



GREEN FUNCTION OF TRIHARMONIC HEAT EQUATION USING ADDITION FORMULA OF FUNCTION e^{-x^3}

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Abstract. In this paper, using the Mellin transform of Airy function we present an integral addition formula for the function e^{-x^3} . In deriving this representation, we make use of the Hankel functions of first kind and apply this representation to get the Green function of triharmonic heat equation in terms of the Airy functions.

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1. INTRODUCTION

As we know for the harmonic heat equation

$$\frac{\partial}{\partial t} u(x, y, t) = \Delta u(x, y, t), \quad t > 0, x, y \in \mathbb{R}, \quad u(x, y, 0) = \delta(x)\delta(y), \quad (1.1)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, the associated Green function (fundamental solution) is given by [17]

$$u(x, y, t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2+y^2}{4t}}. \quad (1.2)$$

Also, the Green function of biharmonic heat equation [5]

$$\frac{\partial}{\partial t} u(x, y, t) + \Delta^2 u(x, y, t) = 0, \quad t > 0, x, y \in \mathbb{R}, \quad u(x, y, 0) = \delta(x)\delta(y), \quad (1.3)$$

has been recently presented in terms of the quartic Lévy stable function given by [18, 23]

$$L_4(\xi, t) = \frac{1}{\pi} \int_0^\infty e^{-tr^4} \cos(r\xi) dr. \quad (1.4)$$

This function is the Green function of generalized heat equation

$$\frac{\partial}{\partial t} u(x, t) + \frac{\partial^4}{\partial x^4} u(x, t) = 0, \quad t > 0, x \in \mathbb{R}, \quad u(x, 0) = \delta(x), \quad (1.5)$$

related to the following fractional exponential operator

$$u(x, t) = e^{-t \frac{\partial^4}{\partial x^4}} \delta(x) = L_4(x, t). \quad (1.6)$$

The main strategy of obtaining the Green function of biharmonic heat equation is applying the two dimensional fractional exponential operator

$$u(x, y, t) = e^{-t \Delta^2} \delta(x) \delta(y) = e^{-t [\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]^2} \delta(x) \delta(y), \quad (1.7)$$

which can be extended for the higher order harmonic heat equations. This is our main motivation for showing the Green function of *triharmonic heat equation*

$$\frac{\partial}{\partial t} u(x, y, t) + \Delta^3 u(x, y, t) = 0, \quad t > 0, x, y \in \mathbb{R}, \quad u(x, y, 0) = \delta(x) \delta(y), \quad (1.8)$$

which its solution can be consequently derived as

$$u(x, y, t) = e^{-t \Delta^3} \delta(x) \delta(y). \quad (1.9)$$

For obtaining the above solution which has been much less developed in the literature, see for example [26], first we get an integral addition formula for the function e^{-x^3} . Next, we present the fundamental solution in terms of the Airy functions.

In this sense, we organize the paper as follows. In Section 2, we recall the definitions of the gamma and reciprocal gamma functions along with its triplication formula. We also mention some algebraic and operational properties of the Airy functions of first and second kinds.

In Section 3, we state an integral relation between the function e^{-x^3} and Airy function on the well-known Hankel contour and then we construct an integral addition formula for function e^{-x^3} . The obtained addition formula is given in terms of the imaginary parts of products of Airy functions and can be simplified with respect to the products of Hankel functions of first kind.

In Section 4, using the exponential differential operators we get the Green function of triharmonic heat equation (1.9) in terms of the Airy functions.

2. PRELIMINARIES

2.1. The Gamma functions

The gamma function and the reciprocal gamma function are defined as

$$\Gamma(s) = \int_0^\infty e^{-\tau} \tau^{s-1} d\tau, \quad \Re(s) > 0, \quad (2.1)$$

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_{\text{Ha}} e^\tau \tau^{-s} d\tau, \quad (2.2)$$

where Ha is the Hankel contour in complex plane with a cut along the negative real semi-axis $\arg \tau = \pi$ and consisting of the following three parts

$$(1) \arg \tau = \pi, \quad |\tau| \geq \epsilon,$$

- (2) $-\pi \leq \arg \tau \leq \pi, \quad |\tau| = \epsilon,$
 (3) $\arg \tau = -\pi, \quad |\tau| \geq \epsilon.$

For this function, the well-known triplication formula is given by [1, p. 256, 6.1.19]

$$\Gamma(s) = \frac{1}{2\pi} 3^{s-\frac{1}{2}} \Gamma\left(\frac{s}{3}\right) \Gamma\left(\frac{s+1}{3}\right) \Gamma\left(\frac{s+2}{3}\right), \quad \Re(s) > 0. \quad (2.3)$$

2.2. The Airy functions

The classical Airy functions of first and second kinds $\text{Ai}(x)$ and $\text{Bi}(x)$ are defined as [3, 6, 9, 12, 20, 27]

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(xt + \frac{t^3}{3}\right) dt, \quad (2.4)$$

$$\text{Bi}(x) = \frac{1}{\pi} \int_0^\infty \left(e^{xt - \frac{t^3}{3}} + \sin\left(xt + \frac{t^3}{3}\right) \right) dt, \quad (2.5)$$

and the Mellin transform of function $\text{Ai}(x)$ is given by [27, p. 96]

$$\mathcal{M}\{\text{Ai}(x); s\} = \int_0^\infty x^{s-1} \text{Ai}(x) dx = 3^{-(s+2)/3} \frac{\Gamma(s)}{\Gamma(\frac{s+2}{3})}, \quad \Re(s) > 0. \quad (2.6)$$

Also, for $|\arg(a)| < \frac{\pi}{3}$ and $|\arg(b)| < \frac{\pi}{3}$ some integral addition formula have been derived in the literature, see [12, 25]

$$\text{Ai}(a+b) = \int_0^1 \text{Ai}'\left(\frac{a}{\xi^{\frac{1}{3}}}\right) \text{Ai}'\left(\frac{b}{(1-\xi)^{\frac{1}{3}}}\right) \frac{d\xi}{\xi^{\frac{2}{3}}(1-\xi)^{\frac{2}{3}}}, \quad (2.7)$$

$$\text{Ai}(a+b) = \frac{ab}{a+b} \int_0^1 \text{Ai}\left(\frac{a}{\xi^{\frac{1}{3}}}\right) \text{Ai}\left(\frac{b}{(1-\xi)^{\frac{1}{3}}}\right) \frac{d\xi}{\xi^{\frac{4}{3}}(1-\xi)^{\frac{4}{3}}}. \quad (2.8)$$

Moreover, the following identities hold for the Airy functions of first and second kinds [27, p. 8], [1, p. 447, 10.4.28]

$$\text{Ai}'\left(xe^{-\frac{\pi i}{3}}\right) = \text{Ai}'\left(-xe^{\frac{2\pi i}{3}}\right) = \frac{e^{-\frac{\pi i}{3}}}{2} [\text{Ai}'(-x) - i \text{Bi}'(-x)], \quad (2.9)$$

$$\text{Ai}'(-x) - i \text{Bi}'(-x) = e^{-\frac{\pi i}{6}} \frac{x}{\sqrt{3}} H_{2/3}^{(1)}\left(\frac{2}{3}x^{\frac{3}{2}}\right), \quad (2.10)$$

where the Hankel function of first kind $H_v^{(1)}(x)$ is defined as [19, p. 914, 8.421(1)]

$$H_v^{(1)}(x) = \frac{2e^{-\frac{\nu\pi i}{2}}}{\pi i} \int_{-\infty}^\infty e^{ix \cosh(u) - \nu u} du, \quad -1 < \Re(\nu) < 1, \quad x > 0. \quad (2.11)$$

3. INTEGRAL ADDITION FORMULA

Lemma 1. For $\Re(s) > 0$, the Mellin transform of function e^{-x^3} is given by

$$\begin{aligned}\mathcal{M}\{e^{-x^3}; s\} &= 2\pi \left[3^{-(4s+1)/6} \frac{1}{\Gamma(\frac{s+1}{3})} \right] \left[3^{-(s+2)/3} \frac{\Gamma(s)}{\Gamma(\frac{s+2}{3})} \right], \\ &= 2\pi \left[3^{-(4s+1)/6} \frac{1}{\Gamma(\frac{s+1}{3})} \right] \mathcal{M}\{\text{Ai}(x); s\}.\end{aligned}\quad (3.1)$$

Proof. According to [19, p. 1131], for the Mellin transform of the function e^{-x^3}

$$\mathcal{M}\{e^{-x^3}; s\} = \frac{1}{3} \Gamma\left(\frac{s}{3}\right), \quad (3.2)$$

and taking into account the relations (2.3) and (2.6), we get the result. \square

Lemma 2. The following representation holds for the function e^{-x^3} and the Airy function

$$e^{-x^3} = -i 3^{-1/6} \int_{\text{Ha}} e^{\tau} \tau^{-1/3} \text{Ai}\left(3^{2/3} \tau^{1/3} x\right) d\tau, \quad (3.3)$$

where Ha is the Hankel contour.

Proof. Taking into account the relation (2.2) for the reciprocal gamma function $\frac{1}{\Gamma(\frac{s+1}{3})}$ and considering the representation (3.1), we can present a simple comparison between the Mellin transform of the Airy function and e^{-x^3} . After a little algebra and using the following property of Mellin transform

$$\mathcal{M}\{f(ax); s\} = \frac{1}{a^s} F(s), \quad a > 0, \quad (3.4)$$

we obtain the result. \square

Theorem 1. For $|\arg(a)| < \frac{\pi}{3}$ and $|\arg(b)| < \frac{\pi}{3}$, the following integral addition formulas hold

$$\begin{aligned}e^{-(a+b)^3} &= \frac{2}{3^{1/6}} \int_0^1 \int_0^\infty \frac{e^{-\tau}}{\sqrt[3]{\tau}} \Im \left[e^{\frac{\pi i}{3}} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} b}{(1-\xi)^{1/3}}\right) \right] \\ &\quad \times \frac{1}{\xi^{\frac{2}{3}} (1-\xi)^{\frac{2}{3}}} d\tau d\xi,\end{aligned}\quad (3.5)$$

$$\begin{aligned}e^{-(a+b)^3} &= 2\sqrt{3} \frac{ab}{a+b} \int_0^1 \int_0^\infty e^{-\tau} \Im \left[\text{Ai}\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} a}{\xi^{1/3}}\right) \text{Ai}\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} b}{(1-\xi)^{1/3}}\right) \right] \\ &\quad \times \frac{1}{\xi^{\frac{4}{3}} (1-\xi)^{\frac{4}{3}}} d\tau d\xi.\end{aligned}\quad (3.6)$$

Proof. First, we use the relation (3.3) to present an addition formula for the function e^{-x^3} with respect to the addition formula of Airy function. Next, we compute the remaining integral on the Hankel contour in three parts by applying Titchmarsh theorem for the inverse Laplace transforms of multi-valued functions [8, 10, 21]

$$\begin{aligned}
e^{-(a+b)^3} &= -i 3^{-1/6} \int_0^1 \int_{\text{Ha}} \left[e^{\tau} \tau^{-1/3} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right] \\
&\quad \times \frac{1}{\xi^{2/3} (1-\xi)^{2/3}} d\tau d\xi \\
&= -i 3^{-1/6} \int_0^1 \int_{\infty}^0 \left[e^{\tau} \tau^{-1/3} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right]_{\tau=r e^{i\pi}} \\
&\quad \times \frac{1}{\xi^{2/3} (1-\xi)^{2/3}} d\tau d\xi \\
&\quad - i 3^{-1/6} \int_0^1 \int_{\pi}^{-\pi} \left[e^{\tau} \tau^{-1/3} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right]_{\tau=r e^{i\theta}} \\
&\quad \times \frac{1}{\xi^{2/3} (1-\xi)^{2/3}} d\tau d\xi \\
&\quad - i 3^{-1/6} \int_0^1 \int_0^{\infty} \left[e^{\tau} \tau^{-1/3} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right]_{\tau=r e^{-i\pi}} \\
&\quad \times \frac{1}{\xi^{2/3} (1-\xi)^{2/3}} d\tau d\xi. \tag{3.7}
\end{aligned}$$

Taking $\epsilon \rightarrow 0$, we obtain the addition formula of function e^{-x^3} in terms of the imaginary part of products of Airy functions as mentioned in (3.5). The result (3.6) is obtained by the same procedure. \square

Remark 1. For simplicity of the imaginary part of products of Airy functions in (3.5), we employ the relations (2.9) and (2.10), respectively, and present two representations as follows

$$\begin{aligned}
&\Im \left[e^{\frac{\pi i}{3}} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} b}{(1-\xi)^{1/3}}\right) \right] \\
&= -\frac{1}{4} \left[\text{Ai}'\left(-\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Bi}'\left(-\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) + \text{Bi}'\left(-\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(-\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right] \\
&\quad - \frac{\sqrt{3}}{4} \left[\text{Ai}'\left(-\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(-\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) - \text{Bi}'\left(-\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Bi}'\left(-\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right], \tag{3.8}
\end{aligned}$$

$$\begin{aligned} & \Im \left[e^{\frac{\pi i}{3}} \text{Ai}' \left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} a}{\xi^{1/3}} \right) \text{Ai}' \left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} b}{(1-\xi)^{1/3}} \right) \right] = \\ & -\frac{1}{8} \frac{3^{5/6} \tau^{2/3} ab}{\xi^{1/3} (1-\xi)^{1/3}} H_v^{(1)} \left(\frac{2}{3^{1/2}} \frac{\tau^{1/2} a^{3/2}}{\xi^{1/2}} \right) H_v^{(1)} \left(\frac{2}{3^{1/2}} \frac{\tau^{1/2} b^{3/2}}{(1-\xi)^{1/2}} \right). \quad (3.9) \end{aligned}$$

4. THE GREEN FUNCTION OF TRIHARMONIC HEAT EQUATION

In this section, using the exponential differential operators and the obtained addition formula in previous section, we intend to obtain the Green function of triharmonic heat equation in terms of the Airy functions. For knowing more applications of the exponential differential operators in the applied mathematics see [2, 4, 7, 11, 13–16, 22, 24].

At the beginning, we show a fundamental lemma for the Green function of generalized heat equation of third order and state a theorem for the Green function of triharmonic heat equation. Although the obtained solution is very formal, but it presents an analytical form for the structure of Green function.

Lemma 3 ([7, 11]). *The Green function of generalized heat equation*

$$\frac{\partial}{\partial t} u(x, t) = \frac{\partial^3}{\partial x^3} u(x, t), \quad t > 0, x \in \mathbb{R}, \quad u(x, 0) = \delta(x), \quad (4.1)$$

is obtained by applying the following exponential operator on the Dirac delta function

$$u(x, t) = e^{t \frac{\partial^3}{\partial x^3}} \delta(x) = \frac{1}{\sqrt[3]{t}} \text{Ai} \left(-\frac{x}{\sqrt[3]{t}} \right). \quad (4.2)$$

Theorem 2. *The Green function of triharmonic heat equation*

$$\frac{\partial}{\partial t} u(x, y, t) + \Delta^3 u(x, y, t) = 0, \quad t > 0, x, y \in \mathbb{R}, \quad u(x, y, 0) = \delta(x)\delta(y), \quad (4.3)$$

is given by

$$\begin{aligned} u(x, y, t) = & -\frac{xy e^{-\frac{\pi i}{3}}}{2^{4/3} \pi^2} \int_0^1 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{-\tau} \tau^{-1/3}}{\xi^{2/3} (1-\xi)^{2/3}} \frac{e^{-2(u+v)/3}}{\sqrt[3]{\cosh^2(u) \cosh^2(v)}} \\ & \times \text{Ai} \left(-\frac{x \sqrt[6]{\frac{3\xi}{t\tau}} e^{-\frac{\pi i}{6}}}{\sqrt[3]{2 \cosh(u)}} \right) \text{Ai} \left(-\frac{y \sqrt[6]{\frac{3(1-\xi)}{t\tau}} e^{-\frac{\pi i}{6}}}{\sqrt[3]{2 \cosh(v)}} \right) du dv d\tau d\xi. \end{aligned}$$

Proof. We begin with the following formal solution of triharmonic heat equation

$$u(x, y, t) = e^{-t \Delta^3} \delta(x)\delta(y) = e^{-t[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]^3} \delta(x)\delta(y), \quad (4.4)$$

and set $a = \sqrt[3]{t} \frac{\partial^2}{\partial x^2}$ and $b = \sqrt[3]{t} \frac{\partial^2}{\partial y^2}$ in relation (3.5) to get the solution as

$$\begin{aligned} e^{-t[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]} \delta(x)\delta(y) &= -\frac{ab}{4 \times 3^{2/3}} \int_0^1 \int_0^\infty e^{-\tau} \sqrt[3]{\tau} \frac{1}{\xi(1-\xi)} \\ &\quad \times \left[H_{2/3}^{(1)} \left(\frac{2}{3^{1/2}} \frac{t^{1/2} \tau^{1/2} \frac{\partial^3}{\partial x^3}}{\xi^{1/2}} \right) \delta(x) \right] \\ &\quad \times \left[H_{2/3}^{(1)} \left(\frac{2}{3^{1/2}} \frac{t^{1/2} \tau^{1/2} \frac{\partial^3}{\partial y^3}}{(1-\xi)^{1/2}} \right) \delta(y) \right] d\tau d\xi. \end{aligned} \quad (4.5)$$

We now apply the integral representation (2.11) and identity (4.1), and simplify the above solution using the following relations

$$\begin{aligned} \left[H_{2/3}^{(1)} \left(\frac{2}{3^{1/2}} \frac{t^{1/2} \tau^{1/2} \frac{\partial^3}{\partial x^3}}{\xi^{1/2}} \right) \delta(x) \right] &= \frac{2e^{-\frac{\pi i}{3}}}{\pi i} \int_{-\infty}^\infty e^{2i \sqrt{\frac{t\tau}{3\xi}} \cosh(u) \frac{\partial^3}{\partial x^3} - 2u/3} \delta(x) du, \\ &= -\frac{2}{\pi} \int_{-\infty}^\infty e^{-2u/3} \frac{\sqrt[6]{\frac{3\xi}{t\tau}}}{\sqrt[3]{2 \cosh(u)}} \\ &\quad \times \text{Ai} \left(-\frac{x \sqrt[6]{\frac{3\xi}{t\tau}} e^{-\frac{\pi i}{6}}}{\sqrt[3]{2 \cosh(u)}} \right) du, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \left[H_{2/3}^{(1)} \left(\frac{2}{3^{1/2}} \frac{t^{1/2} \tau^{1/2} \frac{\partial^3}{\partial y^3}}{(1-\xi)^{1/2}} \right) \delta(y) \right] &= \frac{2e^{-\frac{\pi i}{3}}}{\pi i} \int_{-\infty}^\infty e^{2i \sqrt{\frac{t\tau}{3(1-\xi)}} \cosh(v) \frac{\partial^3}{\partial y^3} - 2v/3} \delta(y) dv, \\ &= -\frac{2}{\pi} \int_{-\infty}^\infty e^{-2v/3} \frac{\sqrt[6]{\frac{3(1-\xi)}{t\tau}}}{\sqrt[3]{2 \cosh(v)}} \\ &\quad \times \text{Ai} \left(-\frac{y \sqrt[6]{\frac{3(1-\xi)}{t\tau}} e^{-\frac{\pi i}{6}}}{\sqrt[3]{2 \cosh(v)}} \right) dv. \end{aligned} \quad (4.7)$$

Finally, by substituting the above relations into (4.5) we obtain the result. \square

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