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AN EXTRAGRADIENT-LIKE PARALLEL METHOD FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS AND SEMIGROUP OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper we propose a parallel iterative hybrid methods for finding a common element of the solution sets of a finite family of pseudomonotone equilibrium problems and the fixed points set of a semigroup-nonexpensive mappings in Hilbert spaces. Under mild conditions, we obtain the strong convergence of the proposed iterative process. Some numerical experiments are given to verify the efficiency the proposed algorithm.

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1. Introduction

Throughout the paper we suppose that $\mathcal{H}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$, C is a closed convex subset in $\mathcal{H}$ and that $f_i : C\times C \to \mathbb{R}$, $i=1,2,\dots,N$, $T(s): C\to C$, $s\geq 0$. Conditions for f_i, $i=1,2,\dots,N$ and $T(s)$ will be detailed later. We are interested in a solution method for the system defined as

$$\text{Find } x^*\in C: \qquad f_i(x^*,y)\geq 0 \quad \forall\, y\in C,\ i=1,2,\dots,N, \tag{1.1}$$

$$x^* = T(s)(x^*) \quad \forall s\geq 0. \tag{1.2}$$

The problem (1.1) can be considered as a system of equilibrium problems, which was first introduced by Blum and Oettli in [2]: Given $f: C\times C\to \mathcal{R}$,

$$\text{find } x^*\in C \text{ such that } f(x^*,y)\geq 0\ \ \forall y\in C. \tag{1.3}$$

The equilibrium problems play an important role in optimization and nonlinear analysis. It is well known that many problems, such as variational inequalities, Nash equilibrium problems, saddle point problems, complementarity problems can be formulated as special cases of equilibrium problems. There are different methods for solving (1.3), see, for example, [1–4], [5], [8], [12–16], [20], [21, 22], and the references cited therein. Among them, we are interested in *Extragradient method* introduced in [9, 20] due to its simplicity and efficiency:

$$\begin{cases} x_0 \in C, \\ y_n = \underset{y\in C}{\operatorname{argmin}}\left\{ \lambda f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}, \\ x_{n+1} = \underset{y\in C}{\operatorname{argmin}}\left\{ \lambda f(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}. \end{cases} \tag{1.4}$$

Under assumptions that the bifunctions f is pseudomonone and Lipschitz-type continuous, the authors proved that the sequence $\{x_n\}$ generated by (1.4) weakly converges to a solution of (1.3).

It is well known that the problems of finding common fixed points of nonexpansive mappings and of nonexpansive semigroups is an important problem in fixed point theory and applications; in particular, in image recovery, convex feasibility problem, and signal processing problem (see e.g. [7], [14]). Iterative approximation methods for these problems in Hilbert or Banach spaces have been studied extensively by many authors; see, for example, [6], [17], [23], [24, 25], and the references therein. Finding a common element of the set of fixed points of nonexpansive mappings or a semigroup of nonexpansive mappings and the set of solutions to a equilibrium problem has been studied extensively in the literature; see, for example, [4], [5], [12], [17], [18], [21, 22], [25] and the references therein. The common approach in these papers is to use a proximal point algorithm for handling the equilibrium problem. For monotone equilibrium problems the subproblems needed to solve in the proximal point method are strongly monotone, and therefore they have a unique solution that can be approximated by available methods. However, for pseudomonone problems the subproblems, in general, may have nonconvex solution set due to the fact that the regularized bifunctions do not inherit any pseudomonotoniciy property from the original one.

In this article, motivated by [4, 11] and inspired by [9, 20], we propose a parallel iterative method for finding a common element of the solution sets of a finite family of pseudomonotone equilibrium problems and the set of fixed points of a nonexpansive semigroup in Hilbert spaces. The main point here is that we combine a parallel splitting-up technique and the extragradient procedure rather than a proximal point algorithm for dealing with a finite family of pseudomonotone equilibrium problems and Mann's iterative algorithms for finding fixed points of nonexpansive mappings. We obtain the strong convergence of iterative processes.

The paper is organized as follows: In Section 2, we collect some definitions and results needed for further investigation. We describe a novel parallel hybrid iterative method in the Section 3. The convergence analysis for the proposed method is detailed in Section 4. Some special cases and illustrative examples are provided in the last section.

2. PRELIMINARIES

In this section, we recall some definitions and results that will be used in the sequel. In what follows by $x_n \rightharpoonup x$ we mean that the sequence $\{x_n\}$ converges to x in the weak topology. Let C be a nonempty closed convex of a Hilbert space $\mathcal{H}$. We recall that mapping $T : C \to C$ is said to be *nonexpansive* on C if

$$\|Tx - Ty\| \le \|x - y\| \text{ for al } x, y \in C.$$

Let $F(T)$ denote the set of fixed points of T. A family $\{T(s) : s \in \mathbb{R}_+\}$ of mappings from C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) for each $s \in \mathbb{R}_+$, $T(s)$ is a nonexpansive mapping on C;
(ii) $T(0)x = x$ for all $x \in C$;
(iii) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$ for all $s_1, s_2 \in \mathbb{R}_+$;
(iv) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into C is continuous.

Let $\mathcal{F} = \bigcap_{s \ge 0} F(T(s))$ be the set of all common fixed points of $\{T(s) : s \in \mathbb{R}_+\}$. We know that $\mathcal{F}$ is nonempty if C is bounded (see [3]).

We begin with the following properties of nonexpansive mappings.

Lemma 1 ([10]). *Let C be a closed convex subset of a Hilbert space $\mathcal{H}$ and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \ne \varnothing$. If a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $x_n - Sx_n \to 0$, then $z = Sz$.*

Lemma 2 ([23]). *Let C be a nonempty bounded closed convex subset of $\mathcal{H}$ and let $\{T(s) : s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on C. Then, for any $h \ge 0$*

$$\lim_{s\to\infty} \sup_{y\in C} \left\| T(h)\Big(\frac{1}{s}\int_0^s T(t)y\,dt\Big) - \frac{1}{s}\int_0^s T(t)y\,dt \right\| = 0$$

Since C is a nonempty closed and convex subset of H, for every $x \in H$, there exists a unique element $P_C x$, defined by

$$P_C x = \arg\min\{\|y - x\| : y \in C\}.$$

The mapping $P_C : H \to C$ is called the metric (orthogonal) projection of H onto C. It is also known that P_C is firmly nonexpansive, or 1-inverse strongly monotone (1-ism), i.e.,

$$\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2 \quad \forall x, y \in H. \tag{2.1}$$

Besides, we have

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \le \|x - y\|^2 \quad \forall x \in C,\ \forall y \in H. \tag{2.2}$$

Moreover, $z = P_C x$ if only if

$$\langle x - z, z - y\rangle \geq 0 \quad \text{for all } y \in C. \tag{2.3}$$

The bifunction $f : C \times C \to \mathbb{R}$ is called *monotone* on C if

$$f(x,y) + f(y,x) \leq 0 \text{ for all } x, y \in C;$$

pseudomonotone on C if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0 \text{ for all } x, y \in C.$$

It is obvious that any monotone bifunction is a pseudomonotone one, but not vice versa.

Throughout this paper we consider bifunctions with the following properties:

$(C1)$ f is pseudomonotone, i.e., for all $x, y \in C$,

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0;$$

$(C2)$ f is Lipschitz-type continuous, i.e., there exist two positive constants c_1, c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x - y\|^2 - c_2\|y - z\|^2 \text{ for all } x, y, z \in C;$$

$(C3)$ f is weakly continuous on $C \times C$;

$(C4)$ $f(x,.)$ is convex, subdifferentiable on C and $f(x,x) = 0$ for every $x \in C$.

The following statements will be needed in the next section.

Lemma 3 ([1])**.** *If the bifunction f satisfies Assumptions $(C1)-(C4)$, then the solution set of equilibrium problems:*

$$\textit{Find } x^* \in C : \ f(x^*, y) \geq 0 \ \textit{for all } y \in C$$

is weakly closed and convex.

Lemma 4 ([8])**.** *Let C be a convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then, x^* is a solution to the following convex problem*

$$\min\{g(x) : x \in C\}$$

if only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(.)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

It is also known that $\mathcal{H}$ satisfies Opial's condition. See the following definition in [19].

Definition 1 ([19])**.** A Banach space X is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in X which converges weakly to x, as $k \to \infty$, then

$$\lim_{n\to\infty} \sup\|x_n - x\| < \lim_{n\to\infty} \sup\|x_n - y\| \quad \forall y \in X, \text{ with } x \neq y.$$

3. Main results

In this section, based on the hybrid method in mathematical programming, projection, extragradient method and Mann's iteration, we propose a parallel iterative method for finding a common element of the solution sets of a finite family of equilibrium problems with pseudomonotone bifunctions $\{f_i\}_{i=1}^N$ and the set of fixed points of nonexpansive semigroup mappings $\{T(s):\ s\geq 0\}$ in a Hilbert space $\mathscr{H}$.
We denote by $Sol(C,f_i)$ the set of (1.1), $i=1,2,\ldots,N$,. In what follows, we assume that the solution set

$$\Omega=\mathscr{F}\bigcap\left(\bigcap_{i=1}^N Sol(C,f_i)\right)$$

is nonempty and each bifunction $f_i, (i=1,\ldots,N)$ satisfies all the conditions $(C1)-(C4)$. Observe that we can choose the same Lipschitz coefficients $\{c_1,c_2\}$ for all bifunctions $f_i, i=1,2,\ldots,N$. Indeed, condition $(C2)$ implies that

$$f_i(x,z)-f_i(x,y)-f_i(y,z)\leq c_{1i}\|x-y\|^2+c_{2i}\|y-z\|^2\leq c_1\|x-y\|^2+c_2\|y-z\|^2,$$

where $c_1=\max\limits_{1\leq i\leq N}c_{1i}$ and $c_2=\max\limits_{1\leq i\leq N}c_{2i}$. Hence,

$$f_i(x,y)+f_i(y,z)\geq f_i(x,z)-c_1\|x-y\|^2-c_2\|y-z\|^2.$$

Further, since $\Omega\neq\varnothing$, by Lemma 3, the sets $Sol(C,f_i), i=1,\ldots,N$ are nonempty, closed and convex, hence the solution set Ω is a nonempty closed and convex subset of C. Thus given any fixed element $x_0\in C$ there exists a unique element $\bar{x}:=P_\Omega x_0$.

Algorithm 1. *Choose positive number* $0<\rho<\min\left(\frac{1}{2c_1},\frac{1}{2c_2}\right)$ *and the positive sequences* $\{\mu_n\}\subset[a,1]$ *for some* $a\in(0,1)$.
Seek a starting point $x_0\in C$ *and set* $n:=0$.
Step 1.

- *Solve the strongly convex programs*

$$y_n^i=argmin\{\rho f_i(x_n,y)+\frac{1}{2}\|x_n-y\|^2:\ y\in C\},$$
$$z_n^i=argmin\{\rho f_i(y_n^i,y)+\frac{1}{2}\|x_n-y\|^2:\ y\in C\},\ i=1,\ldots,N;$$

- *Find positive integer*

$$i_n=\operatorname*{argmax}_{1\leq i\leq N}\{\|z_n^i-x_n\|\},$$

and set $z_n:=z_n^{i_n}$;

Step 2. Compute $u_n=(1-\mu_n)x_n+\mu_nT_nz_n$, *where* T_n *is defined as*

$$T_nx:=\frac{1}{s_n}\int_0^{s_n}T(s)x\,ds,\ \forall x\in C\ \textit{with}\ \lim_{n\to+\infty}s_n=+\infty;$$

$$x_{n+1} = P_{(H_n \cap W_n)} x_0, \text{ where}$$
$$H_n = \{z \in \mathcal{H} : \|u_n - z\| \le \|x_n - z\|\},$$
$$W_n = \{z \in \mathcal{H} : \langle x_n - z, x_0 - x_n \rangle \ge 0\}.$$

Increase n by 1 *and go back to Step* 1.

We now state and prove the convergence of the proposed iteration method.

4. CONVERGENCE RESULTS

In this section, we show the strong convergence of the sequences $\{x_n\}$ and $\{u_n\}$ defined by Algorithm 1 to the common element in a real Hilbert space.

For establishing the strong convergence of $\{x_n\}$ and $\{u_n\}$ in Algorithm 1, we need the following result (see [20]).

Lemma 5. *Suppose that* $x^* \in Sol(C, f_i)$ *and* x_n, y_n^i, z_n^i, $i = 1, \ldots, N$, *are as in Step 1 of Algorithm 1. Then*

$$\|z_n^i - x^*\|^2 \le \|x_n - x^*\|^2 - (1 - 2\rho c_1)\|y_n^i - x_n\|^2 - (1 - 2\rho c_2)\|y_n^i - z_n^i\|^2.$$

Theorem 1. *Let* C *be a nonempty closed convex subset in a real Hilbert space* $\mathcal{H}$, $\{T(s) : s \in \mathbb{R}_+\}$ *be a nonexpansive semigroup on* C, f_i *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying conditions* $(C_1) - (C_4)$. *Suppose that* $\Omega \ne \varnothing$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by the Algorithm 1, where* $\{\mu_n\} \subset [a, 1]$ *for some* $a \in (0, 1)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $p^* = P_\Omega x_0$.

Proof. It is obvious that H_n and W_n are closed and convex for every $n \ge 0$. So that the $\{x_n\}$ is well defined for every $n \ge 0$. Moreover, it is easy seen that T_n is nonexpensive for all $n \ge 0$.

Now we divide the proof into several steps.

Step 1. *Claim that* $\Omega \subset H_n \cap W_n$ *for every* $n \ge 0$.

Indeed, for each $x^* \in \Omega$, by Lemma 5, we have

$$\|z_n^i - x^*\| \le \|x_n - x^*\| \text{ for all } n \ge 0.$$

From the definition of i_n, we have

$$\|z_n - x^*\| \le \|x_n - x^*\| \text{ for all } n \ge 0. \tag{4.1}$$

From the convexity of $\|\cdot\|^2$, the nonexpansiveness of T_n and (4.1) it follows that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \left\|(1-\mu_n)(x_n - x^*) + \mu_n(T_n z_n - x^*)\right\|^2 \\
&\le (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\|T_n z_n - T_n x^*\|^2 \\
&\le (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\|z_n - x^*\|^2 \\
&\le (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 \quad \forall n \ge 0,
\end{aligned} \tag{4.2}$$

which implies $x^* \in H_n$. Hence $\Omega \subset H_n$ for all $n \geq 0$.

Next we show $\Omega \subset W_n$ for all $n \geq 0$. Indeed, when $n = 0$, we have $x_0 \in C$ and $W_0 = H$. Consequently, $\Omega \subset H_0 \cap W_0$. By induction, suppose $\Omega \subset H_m \cap W_m$ for some $m \geq 0$. We have to prove that $\Omega \subset H_{m+1} \cap W_{m+1}$. Since $x_{m+1} = P_{H_m \cap W_m} x_0$, by (2.3), for every $z \in \Omega \subset H_m \cap W_m$, it holds that

$$\langle x_{m+1} - z, x_0 - x_{m+1} \rangle \geq 0,$$

which means that $z \in W_{m+1}$. Note that $\Omega \subset H_n$ for all $n \geq 0$, we can conclude that $\Omega \subset H_n \cap W_n$ for all $n \geq 0$.

Step 2. *Claim that for all* $i = 1, \ldots, N$, *we have*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|x_n - z_n^i\| = \lim_{n\to\infty} \|x_n - y_n^i\| = 0.$$

Indeed, from $x_n = P_{W_n} x_0$ and (2.2), it follows that, for every $u \in \Omega \subset W_n$, we get

$$\|x_n - x_0\|^2 \leq \|u - x_0\|^2 - \|u - x_n\|^2 \leq \|u - x_0\|^2. \tag{4.3}$$

This implies that the sequence $\{x_n\}$ is bounded. From (4.2) and (4.1), it follows that the sequences $\{u_n\}$ and $\{z_n\}$ are also bounded.

Observing that $x_{n+1} = P_{H_n \cap W_n} x_0 \in W_n, x_n = P_{W_n} x_0$, from (2.2) we have

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2. \tag{4.4}$$

Thus, the sequence $\{\|x_n - x_0\|\}$ is nondecreasing, hence there exists the limit of the sequence $\{\|x_n - x_0\|\}$. From (4.4) we obtain

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

Letting $n \to \infty$, we find

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.5}$$

Since $x_{n+1} \in H_n$, it follows that $\|u_n - x_{n+1}\| \leq \|x_{n+1} - x_n\|$. Thus

$$\|u_n - x_n\| \leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\|.$$

The last inequality together with (4.5) implies that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0 \tag{4.6}$$

Moreover, from (4.2), Lemma 5 and the definition of i_n for any fixed $x^* \in \Omega$, we have

$$\begin{aligned}\|u_n - x^*\|^2 &\leq (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \mu_n\left[(1-2\rho c_1)\|y_n^{i_n} - x_n\|^2 + (1-2\rho c_2)\|y_n^{i_n} - z_n\|^2\right].\end{aligned}$$

Therefore

$$\begin{aligned}&a\left[(1-2\rho c_1)\|y_n^{i_n} - x_n\|^2 + (1-2\rho c_2)\|y_n^{i_n} - z_n\|^2\right] \\ &\leq \mu_n\left[(1-2\rho c_1)\|y_n^{i_n} - x_n\|^2 + (1-2\rho c_2)\|y_n^{i_n} - z_n\|^2\right]\end{aligned}$$

$$\le \|x_n - x^*\|^2 - \|u_n - x^*\|^2$$
$$= \left(\|x_n - x^*\| - \|u_n - x^*\|\right)\left(\|x_n - x^*\| + \|u_n - x^*\|\right)$$
$$\le \|x_n - u_n\| \left(\|x_n - x^*\| + \|u_n - x^*\|\right). \tag{4.7}$$

Using the last inequality together with (4.6) and taking into account the boundedness of two sequences $\{u_n\}$ and $\{x_n\}$ as well as the conditions of $\{\mu_n\}$, ρ, we come to the relations

$$\lim_{n\to\infty} \left\| y_n^{i_n} - x_n \right\| = \lim_{n\to\infty} \left\| y_n^{i_n} - z_n \right\| = 0. \tag{4.8}$$

From $\|z_n - x_n\| \le \|z_n - y_n^{i_n}\| + \|y_n^{i_n} - x_n\|$ and (4.8), we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{4.9}$$

By the definition of i_n, we get

$$\lim_{n\to\infty} \left\| z_n^i - x_n \right\| = 0 \tag{4.10}$$

for all $i = 1, \dots, N$. From Lemma 5 and (4.10), we obtain

$$\lim_{n\to\infty} \left\| y_n^i - x_n \right\| = 0$$

for all $i = 1, \dots, N$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some element p.

Step 3. *Claim that $p \in \Omega$.*

From (4.6) and (4.9), we obtain also that $\{u_{n_k}\}$ and $\{z_{n_k}\}$ converges weakly to p. Since $\{u_n\} \subset C$ and C is a closed convex subset in $\mathscr{H}$, we have $p \in C$.

Now, we prove that $p \in \Omega$. To this end, first we show that $p \in \bigcap_{i=1}^{N} Sol(C, f_i)$.

Noting that

$$y_n^i = \operatorname{argmin}\{\rho f_i(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : \ y \in C\},$$

by Lemma 4, we obtain

$$0 \in \partial_2 \left\{ \rho f_i(x_n, y) + \frac{1}{2}\|x_n - y\|^2 \right\} (y_n^i) + N_C(y_n^i).$$

Therefore, there exists $w \in \partial_2 f_i(x_n, y_n^i)$ and $\bar{w} \in N_C(y_n^i)$ such that

$$\rho w + x_n - y_n^i + \bar{w} = 0. \tag{4.11}$$

Since $\bar{w} \in N_C(y_n^i)$, $\langle \bar{w}, y - y_n^i \rangle \le 0$ for all $y \in C$. This together with (4.11) implies that

$$\rho \left\langle w, y - y_n^i \right\rangle \ge \left\langle y_n^i - x_n, y - y_n^i \right\rangle \tag{4.12}$$

for all $y \in C$. Since $w \in \partial_2 f_i(x_n, y_n^i)$,

$$f_i(x_n, y) - f_i(x_n, y_n^i) \geq \left\langle w, y - y_n^i \right\rangle \text{ for all } y \in C. \tag{4.13}$$

From (4.12) and (4.13), we get

$$\rho\Big(f_i(x_n, y) - f_i(x_n, y_n^i)\Big) \geq \left\langle y_n^i - x_n, y - y_n^i \right\rangle \text{ for all } y \in C. \tag{4.14}$$

Since $x_{n_k} \rightharpoonup p$ and $\|x_n - y_n^i\| \to 0$ as $n \to \infty$, we have $y_{n_k}^i \rightharpoonup p$. Letting $n = n_k$ in (4.14), passing to the limit as $k \to \infty$ and using assumptions $(C3)$, we conclude that $f_i(p, y) \geq 0$ for all $y \in C$, $i = 1, 2, \ldots, N$. Thus, $p \in \bigcap_{i=1}^{N} Sol(C, f_i)$.

Now, we prove that $p = T(h)p$ for all $h > 0$. First, we obtain from Step 3 of the algorithm that

$$\begin{aligned} a\|u_n - T_n u_n\| &\leq \mu_n \|u_n - T_n u_n\| \\ &\leq \mu_n\Big(\|u_n - T_n z_n\| + \|T_n z_n - T_n u_n\|\Big) \\ &\leq \|\mu_n u_n - \mu_n T_n z_n\| + \mu_n \|T_n z_n - T_n u_n\| \\ &= \|\mu_n u_n + (1-\mu_n)x_n - u_n\| + \mu_n\|T_n z_n - T_n u_n\| \\ &\leq (1-\mu_n)\|x_n - u_n\| + \mu_n\|z_n - u_n\| \\ &\leq \|u_n - x_n\| + \mu_n\|z_n - x_n\| \end{aligned} \tag{4.15}$$

Taking into account $\|u_n - x_n\| \to 0$ and $\|z_n - x_n\| \to 0$, it follows that

$$\lim_{n\to\infty} \|u_n - T_n u_n\| = 0. \tag{4.16}$$

Note that

$$\begin{aligned} \|T(h)u_n - u_n\| &\leq \left\| T(h)u_n - T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\Big)\right\| \\ &\quad + \left\| T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\Big) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \right\| \\ &\quad + \left\| \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - u_n \right\| \\ &\leq 2\left\| \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - u_n \right\| \\ &\quad + \left\| T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\Big) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \right\|. \end{aligned} \tag{4.17}$$

Since the sequence $\{u_n\}$ is bounded, we can apply Lemma 2 to get

$$\lim_{n\to\infty} \left\| T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\Big) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \right\| = 0, \tag{4.18}$$

for every $h \in (0,\infty)$ and therefore, by (4.16), (4.17) and (4.18), we obtain

$$\lim_{n\to\infty} \|T(h)u_n - u_n\| = 0$$

for each $h > 0$ from which we have by Lemma 1 that p is a fixed point of $T(h)$ for all $h > 0$. Hence $p \in \mathcal{F}$.

Step 4. *The sequence* $\{x_n\}$ *converges strongly to* $p^* := P_\Omega x_0$.

Indeed, from $p^* \in \Omega$ and (4.3), we obtain

$$\|x_n - x_0\| \le \|p^* - x_0\| \quad \forall n \ge 0.$$

The last inequality together with $x_{n_k} \rightharpoonup p$ and the weak lower semicontinuity of the norm $\|.\|$ imply that

$$\|p - x_0\| \le \liminf_{k\to\infty} \|x_{n_k} - x_0\| \le \limsup_{k\to\infty} \|x_{n_k} - x_0\| \le \|p^* - x_0\|.$$

On the other hand, since $p^* := P_\Omega x_0$, we have $\|p - x_0\| \ge \|p^* - x_0\|$. Hence, $p^* = p$ and $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|p^* - x_0\|$. Taking into account $x_{n_k} \rightharpoonup p^*$, we have $x_{n_k} \to p^*$. Finally, suppose that $\{x_{n_j}\}$ is an another weakly convergent subsequence of $\{x_n\}$. By a similar argument as above, we conclude that $\{x_{n_j}\}$ converges strongly to $p^* := P_\Omega x_0$. Therefore, the sequence $\{x_n\}$ generated by the Algorithm 1 converges strongly to $P_\Omega x_0$. Then the strong convergence of the sequences $\{u^n\}$ to p is followed from (4.6). The proof is now completed. □

5. Special cases and illustrative examples

If $N = 1$, Algorithm 1 reduces to the following one for finding a common element in the solution set of pseudomonotone equilibrium problems and the set of the fixed points of a nonexpansive semigroup in Hilbert spaces.

Corollary 1. *Let C be a nonempty closed convex subset in a real Hilbert space $\mathcal{H}$, $\{T(s) : s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on C, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(C_1)-(C_4)$. Suppose that $\Omega = \mathcal{F} \cap Sol(f,C) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{aligned}
& x^0 \in C \text{ chosen arbitrarily},\\
& y_n = argmin\{\rho f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : \ y \in C\},\\
& z_n = argmin\{\rho f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : \ y \in C\},\\
& u_n = (1-\mu_n)x_n + \mu_n T_n z_n,\\
& H_n = \{z \in \mathcal{H} : \ \|u_n - z\| \le \|x_n - z\|\},\\
& W_n = \{z \in \mathcal{H} : \ \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{(H_n \cap W_n)} x_0,
\end{aligned}$$

where $\{\mu_n\} \subset [a,1]$, $a \in (0,1)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $p^* \in \Omega$.

Proof. Taking $N = 1$ in Theorem 1, we get the desired conclusion easily.

Now putting $f_i(x,y) = 0$, $i = 1,2,\dots,N$, for all $x, y \in C$, we obtain the following result for finding a common fixed point of a nonexpansive semigroup $\{T(s) : s \in \mathbb{R}_+\}$ on C. □

Corollary 2. *Let* C *be a nonempty closed convex subset in a real Hilbert space* $\mathcal{H}$, $\{T(s) : s \in \mathbb{R}_+\}$ *be a nonexpansive semigroup on* C *such that* $\mathcal{F} \neq \varnothing$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{aligned} x_0 &\in C \ chosen\ arbitrarily, \\ u_n &= (1-\mu_n)x_n + \mu_n T_n x_n, \\ H_n &= \{z \in \mathcal{H} :\ \|u_n - z\| \le \|x_n - z\|\}, \\ W_n &= \{z \in \mathcal{H} :\ \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{(H_n \cap W_n)} x_0, \end{aligned}$$

where $\{\mu_n\} \subset [a,1]$, $a \in (0,1)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $p^* \in \mathcal{F}$.

If $T(s)x = x$ for all $s > 0$ and $x \in C$, Algorithm 1 reduces to the following one for finding a common element in the solution-set of a finite family of pseudomonotone equilibrium problems in Hilbert spaces.

Corollary 3. *Let* C *be a nonempty closed convex subset in a real Hilbert space* $\mathcal{H}$, f_i *be bifunctions from* $C \times C$ *to* $\mathbb{R}$ *satisfying conditions* $(C_1)-(C_4)$. *Suppose that* $\Omega = \bigcap_{i=1}^{N} Sol(f_i, C) \neq \varnothing$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{aligned} x^0 &\in C \ chosen\ arbitrarily, \\ y_n^i &= argmin\{\rho f_i(x_n, y) + \frac{1}{2}\|x_n - y\|^2 :\ y \in C\},\ i = 1,\dots,N, \\ z_n^i &= argmin\{\rho f_i(y_n^i, y) + \frac{1}{2}\|x_n - y\|^2 :\ y \in C\},\ i = 1,\dots,N, \\ i_n &= \operatorname*{argmax}_{1 \le i \le N}\{\|z_n^i - x_n\|\},\ \ z_n := z_n^{i_n}, \\ u_n &= (1-\mu_n)x_n + \mu_n z_n, \\ H_n &= \{z \in \mathcal{H} :\ \|u_n - z\| \le \|x_n - z\|\}, \\ W_n &= \{z \in \mathcal{H} :\ \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{(H_n \cap W_n)} x_0. \end{aligned}$$

where $\{\mu_n\} \subset [a,1]$ *for some* $a \in (0,1)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $p \in \Omega$.

To illustrate the proposed algorithm, we consider the following examples. The computer used in these experiments had an Intel Boxed Core CPU Q9400 6M Cache, 2.66 GHz, 1333 MHz FSB and 4 GB of memory. The language was MATLAB 2010b.

Example 1. Let $\mathscr{H} = \mathbb{R}^k$ with the inner product $\langle x, y\rangle := x_1y_1 + \cdots + x_ky_k$ for all $x = (x_1, x_2, \cdots, x_k), y = (y_1, y_2, \cdots, y_k) \in \mathscr{H}$. Let $C := [-1,1]^k$ be a k-dimensional box in $\mathscr{H}$. For all $x, y \in C$ and for each $i \in \{1,2,\ldots,N\}$, we define the operator f_i by

$$f_i(x,y) := \sum_{j=1}^{k} \alpha_{ij}(y_j^2 - x_j^2)$$

where $\alpha_{ij} \in (0,1)$ are randomly generated. An elementary computation shows that conditions $(C_1)-(C_4)$ are satisfied for all f_i, $i = 1,2,\ldots,N$. To define a nonexpansive semigroup let us consider the matrix

$$T(s) = \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \ s \in \mathbb{R},$$

and let

$$T(s)x = \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} x$$

$$= \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}.$$

It is easy to verify that $\{T(s) : \ s \geq 0\}$ is a nonexpansive semigroup on C and that the common solution-set is $\Omega = \mathscr{F} \bigcap \big(\bigcap_{i=1}^{N} Sol(C, f_i) \big) = \{(0,0,0,\ldots,0)^T\}$.
We apply Algorithm 1 to solve problem (1.1)-(1.2). Note that the mapping T_n in Algorithm 1 can be found in a closed form:

$$T_n z = (z_1 \frac{1-e^{-s_n}}{s_n}, z_2 \frac{1-e^{-s_n}}{s_n}, z_3, \ldots, z_k)^T.$$

We choose the parameters as follows:

- $\rho = 10$;

- $\mu_n = 0.9 \ \forall n \geq 1$;
- $s_n = n \ \forall n \geq 1$;
- the stopping rule: $\|x_n - x^*\| \leq 5.10^{-3}$, where $x^* = (0,0,0,\ldots,0)^T$ is the unique solution of problem (1.1)-(1.2).

First, we test Algorithm 1 with $k = 6$, $N = 3$, $x_0 = (1,1,1,1,1)^T$. The results are presented in Table 1. The approximate solution is obtained after 198 iterations.

TABLE 1. Iterations of Algorithm 1 in Example 1 with starting point $x_0 = (1,1,1,1,1,1)^T$

Iter(n)	x_n^1	x_n^2	x_n^3	x_n^4	x_n^5	x_n^6	$\|x_n - x^*\|$
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	2.4495
1	0.5810	0.5720	0.6221	0.5845	0.5796	0.6328	1.4593
2	0.3339	0.3348	0.3715	0.3507	0.3485	0.3787	0.8657
3	0.2029	0.1925	0.2287	0.1993	0.1949	0.2360	0.5137
4	0.0879	0.1278	0.1129	0.1499	0.1588	0.1103	0.3109
5	0.3335	0.0252	0.2648	-0.0727	-0.1500	0.3029	0.5491
6	0.1932	0.0471	0.1795	-0.0013	-0.0393	0.2039	0.3390
7	0.1034	0.0715	0.1194	0.0610	0.0514	0.1309	0.2314
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
198	0.0017	0.0017	0.0022	0.0017	0.0024	0.0023	0.0049

Next, we test our algorithm with different choices of k, N and x_0. The results are presented in Table 2.

TABLE 2. Performance of Algorithm 1 in Example 1 with different k, N and x_0

	$x_0 = (1,1,1,\cdots,1)^T$		$x_0 = (-1,-1,-1,\cdots,-1)^T$	
	CPU times	Iter.	CPU times	Iter.
k=6, N=3	32.0823	135	32.7314	140
k=6, N=6	85.4715	171	110.8062	241
k=10, N=6	244.5807	423	231.3017	393
k=20, N=3	364.0974	889	351.2896	862

Example 2. Consider problem (1.1)-(1.2) with $N = 2$

$$\text{find } x^* \in \Omega := EP(C, f_1) \cap EP(C, f_2) \cap F, \tag{5.1}$$

where

$$f_1 : \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}, \quad f_1(x,y) = \|y\|^4 - \|x\|^4 \quad \forall x,y \in \mathcal{R}^3,$$

$$f_2 : \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}, \quad f_2(x,y) = \langle Ax + By + q, y - x \rangle \quad \forall x, y \in \mathcal{R}^3,$$

$$A = \begin{pmatrix} 3 & 0 & 4 \\ 2 & 6 & 3 \\ 3 & 6 & 8 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 2 & 7 & 6 \end{pmatrix}, q = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},$$

F is the set of fixed points of the nonexpansive semigroup $T(t)$ defined by

$$T(t) : \mathcal{R}^3 \to \mathcal{R}^3, \quad T(t)x = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \forall x \in \mathcal{R}^3, \quad t > 0.$$

The feasible set is $C := [0,1]^3 \subset \mathcal{R}^3$. It is easy seen that f_1, f_2 is pseudomonotone and all the conditions of Theorem 1 are satisfied. Moreover, we can check that $\Omega = \{x^*\}$ where $x^* = (0,0,0)^T$ is the unique solution of problem (5.1). We apply Algorithm 1 to problem (5.1). Note that if we choose $s_n = n \quad \forall n \geq 1$, the mappings T_n can be expressed in the form

$$T_n(x) = \frac{1}{n} \begin{pmatrix} x_1 \sin n + x_2(\cos n - 1) \\ x_1(1 - \cos n) + x_2 \sin n \\ n x_3 \end{pmatrix}.$$

Now, we compute the Lipschitz constants of f_i. It is easy seen that f_1 is Lipschitz-type continuous with any constants $c_1, c_2 > 0$. For f_2, we have

$$\begin{aligned} f_2(x,y) + f_2(y,z) - f_2(x,z) &= \langle Ax + By + q, y - x \rangle + \langle Ay + Bz + q, z - y \rangle \\ &\quad - \langle Ax + Bz + q, z - x \rangle \\ &= \langle A(y - x), z - y \rangle + \langle B(y - z), y - x \rangle \\ &\geq -\frac{(\|A\| + \|B\|)}{2} \|x - y\|^2 - \frac{(\|A\| + \|B\|)}{2} \|y - z\|^2. \end{aligned}$$

Hence, $\frac{1}{2c_1} = \frac{1}{2c_2} = \frac{1}{\|A\| + \|B\|} = 0.0418$. The parameters in Algorithm 1 are chosen as follows

- $\rho = 0.04$;
- $\mu = 0.5$;
- $s_n = n \quad \forall n \geq 1$;
- $x_0 = (1,1,1)^T$;
- the stopping rule: $\|x_n - x^*\| \leq 5.10^{-3}$.

The results are presented in Table 3. The approximate solution is obtained after 137 iterations.

CONCLUSION

We have proposed a parallel iterative method for finding a common element in the solution sets of a finite family of pseudomonotone equilibrium problems and the set of fixed points for a semigroup nonexpansive mappings. For handling a finite family

TABLE 3. Iterations of Algorithm 1 in Example 2.

Iter(n)	x_n^1	x_n^2	x_n^3	$\|x_n - x^*\|$
0	1.0000	1.0000	1.0000	1.7321
1	0.8401	0.9462	0.8360	1.5166
2	0.5758	0.8754	0.7388	1.2820
3	0.4044	0.6031	0.6616	0.9823
4	0.3552	0.3659	0.5562	0.7546
5	0.2393	0.3172	0.4439	0.5957
6	0.3476	0.1376	0.3436	0.5078
7	0.1764	0.2634	0.3484	0.4711
···	···	···	···	···
137	-0.0020	0.0010	0.0010	0.0024

of pseudomonotone equilibrium we have used a parallel splitting-up technique and the extragrandient with the Lipschitz-type continuous. The strong convergence of the proposed method has been established by using cutting planes.

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