Abstract. In this paper, we shall introduce the concepts of $F_G$-contraction and $\psi_G$-contraction in uniform space endowed with graph to investigate the existence of a fixed point of mappings satisfying these notions. We shall also introduce a common fixed point theorem for pair of mappings satisfying the notion of $\psi_G$-contraction in uniform space endowed with graph.

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1. INTRODUCTION

Very recently, Wardowski [19] categorized the family of mappings into a new family, denoted by $F$ or $\Phi$, to introduce a new contraction condition, the $F$-contraction. As it is expected, it extends the famous Banach contraction mapping principle. Inspired the result of Wardowski, [19], a number of different approaches has been reported, see [8, 13, 15–17] and related references therein.

On the other hand, Czerwik [9] extended the notion of metric by introducing a new notion, $b$-metric. Indeed, Czerwik [9] modified the triangle inequality that causes to several differences in the related topology, and hence in convergence of a sequence, Hausdorffness of the topology etc. We also mention that Jachymski [10] successively set-up the Banach contraction mapping principle in the frame of a complete metric space endowed with a directed graph, Jachymski [10]. Later, this interesting contribution has been appreciated by several authors [5, 6, 11, 12, 14, 18].

The purpose of this paper is to introduce and discuss some new fixed point theorems and common fixed point theorems for mappings in uniform space endowed with graph satisfying the $F_G$-contraction, and $\psi_G$-contraction. Further, we introduce and use the notion of $E_s$-distance in this paper.

Some basic definitions and fundamental results are recollected for the sake of completeness. Let $X$ be a nonempty set. A nonempty family, $\theta$ of subsets of $X \times X$ is called the uniform structure of $X$, if it satisfies the following properties:
(i) if $G$ is in $\mathfrak{d}$, then $G$ contains the diagonal $\{(x,x)\mid x \in X\};$
(ii) if $G$ is in $\mathfrak{d}$ and $H$ is a subset of $X \times X$ which contains $G$, then $H$ is in $\mathfrak{d};$
(iii) if $G$ and $H$ are in $\mathfrak{d}$, then $G \cap H$ is in $\mathfrak{d};$
(iv) if $G$ is in $\mathfrak{d}$, then there exists $H$ in $\mathfrak{d}$, such that, whenever $(x,y)$ and $(y,z)$ are in $H$, then $(x,z)$ is in $G$;
(v) if $G$ is in $\mathfrak{d}$, then $\{(y,x)\mid (x,y) \in G\}$ is also in $\mathfrak{d}.$

The pair $(X, \mathfrak{d})$ is called a uniform space and the element of $\mathfrak{d}$ is called entourage or neighborhood or surrounding. The pair $(X, \mathfrak{d})$ is called a quasiuniform space (see e.g. [20]) if property (v) is omitted.

Let $\Delta = \{(x,x)\mid x \in X\}$ be the diagonal of a non-empty set $X$. For $V, W \in X \times X,$ we shall use the following setting in the sequel

$$V \circ W = \{(x,y)\mid \text{there exists } z \in X : (x,z) \in W \text{ and } (z,y) \in V\}$$

and

$$V^{-1} = \{(x,y)\mid (y,x) \in V\}.$$ 

For a subset $V \in \mathfrak{d}$, a pair of points $x$ and $y$ are said to be $V$-close if $(x,y) \in V$ and $(y,x) \in V.$ Moreover, a sequence $\{x_n\}$ in $X$ is called a Cauchy sequence for $\mathfrak{d}$, if for any $V \in \mathfrak{d}$, there exists $N \geq 1$ such that $x_n$ and $x_m$ are $V$-close for $n, m \geq N.$ For $(X, \mathfrak{d}),$ there is a unique topology $\tau(\mathfrak{d})$ on $X$ generated by $V(x) = \{y \in X\mid (x,y) \in V\}$ where $V \in \mathfrak{d}.$

A sequence $\{x_n\}$ in $X$ is convergent to $x$ for $\mathfrak{d}$, denoted by $\lim_{n \to \infty} x_n = x$, if for any $V \in \mathfrak{d},$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in V(x)$ for every $n \geq n_0.$ A uniform space $(X, \mathfrak{d})$ is called Hausdorff if the intersection of all the $V \in \mathfrak{d}$ is equal to $\Delta$ of $X$, that is, if $(x,y) \in V$ for all $V \in \mathfrak{d}$ implies $x = y.$ If $V = V^{-1}$ then we shall say that a subset $V \in \mathfrak{d}$ is symmetrical. Throughout the paper, we shall assume that each $V \in \mathfrak{d}$ is symmetrical. For more details, see e.g. [1–4].

Now, we shall recall the notions of $A$-distance and $E$-distance.

**Definition 1.** [2,3] Let $(X, \mathfrak{d})$ be a uniform space. A function $p : X \times X \longrightarrow [0, \infty)$ is said to be an $A$-distance if for any $V \in \mathfrak{d}$ there exists $\delta > 0$ such that if $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ for some $z \in X$, then $(x,y) \in V$.

**Definition 2.** [2,3] Let $(X, \mathfrak{d})$ be a uniform space. A function $p : X \times X \longrightarrow [0, \infty)$ is said to be an $E$-distance if

(i) $p$ is an $A$-distance,
(ii) $p(x,y) \leq p(x,z) + p(z,y), \ \forall x, y, z \in X.$

**Example 1.** [2,3] Let $(X, \mathfrak{d})$ be a uniform space and let $d$ be a metric on $X$. It is evident that $(X, \mathfrak{d}_d)$ is a uniform space where $\mathfrak{d}_d$ is a set of all subsets of $X \times X$ containing a "band" $U_\epsilon = \{(x,y) \in X^2\mid d(x,y) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\mathfrak{d} \subseteq \mathfrak{d}_d$, then $d$ is an $E$-distance on $(X, \mathfrak{d}).$
Lemma 1. [2,3] Let $(X, \vartheta)$ be a Hausdorff uniform space and $p$ be an $A$-distance on $X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ and $\{a_n\}, \{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0. Then, for $x, y, z \in X$, the following results hold:

(a) If $p(x_n, y) \leq a_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) \leq a_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to $z$.

(c) If $p(x_n, x_m) \leq a_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in $(X, \vartheta)$.

Let $p$ be an $A$-distance. A sequence in a uniform space $(X, \vartheta)$ with an $A$-distance is said to be a $p$-Cauchy if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

Definition 3. [2,3] Let $(X, \vartheta)$ be a uniform space and $p$ be an $A$-distance on $X$.

(i) $X$ is $S$-complete if for every $p$-Cauchy sequence $\{x_n\}$, there exists $x$ in $X$ with $\lim_{n \to \infty} p(x_n, x) = 0$.

(ii) $X$ is $p$-Cauchy complete if for every $p$-Cauchy sequence $\{x_n\}$, there exists $x$ in $X$ with $\lim_{n \to \infty} x_n = x$ with respect to $\tau(\vartheta)$.

(iii) $T : X \to X$ is $p$-continuous if $\lim_{n \to \infty} p(x_n, x) = 0$ implies $\lim_{n \to \infty} p(T(x_n), T(x)) = 0$.

Remark 1. Let $(X, \vartheta)$ be a Hausdorff uniform space which is $S$-complete. If a sequence $\{x_n\}$ be a $p$-Cauchy sequence, then we have $\lim_{n \to \infty} p(x_n, x) = 0$. Regarding Lemma 1(b), We derive that $\lim_{n \to \infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ and hence $S$-completeness implies $p$-Cauchy completeness.

Cosenitno et al. [7] modify the $\mathfrak{F}$-family introduced by Wardowski [19], in the setting of $b$-metric spaces as follows:

Definition 4. [7] Let $s \geq 1$ be a real number. Denote by $\mathfrak{F}_s$ the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) $F$ is strictly increasing, that is, for each $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$.

(F2) for each sequence $\{\delta_n\}$ of positive real numbers we have $\lim_{n \to \infty} \delta_n = 0$ if and only if $\lim_{n \to \infty} F(\delta_n) = -\infty$;

(F3) for each sequence $\{\delta_n\}$ of positive real numbers with $\lim_{n \to \infty} \delta_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n \to \infty} \delta_n^k F(\delta_n) = 0$.

(F4) for each sequence $\{\delta_n\}$ of positive real numbers such that $\tau + F(s \delta_n) \leq F(\delta_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^\alpha \delta_n) \leq F(s^{n-1} \delta_{n-1})$ for each $n \in \mathbb{N}$.

Cosenitno et al. [7] also showed that the following functions belong to $\mathfrak{F}_s$:

- $F(x) = x + \ln x$, for each $x > 0$.
- $F(x) = \ln x$, for each $x > 0$. 

2. Main Results

Definition 5. Let \((X, \vartheta)\) be a uniform space. A function \(p : X \times X \to [0, \infty)\) is said to be an \(E_s\)-distance if

(i) \(p\) is an \(A\)-distance,
(ii) \(p(x, y) \leq s[p(x, z) + p(z, y)]\), \(\forall x, y, z \in X\) and for some \(s \geq 1\).

Example 2. Let \((X, \vartheta)\) be a uniform space and let \(d\) be a \(b\)-metric on \(X\). Then clearly, \((X, \vartheta_d)\) is a uniform space where \(\vartheta_d\) is a set of all subsets of \(X \times X\) containing a "band" \(U_\epsilon = \{(x, y) : d(x, y) < \epsilon\}\) for some \(\epsilon > 0\). Moreover, if \(\vartheta \subseteq \vartheta_d\), then \(d\) is an \(E_s\)-distance on \((X, \vartheta)\).

Throughout this section, we assume that \(G = (V, E)\) is a directed graph such that the set of its vertices \(V\) coincides with \(X\) (i.e., \(V = X\)) and the set of its edges \(E\) is such that \(E \supseteq \Delta\), where \(\Delta = \{(x, x) : x \in X\}\). Further, assume that \(G\) has no parallel edges. A mapping \(T : X \to X\) is \(p_G\)-continuous if for each sequence \(x_n\) such that \(x_n \to x\) as \(n \to \infty\), \(p(x_n, x) = 0\), then we have

\[\lim_{n \to \infty} p(Tx_n, Tx) = 0.\]

Definition 6. Let \((X, \vartheta)\) be a uniform space endowed with the graph \(G\) and \(p\) is an \(E_s\)-distance on \(X\). A mapping \(T : X \to X\) is a \(F_G\)-contraction, if there exist \(F \in \mathcal{F}_s\) and \(\tau > 0\), such that, for each \((x, y) \in E\), we have

\[\tau + F(sp(Tx, Ty)) \leq F(p(x, y)),\] (2.1)
whenever \(\min\{p(Tx, Ty), p(x, y)\} > 0\).

Theorem 1. Let \((X, \vartheta)\) be a \(S\)-complete Hausdorff uniform space endowed with the graph \(G\) and \(p\) is an \(E_s\)-distance on \(X\). Let \(T : X \to X\) is an \(F_G\)-contraction satisfying the following conditions:

(i) \(T\) is edge preserving, that is, for \((x, y) \in E\), we have \((Tx, Ty) \in E\);
(ii) there exists \(x_0 \in X\) such that \((x_0, Tx_0) \in E\) and \((Tx_0, x_0) \in E\);
(iii) \(T\) is \(p_G\)-continuous, or, for any sequence \(\{x_n\} \subseteq X\) such that \(x_n \to x\) as \(n \to \infty\) and \((x_n, x_{n+1}) \in E\) for each \(n \in \mathbb{N}\), we have \((x_n, x) \in E\) for each \(n \in \mathbb{N}\).

Then \(T\) has a fixed point.

Proof. By hypothesis (ii), there exists \(x_0 \in X\) such that \((x_0, x_1) = (x_0, Tx_0) \in E\). From (2.1), we have

\[\tau + F(sp(x_1, x_2)) = \tau + F(sp(Tx_0, Tx_1)) \leq F(p(x_0, x_1)).\] (2.2)
As \(T\) is edge preserving, for \((x_0, x_1) \in E\), we have \((x_1, x_2) \in E\), From (2.1), we have

\[\tau + F(sp(x_2, x_3)) = \tau + F(sp(Tx_1, Tx_2)) \leq F(p(x_1, x_2)).\] (2.3)
Continuing in the same way, we get a sequence \( \{x_n\} \subset X \) such that
\[x_n = T x_{n-1}, \ x_{n-1} \neq x_n \text{ and } (x_{n-1}, x_n) \in E \text{ for each } n \in \mathbb{N}.\]

Furthermore,
\[ \tau + F(s p(x_n, x_{n+1})) \leq F(p(x_{n-1}, x_n)) \text{ for each } n \in \mathbb{N}. \]  \(2.4\)

By using the property \( F_4 \), we have
\[ \tau + F(s^n p(x_n, x_{n+1})) \leq F(s^{n-1} p(x_{n-1}, x_n)). \]

Letting \( p_n = p(x_n, x_{n+1}) \), for each \( n \in \mathbb{N} \) and after some simplification, we have
\[ F(s^n p_n) \leq F(p_0) - n \tau \text{ for each } n \in \mathbb{N}. \]  \(2.5\)

Letting \( n \to \infty \) in \(2.5\), we get \( \lim_{n \to \infty} F(s^n p_n) = -\infty \). Thus, by property \( F_2 \), we have \( \lim_{n \to \infty} s^n p_n = 0 \). From \( F_3 \) there exists \( k \in (0, 1) \) such that
\[ \lim_{n \to \infty} (s^n p_n)^k F(s^n p_n) = 0. \]

From \(2.5\) we have
\[ (s^n p_n)^k F(s^n p_n) - (s^n p_n)^k F(p_0) \leq -(s^n p_n)^k n \tau \leq 0 \text{ for each } n \in \mathbb{N}. \]  \(2.6\)

Letting \( n \to \infty \) in \(2.6\), we get
\[ \lim_{n \to \infty} n (s^n p_n)^k = 0. \]  \(2.7\)

This implies that there exists \( n_1 \in \mathbb{N} \) such that \( n(s^n p_n)^k \leq 1 \) for each \( n \geq n_1 \). Thus, we have
\[ s^n p_n \leq \frac{1}{n^{1/k}}, \text{ for each } n \geq n_1. \]  \(2.8\)

To show that \( \{x_n\} \) is a \( p \)-Cauchy sequence, consider
\[ S_n = \sum_{i=n_1}^{n} \frac{1}{i^{1/k}}. \]

Since \( \sum_{i=1}^{\infty} \frac{1}{i^{1/k}} \) is convergent series. Thus, there exists \( S \in [0, \infty) \) such that
\[ \lim_{n \to \infty} S_n = S. \]

Consider \( m, n \in \mathbb{N} \) with \( m > n > n_1 \). By using the triangular inequality and \(2.8\), we have
\[ p(x_n, x_m) \leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m) \]
\[ = \sum_{i=n_1}^{m-1} s^i p_i - \sum_{i=n_1}^{n-1} s^i p_i \]
\[ \leq S_{m-1} - S_{n-1}. \]

Thus, \( \lim_{n,m \to \infty} p(x_n, x_m) = 0 \). In a similar way, we show that \( \lim_{n,m \to \infty} p(x_m, x_n) = 0 \). Thus, \( \{x_n\} \) is a \( p \)-Cauchy sequence. As \( (X, \theta) \) is \( S \)-complete, there exists \( x^* \in X \) such that \( \lim_{n \to \infty} p(x_n, x^*) = 0 \). By condition (iii), when \( T \) is \( p_G \)-continuous,
we have \( \lim_{n \to \infty} p(x_{n+1}, Tx^*) = 0 \). As \( \lim_{n \to \infty} p(x_n, x^*) = 0 \) and \( \lim_{n \to \infty} p(x_n, Tx^*) = 0 \). Thus by Lemma 1-(a) we have \( x^* = Tx^* \). By condition (iii), when we have \( (x_n, x^*) \in E \) for each \( n \in \mathbb{N} \). From (2.1), we have
\[
\tau + F(s p(Tx_n, Tx^*)) \leq F(p(x_n, x^*)).
\]
This implies that \( sp(E_{n+1}, Tx^*) < p(x_n, x^*) \). Letting \( n \to \infty \), we have \( \lim_{n \to \infty} p(x_{n+1}, Tx^*) = 0 \). Again, by Lemma 1-(a) we have \( x^* = Tx^* \). □

Let \( \Psi \) be the family of functions \( \psi : [0, \infty) \to [0, \infty) \) satisfying the following conditions:

\[ \begin{align*}
(\Psi_1) & \quad \psi \text{ is nondecreasing;} \\
(\Psi_2) & \quad \sum_{n=1}^{+\infty} s^n \psi^n(t) < \infty \text{ for all } t > 0, \text{ where } \psi^n \text{ is the } n\text{th iterate of } \psi.
\end{align*} \]

**Definition 7.** Let \( (X, \vartheta) \) be a uniform space endowed with the graph \( G \) and \( p \) is an \( E_{\vartheta} \)-distance on \( X \). A mapping \( T : X \to X \) is a \( \psi_G \)-contraction mapping if for each \( (x, y) \in E \), we have
\[
p(Tx, Ty) \leq \psi(p(x, y))
\]
where \( \psi \in \Psi \).

**Theorem 2.** Let \( (X, \vartheta) \) be a \( S \)-complete Hausdorff uniform endowed with the graph \( G \) and \( p \) is an \( E_{\vartheta} \)-distance on \( X \). Let \( T : X \to X \) is a \( \psi_G \)-contraction mapping satisfying the following conditions:

(i) \( T \) is edge preserving, that is, for \( (x, y) \in E \), we have \( (Tx, Ty) \in E \);

(ii) there exists \( x_0 \in X \) such that \( (x_0, Tx_0) \in E \) and \( (Tx_0, x_0) \in E \);

(iii) \( T \) is \( p_G \)-continuous, or, for any sequence \( \{x_n\} \subseteq X \) such that \( x_n \to x \) as \( n \to \infty \) and \( (x_n, x_{n+1}) \in E \) for each \( n \in \mathbb{N} \), we have \( (x_n, x) \in E \) for each \( n \in \mathbb{N} \).

Then \( T \) has a fixed point.

**Proof.** By hypothesis (ii) of theorem we have \( x_0 \in X \) such that \( (x_0, Tx_0) \in E \). Define the sequence \( \{x_n\} \) in \( X \) by \( x_{n+1} = Tx_n \) for all \( n \in \mathbb{N} \cup \{0\} \). If \( x_{n_0} = x_{n_0+1} \) for some \( n_0 \), then \( x_{n_0} \) is a fixed point of \( T \). So, we can assume that \( x_n \neq x_{n+1} \) for all \( n \). Since \( T \) is edge preserving, we have
\[
(x_0, x_1) = (x_0, Tx_0) \in E \Rightarrow (Tx_0, Tx_1) = (x_1, x_2) \in E.
\]
Inductively, we have
\[
(x_n, x_{n+1}) \in E, \text{ for all } n \in \mathbb{N} \cup \{0\}.
\]
(2.10)
From (2.9) and (2.10), it follows that for all \( n \in \mathbb{N} \cup \{0\} \), we have
\[
p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq \psi(p(x_n, x_{n+1})).
\]
(2.11)
Iteratively, we get
\[ p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \text{ for all } n \in \mathbb{N}. \]

Since \( p \) is an \( E_s \)-distance then for \( m > n \), we have
\[
p(x_n, x_m) \leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m)
\]
\[ \leq s^n \psi^n(p(x_0, x_1)) + s^{n+1} \psi^{n+1}(p(x_0, x_1)) + \cdots + s^{m-1} \psi^{m-1}(p(x_0, x_1)). \]

To show that \( \{x_n\} \) is a \( p \)-Cauchy sequence, consider
\[
S_n = \sum_{k=0}^{n} s^k \psi^k(p(x_0, x_1)).
\]

Thus from (2) we have
\[ p(x_n, x_m) \leq S_{m-1} - S_{n-1}. \tag{2.12} \]

Since \( \psi \in \Psi \), there exists \( S \in [0, \infty) \) such that \( \lim_{n \to \infty} S_n = S \). Thus by (2.12) we have
\[ \lim_{n,m \to \infty} p(x_n, x_m) = 0. \tag{2.13} \]

Since \( p \) is not symmetric then by repeating the same argument we have
\[ \lim_{m \to \infty} p(x_m, x_n) = 0. \]
Hence the sequence \( \{x_n\} \) is a \( p \)-Cauchy in the \( S \)-complete space \( X \). Thus, there exists \( x^* \in X \) such that \( \lim_{n \to \infty} p(x_n, x^*) = 0 \). By condition (iii), when we have \( T \) is \( pG \)-continuous, we get \( \lim_{n \to \infty} p(Tx_n, Tx^*) = 0 \) which implies that \( \lim_{n \to \infty} p(x_{n+1}, Tx^*) = 0 \). Hence we have \( \lim_{n \to \infty} p(x_n, x^*) = 0 \) and \( \lim_{n \to \infty} (x_n, Tx^*) = 0 \). Thus by Lemma 1-(a) we have \( x^* = Tx^* \). By condition (iii), when we have \( (x_n, x^*) \in E \) for each \( n \in \mathbb{N} \), then from \( (2.9) \)
\[ p(x_{n+1}, Tx^*) = p(Tx_n, Tx^*) \leq \psi(p(x_n, x^*)) < p(x_n, x^*). \tag{2.14} \]

Letting \( n \to \infty \) in the above inequality, we have \( \lim_{n \to \infty} p(x_{n+1}, Tx^*) = 0 \). Thus by repeating the same arguments as above we have \( x^* = Tx^* \). □

Example 3. Let \( X = [0, 1] \) be endowed with a graph \( G = (V, E) \) with \( V = X \) and \( E = \{ (x, y) : x, y \in \left[ \frac{1}{n+1} : n \in \mathbb{N} \right] \} \cup \{ (0, 1) \} \cup \{ (x, x) : x \in X \} \), and \( \mathcal{b} \)-metric \( d(x, y) = (x - y)^2 \) with \( s = 2 \). Define \( \delta = \{ U \in \mathcal{U} : \delta > 0 \} \). It is easy to see that \( (X, \delta) \) is a uniform space. Define \( T : X \to X \) by
\[
Tx = \begin{cases} 
0 & \text{if } x = 0 \\
\frac{1}{3n+1} & \text{if } x = \frac{1}{n} : n > 1 \\
\sqrt{x} & \text{otherwise.}
\end{cases} \tag{2.15}
\]

Take \( \psi(t) = \frac{t}{3} \) for all \( t \geq 0 \). It is easy to see that \( T \) is edge preserving and \( \psi \)-contraction. Also for \( x_0 = \frac{1}{2} \) we have \( (x_0, Tx_0) \in E \) and \( (Tx_0, x_0) \in E \). Moreover for any sequence \( \{x_n\} \) in \( X \) with \( x_n \to x \) as \( n \to \infty \) and \( (x_{n-1}, x_n) \in E \) for each \( n \in \mathbb{N} \) we have \( (x_n, x) \in E \) for each \( n \in \mathbb{N} \). Therefore by Theorem 2, \( T \) has a fixed point.
To investigate the uniqueness of a fixed point, we consider the following condition:

\((H)\): For all \(x, y \in \text{Fix}(T)\), there exists \(z \in X\) such that \((z, x) \in E\) and \((z, y) \in E\).

Here, \(\text{Fix}(T)\) denotes the set of all fixed points of \(T\).

The following theorem guarantees the uniqueness of a fixed point.

**Theorem 3.** Adding the condition \((H)\) in the hypothesis of Theorem 2, we obtain the uniqueness of fixed point of \(T\).

**Proof.** Suppose, on the contrary, that \(u, v \in X\) are two distinct fixed points of \(T\). From \((H)\), there exists \(z \in X\) such that \((z, u) \in E\) and \((z, v) \in E\).

By using the fact that \(T\) is edge preserving, from (2.16), we have

\[ T^n z, u \in E \quad \text{and} \quad T^n z, v \in E \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \]

(2.17)

We define the sequence \(\{z_n\}\) in \(X\) by \(z_{n+1} = Tz_n = T^n z_0\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(z_0 = z\). From (2.17) and (2.9), we have

\[ p(z_{n+1}, u) = p(Tz_n, Tu) \leq \psi(p(z_n, u)). \]

(2.18)

for all \(n \in \mathbb{N} \cup \{0\}\). This implies that

\[ p(z_n, u) \leq \psi^n(p(z_0, u)), \quad \text{for all} \quad n \in \mathbb{N}. \]

Letting \(n \to \infty\) in the above inequality, we obtain

\[ \lim_{n \to \infty} p(z_n, u) = 0. \]

(2.19)

Similarly, we have

\[ \lim_{n \to \infty} p(z_n, v) = 0. \]

(2.20)

From (2.19) and (2.20) together with Lemma 1-(a), it follows that \(u = v\). Thus, fixed point of \(T\) is unique. \(\square\)

**Definition 8.** Let \((X, \psi)\) be a uniform space endowed with the graph \(G\). A pair of two self mappings \(T, S : X \to X\) is said to be a \(\psi_G\)-contraction pair if for each \((x, y) \in E\), we have

\[ \max\{p(Tx, Sy), p(Sx, Ty)\} \leq \psi(p(x, y)), \]

(2.21)

where \(\psi \in \Psi\).

**Theorem 4.** Let \((X, \psi)\) be a \(S\)-complete Hausdorff uniform space endowed with the graph \(G\) and \(p\) is an \(E_s\)-distance on \(X\). Suppose that the pair of \(T, S : X \to X\) is \(\psi_G\)-contraction pair satisfying the following conditions.

(i) \((T, S)\) is edge preserving pair; that is, for each \((x, y) \in E\), we have \((Tx, Sy) \in E\) and \((Sx, Ty) \in E\);

(ii) there exists \(x_0 \in X\) such that \((x_0, Tx_0) \in E\) and \((Tx_0, x_0) \in E\);
(iii) for any sequence \{x_n\} in X with \(x_n \to x\) as \(n \to \infty\) and \((x_n, x_{n+1}) \in E\) for each \(n \in \mathbb{N} \cup \{0\}\), then \((x_n, x) \in E\) for each \(n \in \mathbb{N} \cup \{0\}\).

Then \(T\) and \(S\) have a common fixed point.

**Proof.** By hypothesis (ii) of theorem, we have \(x_0 \in X\) such that \((x_0, T x_0) \in E\) and \((T x_0, x_0) \in E\). Since \((T, S)\) is an edge preserving pair, then we can construct a sequence such that

\[ T x_{2n} = x_{2n+1}, \quad S x_{2n+1} = x_{2n+2} \quad \text{and} \quad (x_n, x_{n+1}) \in E, \quad (x_{n+1}, x_n) \in E, \quad \forall n \in \mathbb{N} \cup \{0\}. \]

From (2.21) for all \(n \in \mathbb{N} \cup \{0\}\), we have

\[ p(x_{2n+1}, x_{2n+2}) = p(T x_{2n}, S x_{2n+1}) \]
\[ \leq \max\{p(T x_{2n}, S x_{2n+1}), p(S x_{2n}, T x_{2n+1})\} \]
\[ \leq \psi(p(x_{2n}, x_{2n+1})). \]

Hence, we conclude that

\[ p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n}, x_{2n+1})). \tag{2.22} \]

Similarly, we get

\[ p(x_{2n+2}, x_{2n+3}) = p(S x_{2n+1}, T x_{2n+2}) \]
\[ \leq \max\{p(T x_{2n+1}, S x_{2n+2}), p(S x_{2n+1}, T x_{2n+2})\} \]
\[ \leq \psi(p(x_{2n+1}, x_{2n+2})). \]

Hence, we have

\[ p(x_{2n+2}, x_{2n+3}) \leq \psi(p(x_{2n+1}, x_{2n+2})). \tag{2.23} \]

Thus from (2.22) and (2.23), and by induction, we get

\[ p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.24} \]

Now we show that \(\{x_n\}\) is a \(p\)-Cauchy sequence. Since \(p\) is an \(E_s\)-distance then for \(m \geq n\), we have

\[ p(x_n, x_m) \leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m) \]
\[ \leq s^n \psi^n(p(x_0, x_1)) + s^{n+1} \psi^{n+1}(p(x_0, x_1)) + \cdots + s^{m-1} \psi^{m-1}(p(x_0, x_1)). \]

Now, we shall consider

\[ S_n = \sum_{k=0}^{n} s^k \psi^k(p(x_0, x_1)). \]

Thus, from (2) we have

\[ p(x_n, x_m) \leq S_{m-1} - S_{n-1}. \tag{2.25} \]

Since \(\psi \in \Psi\), there exists \(S \in [0, \infty)\) such that \(\lim_{n \to \infty} S_n = S\). Thus, by (2.25) we have

\[ \lim_{n,m \to \infty} p(x_n, x_m) = 0. \tag{2.26} \]
Since $p$ is not symmetric then by repeating the same argument we have
\[
\lim_{n,m \to \infty} p(x_m, x_n) = 0.
\] (2.27)

Hence the sequence $\{x_n\}$ is a $p$-Cauchy in the $S$-complete space $X$. Thus, there exists $x^* \in X$ such that $\lim_{n \to \infty} p(x_n, x^*) = 0$ which implies $\lim_{n \to \infty} T x_{2n} = \lim_{n \to \infty} S x_{2n+1} = x^*$. By assumption (iii), we have $(x_n, x^*) \in E$. Thus, by using the triangular inequality and (2.21), we have
\[
p(x_n, T x^*) \leq sp(x_n, x_{2n+2}) + sp(x_{2n+2}, T x^*)
\]
\[
= sp(x_n, x_{2n+2}) + sp(S x_{2n+1}, T x^*)
\]
\[
\leq sp(x_n, x_{2n+2}) + s \max\{p(T x_{2n+1}, S x^*), p(S x_{2n+1}, T x^*)\}
\]
\[
\leq sp(x_n, x_{2n+2}) + s \psi(p(x_{2n+1}, x^*))
\] (2.28)

Letting $n \to \infty$ in (2.28), we have $p(x_n, T x^*) = 0$. Hence we have $\lim_{n \to \infty} p(x_n, x^*) = 0$ and $\lim_{n \to \infty} p(x_n, T x^*) = 0$. Thus by Lemma 1-(a) we have $x^* = T x^*$. Analogously, we can drive $x^* = S x^*$. Therefore $x^* = T x^* = S x^*$. \qed

Remark 2. Note that Theorem 4 is valid if one replace condition (ii) with

(ii)*: there exists $x_0 \in X$ such that $(x_0, S x_0) \in E$ and $(S x_0, x_0) \in E$.

Example 4. Let $X = [0, 1]$ be endowed with a graph $G = (V, E)$ with $V = X$ and $E = \{(x, y) : x, y \in \frac{1}{n+1} : n \in \mathbb{N} \} \cup \{0\}$, and dislocated metric space $d(x, y) = \max\{|x-y|, 0\}$. Define $\vartheta = \{U_\epsilon | \epsilon > 0\}$, where $U_\epsilon = \{(x, y) \in X \times X : d(x, y) < d(x, x) + \epsilon\}$. It is easy to see that $(X, \vartheta)$ is a uniform space. Define $T : X \to X$ by
\[
T x = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2n+1} & \text{if } x = \frac{1}{n} : n > 1 \\ \sqrt{x} & \text{otherwise} \end{cases}
\] (2.29)

and $S : X \to X$ by
\[
S x = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2n} & \text{if } x = \frac{1}{n} : n > 1 \\ \sqrt{x} & \text{otherwise} \end{cases}
\] (2.30)

Take $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Further, it is easy to see that $(T, S)$ is edge preserving and $\psi_G$-contraction pair. Also for $x_0 = \frac{1}{2}$ we have $(x_0, T x_0) \in E$ and $(T x_0, x_0) \in E$. Moreover for any sequence $\{x_n\}$ in $X$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N} \cup \{0\}$ we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$. Therefore by Theorem 4, $T$ and $S$ have a common fixed point.

To investigate the uniqueness of a common fixed point, we use the following condition.

(I) For each $x, y \in C F i x(T, S)$, we have $(x, y) \in E$, where $C F i x(T, S)$ is the set of all common fixed points of $T$ and $S$. 


Theorem 5. Adding the condition \((I)\) in the hypothesis of Theorem 4, we obtain the uniqueness of common fixed point of \(T\) and \(S\).

Proof. On the contrary suppose that \(u, v \in X\) are two distinct common fixed points of \(T\) and \(S\). From \((I)\) and (2.21) we have

\[
p(u, v) = \max\{p(Tu, Sv), p(Su, Tv)\} \leq \psi(p(u, v)) < p(u, v),
\]

which is impossible for \(p(u, v) > 0\). Consequently, we have \(p(u, u) = 0\). Analogously, one can show that \(p(u, u) = 0\). Thus we have \(u = v\), which is a contradiction to our assumption. Hence \(T\) and \(S\) have a unique common fixed point. \(\square\)

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REFERENCES


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