Miskolc Mathematical Notes
Vol. 19 (2018), No. 1, pp. 149–156
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2018.2118

COEFFICIENT ESTIMATES FOR A CLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS RELATED TO PSEUDO-STARLIKE FUNCTIONS

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Received 13 October, 2016

Abstract. In this paper we introduce and investigate an interesting subclass $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$, we obtain estimates on the first two Taylor-Maclaurin coefficients a_2 and a_3. The results presented in this paper would generalize and improve some recent work of Joshi et al. [5].

2010 *Mathematics Subject Classification:* 30C45

Keywords: analytic functions, univalent functions, bi-univalent functions, coefficient estimates, pseudo-starlike functions

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions in the class Σ, see [6] (see also [2]). In fact, the aforecited work of Srivastava *et al.* [6] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Xu et al. [7, 8].

Recently, Babalola [1] defined the class $\mathcal{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β as follows:

Suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real. A function $f \in \mathcal{A}$ given by (1.1) belongs to the class $\mathcal{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β in the unit disk $\mathbb{U}$ if and only if

$$\Re\left(\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right) > \beta \qquad (z \in \mathbb{U}).$$

Babalola [1] proved that all pseudo-starlike functions are Bazilevič of type $1 - 1/\lambda$, order $\beta^{1/\lambda}$ and univalent in $\mathbb{U}$.

Motivated by this definition, Joshi *et al.* [5] introduced the following two subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients a_2 and a_3 of functions in each of these subclasses.

Definition 1 ([5]). A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{LB}_\Sigma^\lambda(\alpha)$ if the following conditions are satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \left|\arg\left(\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{1.3}$$

and

$$\left|\arg\left(\frac{w\left(g'(w)\right)^\lambda}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}), \tag{1.4}$$

where $0 < \alpha \leq 1$, $\lambda \geq 1$ and the function $g = f^{-1}$ is given by (1.2).

We call $\mathcal{LB}_\Sigma^\lambda(\alpha)$ the class of strongly λ-bi-pseudo-starlike functions of order α. Also for $\lambda = 1$, we get $\mathcal{LB}_\Sigma^1(\alpha) = \mathcal{S}_\Sigma^*[\alpha]$ the class of strongly bi-starlike functions of order α, introduced and studied by Brannan and Taha [2].

Theorem 1 ([5]). *Let $f(z)$ given by (1.1) be in the class $\mathcal{LB}_\Sigma^\lambda(\alpha)$ $(0 < \alpha \leq 1, \lambda \geq 1)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(2\lambda - 1)(2\lambda - 1 + \alpha)}} \tag{1.5}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(2\lambda-1)^2} + \frac{2\alpha}{3\lambda-1}. \tag{1.6}$$

Definition 2 ([5])**.** A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{LB}_\Sigma(\lambda,\beta)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re\left(\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right) > \beta \quad (z \in \mathbb{U}) \tag{1.7}$$

and

$$\Re\left(\frac{w\left(g'(w)\right)^\lambda}{g(w)}\right) > \beta \quad (w \in \mathbb{U}), \tag{1.8}$$

where $0 \leq \beta < 1$, $\lambda \geq 1$ and the function $g = f^{-1}$ is defined by (1.2).

We call $\mathcal{LB}_\Sigma(\lambda,\beta)$ the class of λ-bi-pseudo-starlike functions of order β. Also for $\lambda = 1$, we get $\mathcal{LB}_\Sigma(1,\beta) = \mathcal{S}^*_\Sigma(\beta)$ the class of bi-starlike functions of order β, introduced and studied by Brannan and Taha [2].

Theorem 2 ([5])**.** *Let $f(z)$ given by (1.1) be in the class $\mathcal{LB}_\Sigma(\lambda,\beta)$ $(0 \leq \beta < 1, \lambda \geq 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\lambda(2\lambda-1)}} \tag{1.9}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(2\lambda-1)^2} + \frac{2(1-\beta)}{3\lambda-1}. \tag{1.10}$$

Here, in our present paper, inspiring by some of the aforecited works (especially [5]), we introduce the following subclass of the analytic function class $\mathcal{A}$, analogously to the definition given by Xu et al. [7].

Definition 3. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min\{\Re(h(z)), \Re(p(z))\} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad h(0) = p(0) = 1.$$

Also let the function f defined by (1.1) be in the analytic function class $\mathcal{A}$. We say that $f \in \mathcal{LB}^{h,p}_\Sigma(\lambda)$ $(\lambda \geq 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \frac{z\left(f'(z)\right)^\lambda}{f(z)} \in h(\mathbb{U}) \quad (z \in \mathbb{U}) \tag{1.11}$$

and

$$\frac{w\left(g'(w)\right)^\lambda}{g(w)} \in p(\mathbb{U}) \quad (w \in \mathbb{U}), \tag{1.12}$$

where the function $g = f^{-1}$ is defined by (1.2).

Setting $\lambda = 1$ in Definition 3, we get the class $\mathcal{LB}_{\Sigma}^{h,p}(1) = \mathcal{B}_{\Sigma}^{h,p}$ introduced and studied by Bulut [3].

Remark 1. There are many choices of the functions h and p which would provide interesting subclasses of the analytic function class $\mathcal{A}$. For example, if we let

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad \text{and} \quad p(z) = \left(\frac{1-z}{1+z}\right)^{\alpha} \qquad (0 < \alpha \leq 1,\, z \in \mathbb{U})$$

or

$$h(z) = \frac{1+(1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1-(1-2\beta)z}{1+z} \qquad (0 \leq \beta < 1,\, z \in \mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in \mathcal{LB}_{\Sigma}^{h,p}(\lambda)$, then

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(\frac{z\,(f'(z))^{\lambda}}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\, \lambda \geq 1,\, z \in \mathbb{U})$$

and

$$\left|\arg\left(\frac{w\,(g'(w))^{\lambda}}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\, \lambda \geq 1,\, w \in \mathbb{U})$$

or

$$f \in \Sigma \quad \text{and} \quad \Re\left(\frac{z\,(f'(z))^{\lambda}}{f(z)}\right) > \beta \qquad (0 \leq \beta < 1,\, \lambda \geq 1,\, z \in \mathbb{U})$$

and

$$\Re\left(\frac{w\,(g'(w))^{\lambda}}{g(w)}\right) > \beta \qquad (0 \leq \beta < 1,\, \lambda \geq 1,\, w \in \mathbb{U}),$$

where the function $g = f^{-1}$ is defined by (1.2). This means that

$$f \in \mathcal{LB}_{\Sigma}^{\lambda}(\alpha) \qquad (0 < \alpha \leq 1,\, \lambda \geq 1)$$

or

$$f \in \mathcal{LB}_{\Sigma}(\lambda, \beta) \qquad (0 \leq \beta < 1,\, \lambda \geq 1).$$

Motivated and stimulated especially by the work of Joshi *et al.* [5], we propose to investigate the bi-univalent function class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ introduced in Definition 3 here and derive coefficient estimates on the first two Taylor-Maclaurin coefficients a_2 and a_3 for a function $f \in \mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ given by (1.1). Our results for the bi-univalent function class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ would generalize and improve the related work of Joshi *et al.* [5].

2. A SET OF GENERAL COEFFICIENT ESTIMATES

In this section we state and prove our general results involving the bi-univalent function class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ given by Definition 3.

Theorem 3. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(2\lambda-1)^2}}\,,\ \sqrt{\frac{|h''(0)|+|p''(0)|}{4\lambda(2\lambda-1)}}\right\} \tag{2.1}$$

and

$$|a_3| \leq \min\{\gamma\,,\ \delta\}, \tag{2.2}$$

where

$$\gamma = \frac{|h'(0)|^2+|p'(0)|^2}{2(2\lambda-1)^2}+\frac{|h''(0)|+|p''(0)|}{4(3\lambda-1)}$$

and

$$\delta = \frac{\begin{cases} \left(2\lambda^2+2\lambda-1\right)|h''(0)|-\left(2\lambda^2-4\lambda+1\right)|p''(0)| & , \quad 1\leq\lambda<1+\frac{\sqrt{2}}{2} \\ \left(2\lambda^2+2\lambda-1\right)|h''(0)|+\left(2\lambda^2-4\lambda+1\right)|p''(0)| & , \quad \lambda\geq 1+\frac{\sqrt{2}}{2} \end{cases}}{4\lambda(2\lambda-1)(3\lambda-1)}.$$

Proof. First of all, we write the argument inequalities in (1.11) and (1.12) in their equivalent forms as follows:

$$\frac{z\,(f'(z))^{\lambda}}{f(z)} = h(z) \quad (z\in\mathbb{U}),$$

and

$$\frac{w\,(g'(w))^{\lambda}}{g(w)} = p(w) \quad (w\in\mathbb{U}),$$

respectively, where $h(z)$ and $p(w)$ satisfy the conditions of Definition 3. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1+h_1z+h_2z^2+\cdots$$

and

$$p(w) = 1+p_1w+p_2w^2+\cdots,$$

respectively. Now, upon equating the coefficients of $\frac{z(f'(z))^{\lambda}}{f(z)}$ with those of $h(z)$ and the coefficients of $\frac{w(g'(w))^{\lambda}}{g(w)}$ with those of $p(w)$, we get

$$(2\lambda-1)\,a_2 = h_1, \tag{2.3}$$

$$\left(2\lambda^2-4\lambda+1\right)a_2^2+(3\lambda-1)\,a_3 = h_2, \tag{2.4}$$

$$-(2\lambda-1)\,a_2 = p_1 \tag{2.5}$$

and

$$\left(2\lambda^2+2\lambda-1\right)a_2^2-(3\lambda-1)\,a_3 = p_2. \tag{2.6}$$

From (2.3) and (2.5), we obtain

$$h_1 = -p_1 \tag{2.7}$$

and

$$2\,(2\lambda-1)^2\,a_2^2 = h_1^2+p_1^2. \tag{2.8}$$

Also, from (2.4) and (2.6), we find that

$$2\lambda\,(2\lambda-1)\,a_2^2 = h_2+p_2. \tag{2.9}$$

Therefore, we find from the equations (2.8) and (2.9) that

$$|a_2|^2 \le \frac{|h'(0)|^2+|p'(0)|^2}{2\,(2\lambda-1)^2}$$

and

$$|a_2|^2 \le \frac{|h''(0)|+|p''(0)|}{4\lambda\,(2\lambda-1)},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.6) from (2.4). We thus get

$$2\,(3\lambda-1)\,a_3-2\,(3\lambda-1)\,a_2^2 = h_2-p_2. \tag{2.10}$$

Upon substituting the value of a_2^2 from (2.8) into (2.10), it follows that

$$a_3 = \frac{h_1^2+p_1^2}{2\,(2\lambda-1)^2}+\frac{h_2-p_2}{2\,(3\lambda-1)}.$$

We thus find that

$$|a_3| \le \frac{|h'(0)|^2+|p'(0)|^2}{2\,(2\lambda-1)^2}+\frac{|h''(0)|+|p''(0)|}{4\,(3\lambda-1)}.$$

On the other hand, upon substituting the value of a_2^2 from (2.9) into (2.10), it follows that

$$a_3 = \frac{\left(2\lambda^2+2\lambda-1\right)h_2+\left(-2\lambda^2+4\lambda-1\right)p_2}{2\lambda\,(2\lambda-1)\,(3\lambda-1)}.$$

We thus obtain

$$|a_3| \le \frac{\begin{cases}\left(2\lambda^2+2\lambda-1\right)|h''(0)|-\left(2\lambda^2-4\lambda+1\right)|p''(0)| & ,1\le\lambda<1+\frac{\sqrt{2}}{2}\\[2ex] \left(2\lambda^2+2\lambda-1\right)|h''(0)|+\left(2\lambda^2-4\lambda+1\right)|p''(0)| & ,\lambda\ge 1+\frac{\sqrt{2}}{2}\end{cases}}{4\lambda\,(2\lambda-1)\,(3\lambda-1)}.$$

This evidently completes the proof of Theorem 3. □

3. Corollaries and consequences

Setting $\lambda = 1$ in Theorem 3, we get following consequence.

Corollary 1 ([3]). *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{B}_{\Sigma}^{h,p}$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{4}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{2}+\frac{|h''(0)|+|p''(0)|}{8}, \frac{3|h''(0)|+|p''(0)|}{8}\right\}.$$

If we set

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad \text{and} \quad p(z) = \left(\frac{1-z}{1+z}\right)^{\alpha} \qquad (0<\alpha\leq 1, z\in\mathbb{U})$$

in Theorem 3, we can readily deduce Corollary 2.

Corollary 2. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{LB}_{\Sigma}^{\lambda}(\alpha)$ $(0<\alpha\leq 1, \lambda\geq 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2\alpha^2}{\lambda(2\lambda-1)}}$$

and

$$|a_3| \leq \begin{cases} \frac{2\alpha^2}{\lambda(2\lambda-1)}, & 1\leq\lambda<1+\frac{\sqrt{2}}{2} \\ \frac{2\alpha^2}{3\lambda-1} & \lambda\geq 1+\frac{\sqrt{2}}{2} \end{cases}.$$

Remark 2. Corollary 2 is an improvement of Theorem 1.

If we set

$$h(z) = \frac{1+(1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1-(1-2\beta)z}{1+z} \qquad (0\leq\beta<1, z\in\mathbb{U})$$

in Theorem 3, we can readily deduce Corollary 3.

Corollary 3. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{LB}_{\Sigma}(\lambda,\beta)$ $(0\leq\beta<1, \lambda\geq 1)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{1-\beta}{\lambda(2\lambda-1)}} & 0\leq\beta\leq\frac{2\lambda+1}{4\lambda} \\ \frac{2(1-\beta)}{(2\lambda-1)} & \frac{2\lambda+1}{4\lambda}\leq\beta<1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \min\left\{\frac{4(1-\beta)^2}{(2\lambda-1)^2} + \frac{1-\beta}{3\lambda-1}, \frac{1-\beta}{\lambda(2\lambda-1)}\right\} & , \quad 1 \leq \lambda < 1+\frac{\sqrt{2}}{2} \\ \frac{1-\beta}{3\lambda-1} & \lambda \geq 1+\frac{\sqrt{2}}{2} \end{cases}.$$

Remark 3. Corollary 3 is an improvement of Theorem 2.

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