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ON A SUMMATION FORMULA FOR THE CLAUSEN'S SERIES $_3F_2$ WITH APPLICATIONS

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Abstract. The aim of this research paper is to establish the following summation formula for the Clausen's series $_3F_2$:

$$ {}_3F_2\left[\begin{matrix}-n, b-a-1, f+1\\ b, f\end{matrix}; 1\right] = \frac{(a)_n(c+1)_n}{(b)_n(c)_n}, $$

where $f = c(1+a-b)/(a-c)$, in *three different ways*. For $c = \frac{1}{2}a$, we have

$$ {}_3F_2\left[\begin{matrix}-n, b-a-1, 2+a-b\\ b, 1+a-b\end{matrix}; 1\right] = \frac{(a)_n\left(1+\frac{1}{2}a\right)_n}{(b)_n\left(\frac{1}{2}a\right)_n}, $$

which is already available in the literarute. Our formula is then applied to obtain two general results, one is the Euler's transformation for the series $_2F_2$ and another is the Kummer–type first transformation for the series $_2F_2$ established recently by Paris by following a different method. The results obtained generalize the related results by Exton.

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1. INTRODUCTION AND RESULTS REQUIRED

In 1997 Exton [3] derived four reduction formulæ for the Kampé de Fériet function and from one of the results, he deduced the following two results:

$$ (1-x)^{-h}{}_3F_2\left[\begin{matrix}h, a, 1+\frac{1}{2}a\\ b, \frac{1}{2}a\end{matrix}; -\frac{x}{1-x}\right] = {}_3F_2\left[\begin{matrix}h, b-a-1, 2+a-b\\ b, 1+a-b\end{matrix}; x\right] \tag{1} $$

and

$$ e^{-x}{}_2F_2\left[\begin{matrix}a, 1+\frac{1}{2}a\\ \frac{1}{2}a\end{matrix}; x\right] = {}_2F_2\left[\begin{matrix}b-a-1, 2+a-b\\ b, 1+a-b\end{matrix}; -x\right]. \tag{2} $$

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It is of interest to compare these results with

$$(1-x)^{-\alpha}{}_2F_1\left[\begin{matrix}\alpha,\gamma-\beta\\ \gamma\end{matrix};-\frac{x}{1-x}\right]={}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right] \tag{3}$$

known as Euler's first transformation [8] and the confluent

$$e^{-x}{}_1F_1\left[\begin{matrix}a\\ b\end{matrix};x\right]={}_1F_1\left[\begin{matrix}b-a\\ b\end{matrix};-x\right] \tag{4}$$

known as Kummer's first theorem [8].

We remark in passing that the result (1) is also recorded in [10] where it is obtained by other method. The result (2) has been obtained by Miller [4] by following two different and transparent ways. On the other hand it should be remarked here that whenever generalized hypergeometric functions reduce to gamma functions, the results are very important from the applications point of view.

It is also well known that the generalized hypergeometric functions ${}_pF_q$ appear ubiquitously as solutions of plethora of problems in mathematics, statistics and mathematical physics. Thus, the results established in this paper should be eventually useful in a wide range of applications.

The aim of this research article is to establish first, a summation formula for the Clausen's series ${}_3F_2$ and then, as an application we derive two results which generalize (1) and (2).

In our present investigations we will require:

(i) Euler's second transformation formula [1]

$${}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right]=(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left[\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma\end{matrix};x\right]; \tag{5}$$

(ii) the Beta-integral transform [2]

$$\int_0^1 x^{c-1}(1-x)^{e-c-1}{}_2F_1\left[\begin{matrix}a,b\\ d\end{matrix};x\right]\mathrm{d}x=\frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)}{}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};1\right], \tag{6}$$

provided $\Re\{c\}>0$, $\Re\{e-c\}>0$, $\Re\{d+e-a-b-c\}>0$;

(iii) Bailey's transform [10, Eq. (2.4.10), p. 60] in summing double series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n-m). \tag{7}$$

2. Main summation formula

Theorem 1. *Let n be a non-negative integer and let $a,b,c\in\mathbb{C}$ such that*

$$f=\frac{c(1+a-b)}{a-c}\notin\mathbb{Z}_0^-:=\{0,-1,-2,\dots\}.$$

Then

$$ {}_3F_2\left[\begin{matrix}-n, b-a-1, f+1\\ b, f\end{matrix}; 1\right] = \frac{(a)_n(c+1)_n}{(b)_n(c)_n}, \tag{8} $$

where $(a)_n = \Gamma(a+n)/\Gamma(n) = a(a+1)\cdots(a+n-1)$ *stands for the Pochhammer symbol.*

First proof. Denoting by S the left hand side of (8), expressing ${}_3F_2$ as a series, we get

$$ S = \sum_{r=0}^{n} \frac{(-n)_r(b-a-1)_r}{(b)_r\, r!} \left\{\frac{(f+1)_r}{(f)_r}\right\} = \sum_{r=0}^{n} \frac{(-n)_r(b-a-1)_r}{(b)_r\, r!} \left\{1+\frac{r}{f}\right\}. $$

Separate now S into two series, and after little adjusment in the second series, summing up the both expression, we arrive at

$$ S = {}_1F_2\left[\begin{matrix}-n, b-a\\ b+1\end{matrix}; 1\right]. $$

Applying Vandermonde's theorem in both hypergeometric series ${}_2F_1$, we have

$$ S = \frac{(a+1)_n}{(b)_n} + \frac{n(a-c)}{bc}\frac{(a+1)_{n-1}}{(b+1)_{n-1}} $$

which, upon a short algebraic simplification yields

$$ S = \frac{(a)_n(c+1)_n}{(b)_n(c)_n}. $$

The proof is complete. □

Second proof. In a two-term transformation formula for the Clausen's series ${}_3F_2$, popularly known as *Kummer–Thomae–Whipple formula* [1], which evidently originates back to Kummer [1] and states that

$$ {}_3F_2\left[\begin{matrix}A, H, C\\ D, E\end{matrix}; 1\right] = \frac{\mathrm{B}(E, D+E-A-H-C)}{\mathrm{B}(E-C, D+E-A-H)}\, {}_3F_2\left[\begin{matrix}C, D-A, D-H\\ D, D+E-A-H\end{matrix}; 1\right] \tag{9} $$

provided $\Re\{E\} > 0$, $\Re\{D+E-A-H-C\} > 0$. Here

$$ \mathrm{B}(s,r) = \int_0^1 x^{s-1}(1-x)^{r-1}\mathrm{d}x \qquad \left(\min\{\Re\{s\}, \Re\{r\}\} > 0\right) \tag{10} $$

stands for the Euler Beta function (Its connection to the familiar Euler Gamma function $\Gamma(\cdot)$ one realizes by $\mathrm{B}(s,r) = \Gamma(s)\Gamma(r)/\Gamma(s+r)$).

In order to derive our summation formula (8), put $A = b-a-1$, $H = f+1$, $C = -n$, $D = f$, and $E = b$ in (9). So, we get after some reduction

$$ S = \frac{(a)_n}{(b)_n}\, {}_3F_2\left[\begin{matrix}-n, f-b+a+1, -1\\ f, a\end{matrix}; 1\right] = \frac{(a)_n}{(b)_n}\left\{1+\frac{n(f-b+a+1)}{af}\right\}. $$

Using now $f = c(1+a-b)/(a-c)$, the last expression transforms into

$$S = \frac{(a)_n}{(b)_n}\left\{1+\frac{n}{c}\right\}.$$

Finally, recalling that $c+n = c(c+1)_n/(c)_n$, we conclude

$$S = \frac{(a)_n(c+1)_n}{(b)_n(c)_n},$$

such that completes the proof. □

Third proof. Consider the Beta-integral transform (6) in the slightly changed form:

$${}_3F_2\left[\begin{matrix}\alpha,\beta,s\\ \gamma,s+r\end{matrix};1\right] = \frac{1}{\mathrm{B}(\rho,\sigma)}\int_0^1 x^{s-1}(1-x)^{r-1}{}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right]\mathrm{d}x$$

provided $\Re\{s\}>0$, $\Re\{r\}>0$, and $\Re\{r+\gamma-\alpha-\beta\}>0$. In this, if we take $\alpha=-n$, $\beta = f+1$, $s = b-a-1$, $\gamma = f$, and $s+r = b$, that is, $r = a+1$, by (5) we get

$$\begin{aligned} S &= \frac{1}{\mathrm{B}(b-a-1,a+1)}\int_0^1 x^{b-a-2}(1-x)^a{}_2F_1\left[\begin{matrix}-n,f+1\\ f\end{matrix};x\right]\mathrm{d}x \\ &= \frac{1}{\mathrm{B}(b-a-1,a+1)}\int_0^1 x^{b-a-2}(1-x)^{a+n-1}{}_2F_1\left[\begin{matrix}-1,f+n\\ f\end{matrix};x\right]\mathrm{d}x \\ &= \frac{1}{\mathrm{B}(b-a-1,a+1)}\int_0^1 x^{b-a-2}(1-x)^{a+n-1}\left\{1-\frac{f+n}{f}x\right\}\mathrm{d}x\,. \end{aligned}$$

Separating the last integral into two parts and calculating the values of both, we get

$$S = \frac{1}{\mathrm{B}(b-a-1,a+1)}\left\{\mathrm{B}(b-a-1,a+n)-\left(1+\frac{n}{f}\right)\mathrm{B}(b-a,a+n)\right\}.$$

Now, straightforward calculations lead the asserted result (8). □

3. Applications

In this section, as applications of our summation formula (8), we derive the following two general formulæ such that generalize Exton's results (1), (2).

Theorem 2. *Let f be the same as before. Then*

$$(1-x)^{-h}{}_3F_2\left[\begin{matrix}h,b-a-1,f+1\\ b,f\end{matrix};-\frac{x}{1-x}\right] = {}_3F_2\left[\begin{matrix}h,a,c+1\\ b,c\end{matrix};x\right] \tag{11}$$

and

$$e^x{}_2F_2\left[\begin{matrix}b-a-1,f+1\\ b,f\end{matrix};-x\right] = {}_2F_2\left[\begin{matrix}a,c+1\\ b,c\end{matrix};x\right]. \tag{12}$$

Proof. Let us denote T the left-hand side of (11). First, expressing ${}_3F_2$ as a series, we conclude

$$T=\sum_{n=0}^{\infty}\frac{(-1)^n(h)_n(b-a-1)_n(f+1)_n}{(b)_n(f)_n\,n!}\,x^n(1-x)^{-h-n}.$$

Using the binomial expansion of $(1-x)^{-h}$ and applying the identity $(h)_n(h+n)_n=(h)_{n+m}$, we arrive at

$$T=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n(h)_{n+m}(b-a-1)_n(f+1)_n}{(b)_n(f)_n\,n!m!}\,x^{n+m}.$$

The consequence of the Bailey's transform (7) of the double-indexed summand will be

$$T=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n\,(h)_m(b-a-1)_n(f+1)_n}{(b)_n(f)_n\,n!(m-n)!}\,x^m.$$

Now, because of

$$(m-n)!=(-1)^n\frac{m!}{(-m)_n},\tag{13}$$

we get

$$T=\sum_{m=0}^{\infty}\frac{(h)_m}{m!}\,x^m\sum_{n=0}^{m}\frac{(b-a-1)_n(f+1)_n(-m)_n}{(b)_n(f)_n\,n!};$$

summing up the inner-most series, we have

$$T=\sum_{m=0}^{\infty}\frac{(h)_m}{m!}\,{}_3F_2\left[\begin{matrix}-m,b-a-1,f+1\\ b,f\end{matrix};1\right]x^m.$$

By the Theorem 1, we deduce

$$T=\sum_{m=0}^{\infty}\frac{(h)_m}{m!}\,x^m\,\frac{(a)_m(c+1)_m}{(b)_m(c)_m}={}_3F_2\left[\begin{matrix}h,a,c+1\\ b,c\end{matrix};x\right]$$

such that is the right-hand side expression in (11).

Denote U the expression on the left in the relation (12) and express both the functions involved, in series. After some little simplification we get

$$U=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m(b-a-1)_m(f+1)_m}{(b)_m(f)_m m!n!}\,x^{m+n}.$$

Making use of the Bailey's transform and the indetity (13), we have

$$U=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m(b-a-1)_m(f+1)_m}{(b)_m(f)_m\,m!(n-m)!}\,x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{n} \frac{(-n)_m (b-a-1)_m (f+1)_m}{(b)_m (f)_m\, m!}.$$

Now, it remains to summing up the inner-most series getting

$$U = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, {}_3F_2\left[\begin{matrix} -n, b-a-1, f+1 \\ b, f \end{matrix}; 1\right]$$

such that, by means of Theorem 1, clearly becomes

$$U = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{(a)_n (c+1)_n}{(b)_n (c)_n} = {}_2F_2\left[\begin{matrix} a, c+1 \\ b, c \end{matrix}; x\right]$$

that is, Theorem 2 is proved. □

Another proof of (11). Having in the mind the earlier convention that T denotes the left side expression of (11), we conclude

$$\begin{aligned} T &= (1-x)^{-h}\, {}_3F_2\left[\begin{matrix} h, b-a-1, f+1 \\ b, f \end{matrix}; -\frac{x}{1-x}\right] \\ &= (1-x)^{-h} \sum_{n=0}^{\infty} \frac{(h)_n\, (b-a-1)_n}{(b)_n\, n!} \frac{(f+1)_n}{(f)_n} \left(-\frac{x}{1-x}\right)^n \\ &= (1-x)^{-h} \sum_{n=0}^{\infty} \frac{(h)_n\, (b-a-1)_n}{(b)_n\, n!} \left(1+\frac{n}{f}\right) \left(-\frac{x}{1-x}\right)^n. \end{aligned}$$

Separating the sum into two parts and summing the first series, we get

$$\begin{aligned} T &= (1-x)^{-h}\, {}_2F_1\left[\begin{matrix} h, b-a-1 \\ b \end{matrix}; -\frac{x}{1-x}\right] \\ &\quad + \frac{1}{f} (1-x)^{-h} \sum_{n=1}^{\infty} \frac{(h)_n (b-a-1)_n}{(b)_n (n-1)!} \left(-\frac{x}{1-x}\right)^n. \end{aligned}$$

Now, changing n into $n+1$ the remaining series T_2, say, becomes

$$\begin{aligned} T_2 &= \frac{1}{f} (1-x)^{-h} \sum_{n=0}^{\infty} \frac{(h)_{n+1} (b-a-1)_{n+1}}{(b)_{n+1}\, n!} \left(-\frac{x}{1-x}\right)^{n+1} \\ &= -\frac{h(b-a-1)x}{fb(1-x)^{h+1}} \sum_{n=0}^{\infty} \frac{(h+1)_n (b-a)_n}{(b+1)_n\, n!} \left(-\frac{x}{1-x}\right)^n, \end{aligned}$$

that is

$$T = (1-x)^{-h}\, {}_2F_1\left[\begin{matrix} h, b-a-1 \\ b \end{matrix}; -\frac{x}{1-x}\right]$$

$$-\frac{h(b-a-1)x}{fb(1-x)^{h+1}}\,{}_2F_1\left[\begin{matrix}h+1,b-a\\b+1\end{matrix};-\frac{x}{1-x}\right].$$

Using (3) and mentioning that $f=\frac{c(1+a-b)}{a-c}$, we deduce

$$\begin{aligned}T&={}_2F_1\left[\begin{matrix}h,a+1\\b\end{matrix};x\right]-\frac{h(b-a-1)x}{fb}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]\\&={}_2F_1\left[\begin{matrix}h,a+1\\b\end{matrix};x\right]+\frac{h(a-c)x}{bc}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]\\&=\left\{{}_2F_1\left[\begin{matrix}h,a+1\\b\end{matrix};x\right]-\frac{hx}{b}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]\right\}\\&\quad+\frac{hax}{bc}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right].\end{aligned}$$

But it can be easily seen that

$$\left\{{}_2F_1\left[\begin{matrix}h,a+1\\b\end{matrix};x\right]-\frac{hx}{b}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]\right\}={}_2F_1\left[\begin{matrix}h,a\\b\end{matrix};x\right].$$

Hence,

$$T={}_2F_1\left[\begin{matrix}h,a\\b\end{matrix};x\right]+\frac{hax}{bc}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]={}_3F_2\left[\begin{matrix}h,a,c+1\\b,c\end{matrix};x\right]$$

such that coincides with the right hand expression in (11). □

Remark 1. Since the result (12) is a confluent limiting case of the (11), so it can be easily deduced by (11).

Remark 2. In (11), if we take $c=\frac{1}{2}a$, we get

$$(1-x)^{-h}\,{}_3F_2\left[\begin{matrix}h,b-a-1,2+a-b\\b,1+a-b\end{matrix};-\frac{x}{1-x}\right]={}_3F_2\left[\begin{matrix}h,a,1+\frac{1}{2}a\\b,\frac{1}{2}a\end{matrix};x\right]$$

which reduces to (1) by replacing x by $-x/(1-x)$.

Remark 3. In (12), taking $c=\frac{1}{2}a$, we get (2).

4. Conclusion

We mention below a result similar to (8) given by Miller [5]

$${}_3F_2\left[\begin{matrix}-n,c+1\\b,c\end{matrix};1\right]=\frac{(b-a-1)_n(f+1)_n}{(b)_n(f)_n},\tag{14}$$

f being the same as before.

For $c = \frac{1}{2}a$, this reduces to

$$ {}_3F_2\left[\begin{matrix} -n, a, 1+\frac{1}{2}a \\ b, \frac{1}{2}a \end{matrix}; 1\right] = \frac{(b-a-1)_n(2+a-b)_n}{(b)_n(1+a-b)_n}, $$

which is a known result in the literature [7].

The result quoted in Remark 2 belongs to Paris [6] who obtained this relation by a different method. Actually, the results mentioned in Remarks 2 and 3 have been also obtained very recently by Rathie and Paris [9] by utilising the summation formula (14) due to Miller [5].

We conclude the paper by remarking that the result (8) and Miller's result (14) are special cases of the following general result given already in the literature [7]

$$ {}_3F_2\left[\begin{matrix} a, b, d+1 \\ c, d \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left\{c-a-b-1+\frac{ab}{d}\right\} $$

such that can be easily transformed into Beta function expression, reads as follows:

$$ {}_3F_2\left[\begin{matrix} a, b, d+1 \\ c, d \end{matrix}; 1\right] = \frac{\mathrm{B}(c, c-a-b-1)}{\mathrm{B}(c-a, c-b-1)} \left\{1+\frac{a(b-d)}{d(c-b-1)}\right\}. $$

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