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Laws and identities for some upper triangular matrices

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LAWS AND IDENTITIES FOR SOME UPPER TRIANGULAR MATRICES

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Abstract. J. C. Robson has investigated the ideal I_n of all polynomials in the free associative algebra $R\langle x \rangle$ over a non-commutative ring R generated by x and the n^2 entries of an $n \times n$ matrix $\alpha = (a_{ij})$, which are satisfied by α . He found four cubics generating the ideal for $n = 2$ and proved its finite generation for any n . Ts. Rashkova has considered the ideal I_2 for matrix algebras with involution over a noncommutative ring and over a field of characteristic zero. In the paper the ideal I_3 is described for some special upper triangular matrices over a field of characteristic 0. The T -ideal $T(U_2(G))$ is investigated as well for G denoting the infinite dimensional Grassmann algebra.

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1. INTRODUCTION

The Cayley–Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $M_n(K)$ over a field K of characteristic zero.

The structure theory of semisimple rings and quantum matrices for example show the importance of matrices over non-commutative rings in the theory of PI-algebras and other branches of algebra as well.

J. C. Robson investigated in [12–14] the ideal I_n of all polynomials (including nonmonics) in the free associative algebra $R\langle x \rangle$ over a non-commutative ring R generated by x and the n^2 entries of an $n \times n$ matrix $\alpha = (a_{ij})$, which are satisfied by α . Those polynomials we call the laws over R of a non-commutative $n \times n$ matrix α . These are not polynomial identities since the entries of α are allowed as coefficients in the laws and they vary with the choice of α .

Robson showed that I_n is an insertive ideal (meaning that its homogeneous elements are closed under inner multiplication by a constant a_{ij} in some fixed position) and as such is finitely generated (see [13] and [12, Theorem 2.3]).

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The minimal degree of polynomials in I_n remains unknown. However, for the case $n = 2$, Robson [12, Proposition 3.2] found four polynomials of degree 3 (least possible) in I_2 and Pearson showed in [10, Corollary] that these four Robson cubics do indeed generate I_2 as an insertive ideal.

2. RESULTS

The first study of the non-commutative case for $n = 3$ was done in [9]. The results there provide further evidence of the tantalizing complexity of even these small matrices. Each of the four found laws of degree 7 has 1156 terms. Thus, special cases of 3×3 matrices even over a field are of interest.

In [11] we study the ideal I_3 for some special 3×3 upper triangular matrices considered in [7]. These algebras could be endowed with involution $*$ and their $*$ -codimensions have important properties.

Here we give a complete answer for the existing laws in these algebras and investigate the T -ideal $T(U_2(G))$ of the 2×2 upper triangular matrices over G , where G stands for the infinite dimensional Grassmann algebra.

2.1. Laws for upper triangular matrices over a field

2.1.1. Special upper triangular matrices

In [7] D. La Mattina and P. Misso study some associative algebras with involution generating $*$ -varieties of algebras with linear or linearly bounded sequences of $*$ -codimensions. Questions concerning laws for them were discussed in [11]. Here we give the complete answers.

Let K be a field of characteristic zero and

$$M_2(K) = \left\{ x = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in K \right\}.$$

All coefficients in the polynomials below mean the corresponding scalar matrix, i. e., $a = aE$ for example.

Theorem 1. *The only Robson cubic for a matrix from $M_2(K)$ in the general case is $r(x) = (x - a)^3$. For $b = 0$, $c \neq 0$ a law of minimal degree is $(x - a)^2$ and for $b = c = 0$ it is $x - a$.*

Proof. For a matrix x from the considered algebra we prove directly that $(x - a)^3 = 0$. Due to [8, Lemma 2, p.432] considering the subring T of the diagonal elements of a generic upper triangular matrix A and $w(x) \in T\langle x \rangle$, then $w(A) = 0$ iff $w(x) \in \langle x - a_{11} \rangle \langle x - a_{22} \rangle \cdots \langle x - a_{nn} \rangle$, where $\langle f \rangle$ is the ideal generated by f . In the cited Lemma T is a subring of a non-commutative ring. In Theorem 1 the elements of the matrix are elements of a field and then $T\langle x \rangle = K\langle x \rangle$ and the possible

laws of degree less or equal to 3 could be given with the linear combination

$$A = \alpha_1 + \alpha_2 x + \alpha_3(x-a) + \alpha_4 x^2 + \alpha_5(x-a)^2 + \alpha_6 x(x-a).$$

Equating to zero the corresponding entries we get the system

$$\begin{aligned}\alpha_1 + a\alpha_2 + a^2\alpha_4 &= 0, \\ b\alpha_2 + b\alpha_3 + 2ab\alpha_4 + ab\alpha_6 &= 0, \\ c\alpha_2 + c\alpha_3 + (2ac + b^2)\alpha_4 + b^2\alpha_5 + (ac + b^2)\alpha_6 &= 0.\end{aligned}$$

For $b \neq 0$ the system has a solution

$$\begin{aligned}\alpha_5 &= -\alpha_4 - \alpha_6, \\ \alpha_1 &= a\alpha_3 + a^2\alpha_4 + a^2\alpha_6, \\ \alpha_2 &= -\alpha_3 - 2a\alpha_4 - a\alpha_6.\end{aligned}$$

In this case A , is identically equal to zero. Considering $b = 0$, we get the rest of the statement. $\square$

Let us put

$$M_3(K) = \left\{ x = \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in K \right\}.$$

Theorem 2. *The only Robson cubic for a matrix from $M_3(K)$ in the general case is $r(x) = (x-a)^2x$. For the three cases $b \neq 0, c = d = 0$; $d \neq 0, b = c = 0$, and $b = c = d = 0$ a law of minimal degree is $(x-a)x$.*

Proof. Directly we calculate that for a matrix x from the considered algebra we have $(x-a)^2x = 0$. Then, due to Lemma 2 from [8, p. 432], considerations analogous to the proof of Theorem 1 lead us to the linear combination

$$A = \alpha_1 + \alpha_2 x + \alpha_3(x-a) + \alpha_4 x^2 + \alpha_5(x-a)^2 + \alpha_6 x(x-a)$$

of all possible laws of degree ≤ 3 . Equating to zero the corresponding entries, we obtain the system

$$\begin{aligned}\alpha_1 + a\alpha_2 + a^2\alpha_4 &= 0, \\ b\alpha_2 + b\alpha_3 + ab\alpha_4 - ab\alpha_5 &= 0, \\ c\alpha_2 + c\alpha_3 + (2ac + bd)\alpha_4 + bd\alpha_5 + (ac + bd)\alpha_6 &= 0, \\ \alpha_1 - a\alpha_3 + a^2\alpha_5 &= 0, \\ d\alpha_2 + d\alpha_3 + ad\alpha_4 - ad\alpha_5 &= 0.\end{aligned}$$

For $b \neq 0, d \neq 0$ the system has a solution

$$\begin{aligned}\alpha_6 &= -\alpha_4 - \alpha_5, \\ \alpha_1 &= a\alpha_3 - a^2\alpha_5, \\ \alpha_2 &= -\alpha_3 - a\alpha_4 + a\alpha_5.\end{aligned}$$

In this case A is identically equal to zero.

The three cases $b \neq 0, c = d = 0$; $d \neq 0, b = c = 0$, and $b = c = d = 0$ give the law $(x - a)x$. $\square$

Let us put

$$M_4(K) = \left\{ x = \begin{pmatrix} 0 & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix} : a, b, c, d \in K \right\}.$$

Theorem 3. *The only Robson cubic for a matrix from $M_4(K)$ in the general case is $r(x) = (x - a)x^2$. For $c = d = 0$ or $b = c = 0$ a law of minimal degree is $(x - a)$.*

Proof. It follows the above pattern of proof. Directly we get $(x - a)x^2 = 0$. The corresponding system is

$$\begin{aligned}\alpha_1 + a\alpha_2 + a^2\alpha_4 &= 0, \\ b\alpha_2 + b\alpha_3 + ab\alpha_4 - ab\alpha_5 &= 0, \\ c\alpha_2 + c\alpha_3 + bd\alpha_4 + (bd - 2ac)\alpha_5 + (bd - ac)\alpha_6 &= 0, \\ \alpha_1 - a\alpha_3 + a^2\alpha_5 &= 0, \\ d\alpha_2 + d\alpha_3 + ad\alpha_4 - ad\alpha_5 &= 0.\end{aligned}$$

For $b \neq 0, d \neq 0$ the system has the same solution as in the previous theorem and A is identically equal to zero.

The two cases $c = d = 0$ and $b = c = 0$ give the law $(x - a)x$. $\square$

2.1.2. The general upper triangular case

Now we consider the general case, namely, the algebra $U_3(K, *)$ of the upper triangular matrices of order 3 with the involution $*$ reflection along the second diagonal. We could find the analogues of the Robson cubics for the $*$ -symmetric matrices and for the $*$ -skew-symmetric matrices as well.

Theorem 4. *The only law of minimal degree for a symmetric matrix $x \in U_3^+(K, *)$, where*

$$U_3^+(K, *) = \left\{ x = \begin{pmatrix} a & b & d \\ 0 & c & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in K \right\},$$

is $(x - a)^2(x - c)$. For $b = d = 0$ it is $(x - a)(x - c)$.

Proof. Directly we calculate that $(x-a)^2(x-c) = 0$. Then as explained in the above proofs we have to form the linear combination

$$A = \alpha_1 + \alpha_2(x-a) + \alpha_3(x-a)^2 + \alpha_4(x-c) + \alpha_5(x-c)^2 + \alpha_6(x-a)(x-c)$$

of all possible laws of degree ≤ 3 . Equating to zero the corresponding entries we get the system

$$\begin{aligned}\alpha_1 + (a-c)\alpha_4 + (a-c)^2\alpha_5 &= 0, \\ b\alpha_2 + b(c-a)\alpha_3 + b\alpha_4 + b(a-c)\alpha_5 &= 0, \\ d\alpha_2 + b^2\alpha_3 + d\alpha_4 + (2d(a-c) + b^2)\alpha_5 + (d(a-c) + b^2)\alpha_6 &= 0, \\ \alpha_1 + (c-a)\alpha_2 + (c-a)^2\alpha_3 &= 0, \\ b\alpha_2 + b(c-a)\alpha_3 + b\alpha_4 + b(a-c)\alpha_5 &= 0, \\ \alpha_1 + (a-c)\alpha_4 + (a-c)^2\alpha_5 &= 0.\end{aligned}$$

For $b \neq 0, d \neq 0$ the system has a solution

$$\begin{aligned}\alpha_6 &= -\alpha_3 - \alpha_5, \\ \alpha_1 &= -(a-c)\alpha_4 - (a-c)^2\alpha_5, \\ \alpha_2 &= -(-a+c)\alpha_3 - \alpha_4 - (a-c)\alpha_5.\end{aligned}$$

In this case A , is identically equal to zero. For $b = d = 0$, we obtain the quadratic law $(x-a)(x-c)$. $\square$

We point that in the general case $(x-a)^2(x-c) = 0$ is in fact the Cayley–Hamilton theorem in a factor form, i. e., $x^3 - (2a+c)x^2 + a(a+2c)x - a^2c = 0$.

Theorem 5. *The only law of minimal degree for a skew-symmetric matrix $x \in U_3^-(K, *)$, where*

$$U_3^-(K, *) = \left\{ x = \begin{pmatrix} a & b & 0 \\ 0 & c & -b \\ 0 & 0 & -a \end{pmatrix} : a, b, c \in K \right\},$$

is $(x-a)(x-c)(x+a)$. If $a = b = 0$, then a law is $x(x-c)$.

Proof. The law $(x-a)(x-c)(x+a) = 0$ is checked directly. Then we form the linear combination

$$\begin{aligned}A &= \alpha_1 + \alpha_2(x-a) + \alpha_3(x-a)^2 + \alpha_4(x-c) \\ &\quad + \alpha_5(x-c)^2 + \alpha_6(x+a) + \alpha_7(x+a)^2 \\ &\quad + \alpha_8(x-a)(x-c) + \alpha_9(x-a)(x+a) + \alpha_{10}(x-c)(x+a)\end{aligned}$$

of all possible laws of degree ≤ 3 . Equating to zero the corresponding entries, we obtain the system

$$\begin{aligned}
&\alpha_1 + (a-c)\alpha_4 + (a-c)^2\alpha_5 + 2a\alpha_6 + 4a^2\alpha_7 + 2a(a-c)\alpha_{10} = 0, \\
&b\alpha_2 + b(c-a)\alpha_3 + b\alpha_4 + b(a-c)\alpha_5 + b\alpha_6 \\
&\quad + b(c+3a)\alpha_7 + b(c+a)\alpha_9 + 2ab\alpha_{10} = 0, \\
&-b^2\alpha_3 - b^2\alpha_5 - b^2\alpha_7 - b^2\alpha_8 - b^2\alpha_9 - b^2\alpha_{10} = 0, \\
&\alpha_1 + (c-a)\alpha_2 + (c-a)^2\alpha_3 + (c+a)\alpha_6 + (c+a)^2\alpha_7 + (c-a)(c+a)\alpha_9 = 0, \\
&-b\alpha_2 - b(c-3a)\alpha_3 - b\alpha_4 + b(a+c)\alpha_5 - b\alpha_6 \\
&\quad - b(c+a)\alpha_7 + 2ab\alpha_8 - b(c-a)\alpha_9 = 0, \\
&\alpha_1 - 2a\alpha_2 + 4a^2\alpha_3 - (a+c)\alpha_4 + (a+c)^2\alpha_5 + 2a(a+c)\alpha_8 = 0.
\end{aligned}$$

Its solution is

$$\begin{aligned}
\alpha_1 &= -2a(a-c)\alpha_{10} - (a-c)\alpha_4 - (a-c)^2\alpha_5 - 2a\alpha_6 - 4a^2\alpha_7, \\
\alpha_8 &= -\alpha_{10} - \alpha_3 - \alpha_5 - \alpha_7 - \alpha_9, \\
\alpha_2 &= -2a\alpha_{10} - (-a+c)\alpha_3 - \alpha_4 - (a-c)\alpha_5 - \alpha_6 - (3a+c)\alpha_7 - (a+c)\alpha_9.
\end{aligned}$$

In this case A , is identically equal to zero. The case $a = b = 0$ leads one directly to the validity of the law $x(x-c)$. $\square$

The law in the general case illustrates the Cayley–Hamilton theorem in a factor form, namely $x^3 - cx^2 - a^2x + a^2b = 0$. All the computations are made using the computer algebra system *Mathematica*.

2.2. Laws and identities for upper triangular matrices over the Grassmann algebra

2.2.1. Preliminaries

We consider the matrix algebra of the 2×2 upper triangular matrices $U_2(G)$ over the Grassmann algebra G .

We recall the definition of the infinite dimensional Grassmann algebra G , namely,

$$G = G(V) = K\langle v_1, v_2, \dots \mid v_i v_j + v_j v_i = 0, i, j = 1, 2, \dots \rangle.$$

The algebra G' (without 1) has a basis $v_{i_1} v_{i_2} \dots v_{i_k}$, where $1 \leq i_1 < i_2 < \dots < i_k$. The elements v_i are called generators of G' while the elements $v_{i_1} v_{i_2} \dots v_{i_k}$ for $1 \leq i_1 < i_2 < \dots < i_k$ are called basic monomials of G' . For $G = G' \cup 1$, a generator is 1 as well. The algebras G and G' are PI-equivalent (they satisfy one and the same identities). It is easy to be seen that $G' = J(G)$ for $J(G)$ being the Jacobson radical of the algebra.

The algebra G is in the mainstream of recent research in PI theory. Its importance is connected mainly with the structure theory for the T -ideals of identities of associative algebras developed by Kemer in [5]. Kemer proved [5, Theorem 1.2] that any T -prime T -ideal can be obtained as the T -ideal of identities of one of the following algebras: $M_n(K)$, $M_n(G)$ and $M_{n,u}(G)$, the latter being the algebra of $n \times n$ supermatrices over $G = G_0 \oplus G_1$ with G_0 blocks (with entries of even degree) of sizes $u \times u$ and $(n-u) \times (n-u)$ and with G_1 blocks (with entries of odd degree) of sizes $u \times (n-u)$ and $(n-u) \times u$.

Well known facts concerning the algebra G are the following:

Proposition 1 ([6, Corollary, p. 437]). *The T -ideal $T(G)$ is generated by the identity $[x_1, x_2, x_3] = 0$.*

Proposition 2 ([4, Exercise 5.3]). *For $G_k = G(V_k)$ over k -dimensional vector space V_k all identities follow from the identity $[x_1, x_2, x_3] = 0$ and the standard identity $S_{2p}(x_1, \dots, x_{2p}) = \sum_{\sigma \in \text{Sym}(2p)} (-1)^\sigma x_{\sigma(1)} \dots x_{\sigma(2p)} = 0$, where p is the minimal integer with $2p > k$.*

Remark 1. In the monograph [4, Exercise 5.3] one could see that the identity $[x_1, x_2] \dots [x_{2p+1}, x_{2p+2}] = 0$ on G_{2p+1} is equivalent to the standard identity of degree $2p+2$.

Remark 2. It could be seen [4, Exercise 5.8] that the T -ideal of the algebra $M_2(K)$ from Theorem 1 is generated by the identities $[x_1, x_2, x_3] = 0$ and $S_4(x_1, \dots, x_4) = 0$. Nevertheless, the algebra $M_2(K)$ is not isomorphic to the Grassmann algebra G_2 of the two-dimensional vector space.

For the rest of the paper we will use capital letters for the matrices with entries from the Grassmann algebra.

2.2.2. The T -ideal $T(U_2(G))$

Theorem 6. *The identity $[X_1, X_2, X_3][X_4, X_5, X_6] = 0$ holds on the algebra $U_2(G)$.*

Proof. As the considered polynomial is multilinear we could rely on [16, Remark 3.1] stating that it is an identity on $M_2(G)$ if and only if for every choice of the matrix units e_{a_i, b_i} and either $v_i^* = v_i$ or $v_i^* = 1$, the substitution $x_i \rightarrow e_{a_i, b_i} v_i^*$ in the polynomial gives zero.

We take the matrices $X_i = \begin{pmatrix} a_{1i} & b_{1i} \\ 0 & c_{1i} \end{pmatrix}$ for $i = 1, 2, 3$ belonging to $U_2(G)$ with entries being generators of G . It is easy to see that

$$[X_1, X_2, X_3] = \begin{pmatrix} [a_{11}, a_{12}, a_{13}] & * \\ 0 & [c_{11}, c_{12}, c_{13}] \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

The same form has the matrix $[X_4, X_5, X_6]$. As the only possibly non-zero entry of the considered matrices is the (1,2)-entry, the multiplication gives 0. $\square$

Remark 3. Theorem 6 could be reformulated as follows. Let $f(x, y, z) = [x, y, z]$. The 2×2 upper triangular matrices over the Grassmann algebra satisfy the identity $f^2 = 0$ while their entries satisfy the identity $f = 0$.

In [3, Theorem 3.1], Domokos gave a compact form of a theorem of Szigeti from [15], namely,

Proposition 3. *For any 2×2 matrix X over a K -algebra S satisfying the identity $[x_1, x_2, x_3] = 0$ we have that*

$$X^4 - 2X^3(\operatorname{tr} X) + X^2(2\operatorname{tr}^2 X - \operatorname{tr} X^2) + X\left(\frac{1}{2}\operatorname{tr} X \circ \operatorname{tr} X^2 - \operatorname{tr}^3 X\right) \\ + \frac{1}{4}\left(\operatorname{tr}^4 X + \operatorname{tr}^2 X^2 + \frac{1}{2}\operatorname{tr}^2 X \operatorname{tr} X^2 - \frac{5}{2}\operatorname{tr} X^2 \operatorname{tr}^2 X + 2[\operatorname{tr} X^3, \operatorname{tr} X]\right)E$$

and

$$X^4 - 2(\operatorname{tr} X)X^3 + (2\operatorname{tr}^2 X - \operatorname{tr} X^2)X^2 + \left(\frac{1}{2}\operatorname{tr} X \circ \operatorname{tr} X^2 - \operatorname{tr}^3 X\right)X \\ + \frac{1}{4}\left(\operatorname{tr}^4 X + \operatorname{tr}^2 X^2 - \frac{5}{2}\operatorname{tr}^2 X \operatorname{tr} X^2 + \frac{1}{2}\operatorname{tr} X^2 \operatorname{tr}^2 X - 2[\operatorname{tr} X^3, \operatorname{tr} X]\right)E$$

are equal to zero in $S^{2 \times 2}$.

In [15], Szigeti developed a new theory of determinants of $n \times n$ matrices over rings satisfying the polynomial identity of m -Lie nilpotency

$$[[[\cdots[x_1, x_2], x_3], \cdots], x_m], x_{m+1}] = 0.$$

As the Grassmann algebra is 2-Lie nilpotent the defined in [15] right m -adjoint of a matrix, the right m -determinant of a matrix rd_m and the right m -characteristic polynomial $p(x)$ of a matrix and their properties could be interpreted for the matrix algebra $U_2(G)$.

Proposition 4 ([15, Theorem 4.2]). *If $p(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_d x^d$ is the right m -characteristic polynomial of a $n \times n$ matrix $A \in M_n(R)$ over a m -Lie nilpotent ring R then the left substitution of A into $p(x)$ is zero: $(A)p = E\lambda_0 + A\lambda_1 + \cdots + A^d\lambda_d = 0$.*

Again in [15], Szigeti pointed out the identity of “algebraicity” for matrices over the Grassmann algebra.

Proposition 5 ([15, Theorem 5.1]). *The polynomial identity*

$$S_{2n^2}([Y^{2n^2}, Z], [Y^{2n^2-1}, Z], \dots, [Y^2, Z], [Y, Z]) = 0$$

holds on $M_n(G)$ for any two matrices Y and Z .

Now we give some laws and identities for the upper triangular matrices over the Grassmann algebra G .

Theorem 7. *Let the matrix $X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ belong to $U_2(J(G))$ for a, b, c being basic monomials of G' .*

(I) *The following laws are valid for X :*

$$\begin{aligned} (X - a)(X - c) &= 0, \\ X^3(\operatorname{tr} X) &= 0, \\ (\operatorname{tr} X)X^3 &= 0. \end{aligned}$$

(II) *Two identities hold for any matrices X and Y of the considered type, namely $X^2Y^2 = 0$ and $(X^2Y)^2 = 0$.*

Thus any matrix X is nilpotent of index 4. A matrix X with $\operatorname{tr} X = 0$ is nilpotent of index 3.

Proof. (I) Direct calculations give the validity of the three stated laws for a matrix X .

(II) Again applying direct calculations we get that the only non-zero entries in X^2 , Y^2 and X^2Y are the $(1, 2)$ entries and the corresponding multiplication gives zero. $\square$

We see that both Proposition 3 and Proposition 4 for $n = 2$ are compatible with Theorem 7 as for such a matrix $A \in U_2(G)$ we have $\operatorname{rdet}_2 A = 0$ and $p(x) = \operatorname{rdet}_2(A - Ex) = x^4 - 2\operatorname{rdet} Ax^3$. Thus $A^4 - 2(\operatorname{tr} A)A^3 = 0$.

Corollary 1. *An identity of degree 9 holds for any two matrices Y and Z from $U_2(G)$ with entries being basic monomials, namely, $S_3([Y^3, Z], [Y^2, Z], [Y, Z]) = 0$.*

Proof. Applying Proposition 5 and the index of nilpotency of the matrix Y . $\square$

Remark 4. If we consider the Grassmann algebra over a finite dimensional space, the corresponding identities have much smaller degrees.

Theorem 8. *The following two assertions hold:*

- (I) *On the algebra $U_2(J(G_2))$ we get the identity $XYZ = 0$ (respectively, $X^3 = 0$).*
- (II) *Any matrix $X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ from $U_2(J(G_2))$ satisfies the law $(\operatorname{tr} X)X^2 = 0$, respectively, $X^2(\operatorname{tr} X) = 0$.*

Proof. In the considered algebra the square of every element is zero. The identity and the law are proved directly. $\square$

Multiplying two linear combinations of the basic elements e_1 , e_2 and e_1e_2 we get only αe_1e_2 . Its product with any other linear combination of e_1 , e_2 and e_1e_2 gives zero.

Analogous considerations are valid for the linear combinations of the basic elements of any finite dimensional Grassmann algebra (over a n -dimensional vector

space). The multiplication of n linear combinations will result in $\alpha e_1 e_2 \dots e_n$ and the result of the next multiplication will be zero. Thus, we come to the following

Corollary 2. *The matrix algebra $U_k(J(G_n))$ (respectively, $M_k(J(G_n))$) is nilpotent of class $\leq n + 1$.*

Corollary 3. *The polynomial identity $S_{n-1}([Y^{n-1}, Z], [Y^{n-2}, Z], \dots, [Y, Z]) = 0$ holds on $M_k(J(G_n))$ for $n \geq 3$ and $k \geq 2$.*

As the algebra $U_2(G)$ is a subalgebra of $M_2(G)$ we turn to the Hall identity, the four degree standard identity for $M_2(K)$ and the product commutator identity for $U_2(K)$ considering the Grassmann algebra G instead of the field K . We get

Theorem 9. *The polynomials $[[x_1, x_2]^2, x_1]$, $S_4(x_1, x_2, x_3, x_4)$, and $[x_1, x_2][x_3, x_4]$ are not identities for the algebra $U_2(G)$.*

Proof. A counter example for the validity of the first and the third identity gives the matrices

$$X_1 = \begin{pmatrix} e_1 & e_2 \\ 0 & e_3 \end{pmatrix}$$

and

$$X_2 = \begin{pmatrix} e_4 e_5 + e_6 & 1 \\ 0 & e_1 + e_3 \end{pmatrix}.$$

We get that the $(1, 2)$ -entry of $[x_1, x_2]^2$ is nonzero, namely,

$$2e_1 e_3 e_6 + 2e_1 e_2 e_4 e_5 e_6 - 2e_1 e_2 e_3 e_4 e_5 + 4e_1 e_2 e_3 e_6.$$

The $(1, 2)$ -entry of $[[x_1, x_2]^2, x_1]$ is $-2e_1 e_2 e_3 e_4 e_5 e_6$. For the second statement, we rely on the general case considered later. $\square$

In [1, 2], a connection is given between the identities on $M_n(K)$ and those on $M_n(G)$.

Proposition 6 ([2, Proposition 2.1]). *Let $f_1, \dots, f_d \in K\langle x_1, \dots, x_m \rangle$ be elements of the T -ideal of identities of $M_n(K)$. If $d > \frac{1}{2}n^2m$, then $f_1 f_2 \dots f_d = 0$ is an identity on $M_n(G)$.*

Remark 5. Applying the result to $M_2(G)$ and the standard polynomial S_4 we get that $S_4^9 = 0$ is an identity on $M_2(G)$. However, this is not the best possible result. Really on $M_2(G)$ we get the identities $S_4^5 = 0$ and $[[x, y]^2, x]^5 = 0$. Thus we get two identities of degree 20 and 25, respectively, for $U_2(G)$.

The above Proposition 6 has an analogue for the upper triangular matrices U_n .

Theorem 10. *Let $f_1, \dots, f_d \in K\langle x_1, \dots, x_m \rangle$ be elements of the T -ideal of identities of $U_n(K)$. If $d > \frac{1}{4}n(n+1)m$, then $f_1 f_2 \dots f_d = 0$ is an identity on $U_n(G)$.*

Proof. It follows the proof of Proposition 6 taking into account the dimension of $U_n(K)$ and the fact that the relatively free algebra $F(U_n(K))$ has as a basis the monomials

$$x_1^{a_1} \cdots x_m^{a_m} [x_{i_{11}}, x_{i_{21}}, \dots, x_{i_{p_1 1}}] \cdots [x_{i_{1r}}, x_{i_{2r}}, \dots, x_{i_{p_r r}}],$$

where the number r of participating commutators is $\leq n - 1$ and the indices in each commutator $[x_{i_{1s}}, x_{i_{2s}}, \dots, x_{i_{p_s s}}]$ satisfy the relations $i_{1s} > i_{2s} \leq \dots < i_{p_s s}$. $\square$

Proposition 7 ([1, Lemma, p. 1509]). *The algebra $M_n(G)$ satisfies the identity S_{2n}^k for some $k > 1$ but satisfies neither S_{2n} nor identities of the form S_m^k for any k when $m < 2n$.*

Theorem 11. *The algebra $U_n(G)$ does not satisfy the identity $S_{2n} = 0$.*

Proof. We have

$$S_{2n}(e_{11}, e_{12}, e_{22}, e_{23}, \dots, e_{n-1, n-1}, e_{n-1, n}, e e_{nn}, f e_{nn}) = 2e f e_{1n} \neq 0$$

for $e, f \in G$ such that $ef = -fe \neq 0$. $\square$

In [16, Proposition 4.1 and Corollary 6.1], Vishne described an efficient way to use the S_n -module structure in the computation of the multilinear identities of degree n of a given algebra. He used the method to show that the minimal degree of an identity for $M_2(G)$ is 8 and gave explicit identities of degree 8. He described a class of identities for $M_2(G)$, namely,

Proposition 8 ([16, Corollary 4.3]). *Let f be a multilinear polynomial of degree 8. If $\text{tr } f(x_{\sigma(1)}, \dots, x_{\sigma(8)}) = 0$ for every $x_1, \dots, x_8 \in M_2(G)$, then f is an identity of $M_2(G)$.*

We use the notation $A_n = \sum_{\sigma \in \text{Sym}(n)} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$. By G'_0 we denote the even part of G' and by G'_1 its odd part.

Proposition 9. *The following identities hold:*

- (1) *On $U_2(G'_1)$ we have $A_k^2 = 0$ for every integer k .*
- (2) *On $U_2(G'_0)$ we have $S_k^2 = 0$ for every integer k .*

Proof. For $x_1, x_2 \in U_2(G'_1)$ and $A_2 = (a_{ij})$ we get $a_{11} = a_{21} = a_{22} = 0$. Thus $A_2^2 = 0$. Then we use induction. Let $A_{k-1}(x_1, \dots, x_{k-1})$ have only one nonzero entry, namely the $(1, 2)$ -entry. We have

$$\begin{aligned} A_k(x_1, \dots, x_k) &= A_{k-1}(x_1, \dots, x_{k-1})x_k + A_{k-1}(x_1, \dots, x_{k-2}, x_k)x_{k-1} \\ &\quad + A_{k-1}(x_1, \dots, x_{k-3}, x_{k-1}, x_k)x_{k-2} + \cdots \\ &\quad + A_{k-1}(x_1, x_3, \dots, x_k)x_2 + A_{k-1}(x_2, \dots, x_k)x_1. \end{aligned}$$

The multiplication by x_i keeps the three zero entries in every summand. So for $A_k = (b_{ij})$ we have $b_{11} = b_{21} = b_{22} = 0$ and thus $A_k^2 = 0$.

The arguments for $S_2^2 = 0$ are similar as for $x_1, x_2 \in U_2(G'_0)$ and for $S_2 = (c_{ij})$ we have $c_{11} = c_{21} = c_{22} = 0$. The recursive formulas

$$S_k(x_1, \dots, x_k) = \sum_{i=1}^k x_i (-1)^{k-1} S_{k-1}(x_{i+1}, \dots, x_k, x_1, \dots, x_{i-1})$$

for k even and

$$S_k(x_1, \dots, x_k) = \sum_{i=1}^k x_i S_{k-1}(x_{i+1}, \dots, x_k, x_1, \dots, x_{i-1})$$

for k odd show that for $S_k = (d_{ij})$ we have $d_{11} = d_{21} = d_{22} = 0$ and, therefore, $S_k^2 = 0$. $\square$

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