Miskolc Mathematical Notes
Vol. 18 (2017), No. 1, pp. 427–429
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.2117

A NOTE ON A CHARACTERIZATION THEOREM FOR A CERTAIN CLASS OF DOMAINS

SHAFIQ UR REHMAN

Received 12 October, 2016

Abstract. We have introduced and studied in [2] the class of *Globalized multiplicatively pinched-Dedekind domains* (*GMPD domains*). This class of domains could be characterized by a certain factorization property of the non-invertible ideals, (see [2, Theorem 4]). In this note a simplification of the characterization theorem [2, Theorem 4] is provided in more general form.

2010 *Mathematics Subject Classification:* 13A15; 13F05

Keywords: invertible ideal, Prüfer domain, h-local domain

Let D be an integral domain. By an *MNI ideal* of D we mean an ideal of D which is maximal among the nonzero noninvertible ideals of D. By [6, Exercise 36, page 44], every MNI ideal is a prime ideal. Moreover, using standard Zorn's Lemma arguments, one can show that every nonzero non-invertible ideal is contained in some MNI ideal. D is said to be *h-local* provided every nonzero ideal of D is contained in at most finitely many maximal ideals of D and each nonzero prime ideal of D is contained in a unique maximal ideal of D. D is called a *pseudo-valuation domain (PVD)* if D is quasi-local with maximal ideal M and $M : M$ is a valuation domain with maximal ideal M, cf. [5] and [1, Proposition 2.5]. A *two-generated* domain is a domain whose ideals are two-generated. Let D be a quasi-local domain with maximal ideal M. By [5, Theorems 2.7 and 3.5], D is a two-generated PVD if and only if D is a field, a DVR, or a Noetherian domain such that its integral closure D' is a DVR with maximal ideal M and D'/M is a quadratic field extension of D/M.

In [2], we introduced and study the class of *Globalized multiplicatively pinched-Dedekind domains* (*GMPD domains*). A domain D is called a *globalized multiplicatively pinched-Dedekind domain* (*GMPD domain*) if D is h-local and for each maximal ideal M, D_M is a two-generated PVD, or a valuation domain with value group $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{R}$, cf. [2, Definition 2]. A Dedekind domain is a GMPD domain and the integrally closed Noetherian GMPD domain are exactly the Dedekind domains. This class of domains could be characterized by a certain factorization property of the non-invertible ideals. An h-local domain D is a GMPD domain if and only if every two MNI ideals are comaximal and every nonzero non-invertible ideal I of D can be written as $I = JP_1 \cdots P_k$ for some invertible ideal J and distinct MNI ideals $P_1, ..., P_k$ uniquely determined by I ([2, Theorem 4, Remark 5]). In this note a more general simplification of this characterization theorem in provided (Theorem 2).

Throughout this note all rings are (commutative unitary) integral domains. For a domain D, D' (resp. $\bar{D}$) denotes the integral closure (resp. complete integral closure) of D. Any unexplained material is standard like in [4] or [6].

Lemma 1. *Every two distinct MNI ideals of a domain D are comaximal.*

Proof. Deny. Let $P_1 \neq P_2$ be the MNI ideals of D both contained in the maximal ideal M. Then M is invertible and so $P_i \subsetneq M$ implies that $P_i \subsetneq \cap_{n \geq 1} M^n = Q$. The ideal Q is invertible and prime. Indeed, if $ab \in Q$ with both $a, b \notin Q$, then there exist integers k, l and the ideals U, V such that $(a) = M^k U$ with $U \nsubseteq M$ and $(b) = M^l V$ with $V \nsubseteq M$. Since $ab \in M^{k+l+1}$, so $(ab) = M^{k+l+1} N$ for some ideal N. Combining, we get that $M^{k+l} UV = M^{k+l+1} N$. This implies that $UV = MN$ which is not possible because U, V are not contained in M. Hence Q is prime. As $P_i \subsetneq Q$, so Q is invertible. Since any two invertible prime ideals are not comparable, so $Q = M$. This implies that $M = M^2$ and hence $M = D$, a contradiction. □

Recall [3, Section 5.1] that a domain D has pseudo-Dedekind factorization if for each nonzero non-invertible ideal I, there is an invertible ideal B (which might be D) and finitely many pairwise comaximal primes $P_1, P_2, ..., P_n$ such that $I = BP_1 P_2 \cdots P_n$ (the requirement that $n > 0$ comes for free).

Theorem 1. *Let D be a domain such that every nonzero non-invertible ideal I of D can be written as $I = JP_1 \cdots P_k$ for some invertible ideal J and distinct MNI ideals $P_1, ..., P_k$. Then D is h-local.*

Proof. By [6, Exercise 36, page 44], every MNI ideal is a prime ideal and by Lemma 1, $P_1, ..., P_k$ are pairwise comaximal. Hence D has pseudo-Dedekind factorization, cf. [3, Section 5.1]. Now Apply [3, Corollary 5.2.14]. □

Theorem 2. *A domain D is a GMPD domain if and only if every nonzero non-invertible ideal I of D can be written as $I = JP_1 \cdots P_k$ for some invertible ideal J and distinct MNI ideals $P_1, ..., P_k$.*

Proof. Apply Theorem 1 and [2, Theorem 4]. □

ACKNOWLEDGEMENT

This research is supported by the Higher Education Commission of Pakistan.

REFERENCES

[1] D. F. Anderson and D. E. Dobbs, "Pairs of rings with the same prime ideals." *Canad. J. Math.*, vol. 32, pp. 362–384, 1980, doi: 10.4153/CJM-1980-029-2.

[2] T. Dumitrescu and S. U. Rahman, "A class of pinched domains II," *Comm. Alg.*, vol. 39, pp. 1394–1403, 2011, doi: 10.1080/00927871003705591.

[3] M. Fontana, E. Houston, and T. Lucas, *Factoring Ideals in Integarl domains*. Lecture Notes of the Unione Mathematica Italiana, Vol. 14, Springer-Verlag Berlin Heidelberg, 2013. doi: 10.1007/978-3-642-31712-5.

[4] R. Gilmer, *Multiplicative Ideal Theory*. New York: Marcel Dekker, 1972.

[5] J. R. Hedstrom and E. G. Houston, "Pseudo-valuation domains," *Pacific J. Math.*, vol. 75, pp. 137–147, 1978, doi: 10.2140/pjm.1978.75.137.

[6] I. Kaplansky, *Commutative Rings*. Chicago and London: The University of Chicago Press, 1974.

Author's address

Shafiq Ur Rehman
COMSATS Institute of Information Technology, Attock Campus, Pakistan
Current address: ASSMS, GC University Lahore, Pakistan
E-mail address: shafiq_ur_rahman2@yahoo.com, shafiq@ciit-attock.edu.pk