1}.$$

Together with Lemma 4, we have

$$\alpha^{2(n+r)} - 2r < RS < \alpha^{2(n+r)+1-0.83}. \tag{3.5}$$

The inequalities

$$\alpha^{3n-4} < u_{3n-3} < \frac{u_{3n} - u_{3n-3} - u_3}{A^3 - 3A - 2} < \frac{A^2 - 3A - 1}{A^2 - 3A - 2} u_{3n-3} < \alpha^{3n-3.89},$$

$$\alpha^{n-2} < 3u_{n-1} < \frac{3}{A-2}(u_n - u_{n-1} - 1) < \frac{3(A-1)}{A-2} u_{n-1} < \alpha^{n-1.83+1.15}$$

and

$$\alpha^{2n} < u_{2n+1} < \alpha^{2n+0.17}$$

yield that

$$\begin{aligned} &\alpha^{n-0.68}\left(\alpha^{2n-3.32} - 1 + \alpha^{n+0.68}\right) \\ &= \alpha^{3n-4} - \alpha^{n-0.68} + \alpha^{2n} < LS \\ &< \alpha^{3n-3.89} - \alpha^{n-2} + \alpha^{2n+0.17} \\ &= \alpha^{n-2}\left(\alpha^{2n-2.89} - 1 + \alpha^{n+2.17}\right). \end{aligned}$$

Since $\alpha^{2n-4.1} < \alpha^{2n-3.32} - 1$ and $\alpha^{2n-2.89} - 1 < \alpha^{2n-2.88}$ hold, then we obtain

$$\alpha^{n-0.68}\left(\alpha^{2n-4.1} + \alpha^{n+0.68}\right) < LS < \alpha^{n-2}\left(\alpha^{2n-2.88} + \alpha^{n+2.17}\right).$$

By the inequalities $\alpha^{2n}\alpha^{n-4.78} < \alpha^{n-0.68}\left(\alpha^{n+0.68}\left(\alpha^{n-4.78} + 1\right)\right) = \alpha^{n-0.68}\left(\alpha^{2n-4.1} + \alpha^{n+0.68}\right)$ and $\alpha^{n-2}\left(\alpha^{2n-2.89} + \alpha^{n+2.17}\right) = \alpha^{n-2}\left(\alpha^{n+2.17}\left(\alpha^{n-5.06} + 1\right)\right) < \alpha^{n-2+n+2.17+n-1.9}$, then

$$\alpha^{3n-4.78} < LS < \alpha^{3n-1.73} \tag{3.6}$$

follows. Together with inequalities (3.5) and (3.6), the condition

$$\max\{3n - 4.78, 2n + 2r\} < \min\{3n - 1.73, 2n + 2r + 0.17\}$$

gives that $1.73 < n - 2r < 4.95$. So, the possible cases are $n - 2r = i$ where $i \in \{2, 3, 4\}$. When we replace $n = 2r + i$ in the equation (1.6), we get that

$$u_1^3 + u_2^3 + \cdots u_{2r+i-1}^3 = u_{2n+2}^2 + u_{2n+3}^2 + \cdots + u_{3r+i}^2.$$

Obviously, the left hand side is less than the right hand side. Therefore, we arrive at a contradiction. Hence, the proof is completed. □

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