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A CORRECTION TO APPROXIMATION OF GENERALIZED HOMOMORPHISMS IN QUASI–BANACH ALGEBRAS

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Abstract. Eshaghi et. al [Approximation of generalized homomorphisms in quasi–Banach algebras, An. St. Univ. Ovidius Constanta, 17(2), (2009), 203–214] defined the notion of generalized homomorphisms in quasi–Banach algebras. They investigated generalized homomorphisms from quasi–Banach algebras to p–Banach algebras and proved the generalized Hyers–Ulam–Rassias stability. In this paper, we show that their results only hold for Banach algebras and then we correct and prove the results for p–Banach algebras.

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1. Introduction and preliminaries

We know that a $\mathbb{C}$-linear mapping $f : A \to B$ is called a homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in A$ so that every homomorphism is a generalized homomorphism, but the converse is false, in general.

Eshaghi et. al [1] investigated generalized homomorphisms from quasi–Banach algebras to p–Banach algebras and proved the generalized Hyers–Ulam–Rassias stability. In this paper, we verify that the results presented in [1] hold for Banach algebras. Then, we correct their results and prove the results for p–Banach algebras. We remark that the presenting results in this paper hold for p–Banach algebras, where $0 < p \leq 1$, in general. The stability problems of several functional equations have been extensively investigated by a number of authors in p–Banach algebras and there are many interesting results concerning this problem (see [2,3,5] and references therein).

Let X be a real linear space. A quasi–norm is a real–valued function on X satisfying the following conditions:

(i) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$,
(ii) $||\lambda x|| = |\lambda| ||x||$ for all $\lambda \in \mathbb{R}$ and all $x \in X$,
(iii) there is a constant $K \geq 1$ such that $||x + y|| \leq K(||x|| + ||y||)$ for all $x, y \in X$.

The pair $(X, ||\cdot||)$ is called a quasi–normed space if $||\cdot||$ is a quasi–norm on X. The smallest possible K is called the modulus of concavity of $||\cdot||$. A quasi–Banach space is a complete quasi–normed space. Indeed, by a quasi–Banach space we mean a quasi–normed space in which every $||\cdot||$–Cauchy sequence in X converges. This class includes Banach spaces and the most significant class of quasi–Banach spaces which are not Banach spaces. A quasi–norm $||\cdot||$ is called a p–norm $(0 < p \le 1)$ if

$$||x+y||^p \le ||x||^p + ||y||^p \tag{1.1}$$

for all $x, y \in X$. In this case, a quasi–Banach space is called a p–Banach space.

Let $(A, ||\cdot||)$ be a quasi–normed space. The quasi–normed space $(A, ||\cdot||)$ is called a quasi–normed algebra if A is an algebra and there is a constant $K > 0$ such that $||xy|| \le K||x||||y||$ for all $x, y \in A$. A quasi–Banach algebra is a complete quasi–normed algebra. If the quasi–norm $||\cdot||$ is a p–norm, then the quasi–Banach algebra is called a p–Banach algebra. Eshaghi et. al [1] defined the notion of generalized homomorphisms in quasi–Banach algebra as follows:

Definition 1. Let A be a quasi–Banach algebra with quasi–norm $||\cdot||_A$ and let B be a p-Banach algebra with p–norm $||\cdot||_B$. A $\mathbb{C}$-linear mapping $f : A \to B$ is called a generalized homomorphism if there exists a homomorphism $h : A \to B$ such that $f(xy) = f(x)h(y)$ for all $x, y \in A$.

Then, they investigated generalized homomorphisms from quasi–Banach algebras to p–Banach algebras associated with the following functional equation

$$rf(\frac{x+y}{r}) = f(x) + f(y)$$

and proved the generalized Hyers–Ulam–Rassias stability and superstability of generalized homomorphisms in quasi–Banach algebras. In this paper, we prove that their results only hold for Banach algebras and then we correct their results and confirm the results for p–Banach algebras.

2. Main problems

Following [1] throughout this paper, assume that A is a quasi–Banach algebra with quasi–norm $||\cdot||_A$ and that B is a p–Banach algebra with p–norm $||\cdot||_B$. In addition, we assume r to be a constant positive integer.

We will use the following lemma in this section.

Lemma 1 ([4]). *Let X and Y be linear spaces and let $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and $\mu \in \mathbb{T}^1 := \{z \in \mathbb{C} : |z| = 1\}$. Then the mapping f is $\mathbb{C}$-linear.*

The following two theorems proved in [1, Theorems 2.2 and 2.5]. In these theorems, the authors want to prove the generalized Hyers–Ulam–Rassias stability of generalized homomorphisms from quasi–Banach algebras to p–Banach algebras.

Theorem 1. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A^4 \to \mathbb{R}^+$ such that*

$$||rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)||_B \leq \varphi(a,b,c,d), \tag{2.1}$$

$$||g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)||_B \leq \varphi(a,b,c,d), \tag{2.2}$$

and

$$\tilde{\varphi}(a,b,c,d) := \sum_{i=0}^{\infty} \frac{\varphi(2^i a, 2^i b, 2^i c, 2^i d)}{2^i} < \infty$$

for all $a,b,c,d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \to B$ such that

$$||f(a) - h(a)||_B \leq \frac{1}{2}\tilde{\varphi}(a,a,0,0)$$

for all $a \in A$.

Theorem 2. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A^4 \to \mathbb{R}^+$ satisfying the inequalities* (2.1), (2.2) *and*

$$\tilde{\varphi}(a,b,c,d) := \sum_{i=1}^{\infty} 2^i \varphi(\frac{a}{2^i}, \frac{b}{2^i}, \frac{c}{2^i}, \frac{d}{2^i}) < \infty$$

for all $a,b,c,d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \to B$ such that

$$||f(a) - h(a)||_B \leq \frac{1}{2}\tilde{\varphi}(a,a,0,0)$$

for all $a \in A$.

In the following theorems we correct the results and prove the generalized Hyers–Ulam–Rassias stability of generalized homomorphisms from quasi–Banach algebras to p–Banach algebras.

Theorem 3. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A^4 \to \mathbb{R}^+$ such that*

$$||rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)||_B \leq \varphi(a,b,c,d), \tag{2.3}$$

$$||g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)||_B \leq \varphi(a,b,c,d), \tag{2.4}$$

and

$$\tilde{\varphi}(a,b,c,d) := \Big(\sum_{i=0}^{\infty} \frac{\varphi(2^i a, 2^i b, 2^i c, 2^i d)^p}{2^{ip}}\Big)^{\frac{1}{p}} < \infty \tag{2.5}$$

for all $a,b,c,d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \to B$ such that

$$||f(a) - h(a)||_B \leq \frac{1}{2}\tilde{\varphi}(a,a,0,0) \tag{2.6}$$

for all $a \in A$.

Proof. Setting $a = b$, $c = d = 0$ and $r = \mu = 1$ in (2.3) and dividing both sides of the resulting inequality by 2, we obtain

$$||\frac{f(2a)}{2} - f(a)||_B \leq \frac{\varphi(a,a,0,0)}{2} \tag{2.7}$$

for all $a \in A$. Then, we have

$$||\frac{f(2a)}{2} - f(a)||_B^p \leq \frac{\varphi(a,a,0,0)^p}{2^p} \tag{2.8}$$

for all $a \in A$. Replacing a in (2.7) by $2a$ and dividing both sides of the resulting inequality by 2, we get

$$||\frac{f(2^2a)}{2^2} - \frac{f(2a)}{2}||_B \leq \frac{\varphi(2a,2a,0,0)}{2^2} \tag{2.9}$$

for all $a \in A$. Then, we have

$$||\frac{f(2^2a)}{2^2} - \frac{f(2a)}{2}||_B^p \leq \frac{\varphi(2a,2a,0,0)^p}{2^{2p}} \tag{2.10}$$

for all $a \in A$. Applying (2.8), (2.10), and (1.1) we detect that

$$||\frac{f(2^2a)}{2^2} - f(a)||_B^p \leq \frac{\varphi(a,a,0,0)^p}{2^p} + \frac{\varphi(2a,2a,0,0)^p}{2^{2p}} \tag{2.11}$$

for all $a \in A$. By induction on n we conclude that

$$||\frac{f(2^n a)}{2^n} - f(a)||_B^p \leq \frac{1}{2^p}\sum_{i=0}^{n-1}\frac{\varphi(2^i a, 2^i a, 0,0)^p}{2^{ip}} \tag{2.12}$$

for all $a \in A$ and all non-negative integers n. Consequently,

$$||\frac{f(2^{n+m}a)}{2^{n+m}} - \frac{f(2^m a)}{2^m}||_B \leq \frac{1}{2}\Big(\sum_{i=m}^{n+m-1}\frac{\varphi(2^i a, 2^i a, 0,0)^p}{2^{ip}}\Big)^{\frac{1}{p}} \tag{2.13}$$

for all non-negative integers n and m with $n \geq m$ and all $a \in A$. It follows from (2.5) and (2.13) that the sequence $\{\frac{f(2^n a)}{2^n}\}$ is Cauchy in B for all $a \in A$. Since B is a p–Banach algebras, this sequence is convergent in B for all $a \in A$. Define the mapping

$$h(a) := \lim_{n\to\infty}\frac{f(2^n a)}{2^n}. \tag{2.14}$$

Setting $c = d = 0, r = 1$ and replacing a,b by $2^n a, 2^n b$, respectively, in (2.3) and dividing both sides of (2.3) by 2^n and taking the limit as $n \to \infty$ we deduce

$$h(\mu a + \mu b) = \mu h(a) + \mu h(b) \tag{2.15}$$

for all $a,b \in A$ and $\mu \in \mathbb{T}^1$. So the mapping h is $\mathbb{C}$-linear by Lemma 1. Note that inequality (2.6) follows from (2.12) and (2.14). To show that h is unique, let k be another $\mathbb{C}$-linear mapping satisfying (2.6). From (2.6) we conclude that

$$\begin{aligned}||h(a)-k(a)||_B^p &= \frac{1}{2^{np}}||h(2^n a)-k(2^n a)||_B^p \\ &\le \frac{1}{2^{np}}(||h(2^n a)-f(2^n a)||_B^p + ||f(2^n a)-k(2^n a)||_B^p \\ &\le \frac{1}{2^{np}}\frac{2}{2^p}\tilde{\varphi}(2^n a, 2^n a, 0, 0)^p \\ &= \frac{2}{2^p}\sum_{i=n}^{\infty}\frac{\varphi(2^i a, 2^i a, 0, 0)^p}{2^{ip}}\end{aligned}$$

for all $a \in A$. The right hand side tends to zero as $n \to \infty$. The rest of the proof is similar to that of [1, Theorem 2.2] and we omit it. □

Theorem 4. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A^4 \to \mathbb{R}^+$ satisfying the inequality (2.3) and (2.4) and*

$$\tilde{\varphi}(a,b,c,d) := \Big(\sum_{i=1}^{\infty} 2^{ip}\varphi(\frac{a}{2^i},\frac{b}{2^i},\frac{c}{2^i},\frac{d}{2^i})^p\Big)^{\frac{1}{p}} < \infty \tag{2.16}$$

for all $a,b,c,d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \to B$ such that

$$||f(a)-h(a)||_B \le \frac{1}{2}\tilde{\varphi}(a,a,0,0) \tag{2.17}$$

for all $a \in A$.

Proof. Setting $a = b$, $c = d = 0$ and $r = \mu = 1$ in (2.3), we obtain

$$||f(2a)-2f(a)||_B \le \varphi(a,a,0,0) \tag{2.18}$$

for all $a \in A$. Replacing a in (2.18) by $\frac{a}{2}$, we get

$$||f(a)-2f(\frac{a}{2})||_B \le \varphi(\frac{a}{2},\frac{a}{2},0,0) \tag{2.19}$$

for all $a \in A$. Replacing a in (2.19) by $\frac{a}{2}$ and multiplying both sides of the resulting inequality by 2, we detect

$$||2f(\frac{a}{2})-2^2 f(\frac{a}{2^2})||_B \le 2\varphi(\frac{a}{2^2},\frac{a}{2^2},0,0) \tag{2.20}$$

for all $a \in A$. Using (2.19), (2.20), and (1.1) we conclude that

$$||f(a) - 2^2 f(\frac{a}{2^2})||_B^p \leq \varphi(\frac{a}{2}, \frac{a}{2}, 0, 0)^p + 2^p \varphi(\frac{a}{2^2}, \frac{a}{2^2}, 0, 0) \tag{2.21}$$

for all $a \in A$. By induction on n we deduce that

$$||f(a) - 2^n f(\frac{a}{2^n})||_B^p \leq \frac{1}{2^p} \sum_{i=1}^{n} 2^{ip} \varphi(\frac{a}{2^i}, \frac{a}{2^i}, 0, 0)^p \tag{2.22}$$

for all $a \in A$ and all non-negative integers n. Hence,

$$||2^m f(\frac{a}{2^m}) - 2^{n+m} f(\frac{a}{2^{n+m}})||_B^p \leq \frac{1}{2^p} \sum_{i=m+1}^{m+n} 2^{ip} \varphi(\frac{a}{2^{i+m}}, \frac{a}{2^{i+m}}, 0, 0)^p \tag{2.23}$$

for all non-negative integers n and m with $n \geq m$ and all $a \in A$. It follows from (2.16) and (2.23) that the sequence $\{2^n f(\frac{a}{2^n})\}$ is Cauchy in B for all $a \in A$ so that this sequence is convergent in B. Define the mapping

$$h(a) := \lim_{n\to\infty} 2^n f(\frac{a}{2^n}). \tag{2.24}$$

The rest of the proof is similar to that of Theorem 3 and we omit it. $\square$

Corollary 1. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist constants $\epsilon > 0, \alpha \neq 1$ and a mapping $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ such that*

$$||rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)||_B$$
$$\leq \epsilon(||a||_A^\alpha + ||b||_A^\alpha + ||c||_A^\alpha + ||d||_A^\alpha),$$

$$||g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)||_B \leq \epsilon(||a||_A^\alpha + ||b||_A^\alpha + ||c||_A^\alpha + ||d||_A^\alpha)$$

for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \to B$ such that

$$||f(a) - h(a)||_B \leq \frac{\epsilon}{|1 - 2^{p(\alpha-1)}|^{\frac{1}{p}}} ||a||_A^\alpha. \tag{2.25}$$

for all $a \in A$.

Proof. Define $\varphi(a, b, c, d) := \epsilon(||a||_A^\alpha + ||b||_A^\alpha + ||c||_A^\alpha + ||d||_A^\alpha)$. If $0 \leq \alpha < 1$, then Theorem 3 entails that

$$\tilde{\varphi}(a, a, 0, 0) = \frac{2\epsilon}{(1 - 2^{p(\alpha-1)})^{\frac{1}{p}}} ||a||_A^\alpha.$$

Consequently,

$$||f(a) - h(a)||_B \leq \frac{\epsilon}{(1 - 2^{p(\alpha-1)})^{\frac{1}{p}}} ||a||_A^\alpha. \tag{2.26}$$

If $\alpha > 1$, then by applying Theorem 4 we find that

$$\tilde{\varphi}(a,a,0,0) = \frac{2\epsilon}{(2^{p(\alpha-1)}-1)^{\frac{1}{p}}}||a||_A^{\alpha}.$$

Hence,

$$||f(a)-h(a)||_B \leq \frac{\epsilon}{(2^{p(\alpha-1)}-1)^{\frac{1}{p}}}||a||_A^{\alpha}. \tag{2.27}$$

From inequalities (2.26) and (2.27) we conclude inequality (2.25). □

3. Conclusions

We conclude that

(i) in Theorems 3, 4, if we take $p = 1$, then we obtain [1, Theorems 2.2, 2.5], respectively,

(ii) in Corollary 1, if we take $p = 1$, then we deduce [1, Corollary 2.3],

(iii) in Corollary 1, if we take $p = 1$, $\epsilon = \frac{\delta}{2}$, and $\alpha = 0$, then we recover [1, Corollary 2.4].

Indeed, the results presented in [1] hold for 1–Banach algebras. We know that 1–Banach algebras exactly coincide with Banach algebras and so the results of [1] only hold for Banach algebras. Our presented results in this paper hold for p–Banach algebras where $0 < p \leq 1$.

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