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ON THE HYPER-SUMS OF POWERS OF INTEGERS

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Abstract. The main object of this paper is to study an old problem concerning the hyper-sums of powers of integers. First, we establish some important properties of this problem (generating function, explicit formula, congruence). Finally, an explicit formula for the hyper-sums of powers of integers involving the generalized Bernoulli polynomials are also given.

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1. INTRODUCTION

Following the usual notations (see [4]), the falling factorial $x^{\underline{n}}$ $(x \in \mathbb{C})$ is defined by $x^{\underline{0}} = 1,\ x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ for $n > 0$ and, the rising factorial denoted by $x^{\overline{n}}$, is defined by $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ with $x^{\overline{0}} = 1$. The (signed) Stirling numbers of the first kind $s(n,k)$ are the coefficients in the expansion

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)\, x^k.$$

The Stirling numbers of the second kind, denoted $\left\{ {n \atop k} \right\}$ are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}.$$

The Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ count the number of ways to partition a set of n elements into exactly k nonempty subsets.

The hyper-sums (sums of sums) of powers of integers $S_p^{(r)}(n)$ ($p \geq 0$) (or the r-fold summation of pth powers) are defined recursively as

$$\begin{cases} S_0^{(r)}(n) = \binom{n+r}{r+1}, \; r \geq 0, \\ S_p^{(0)}(n) = \sum_{i=1}^{n} i^p \;\; ; \;\; S_p^{(r)}(n) = \sum_{j=1}^{n} S_p^{(r-1)}(j), \; n, r, p \geq 1. \end{cases}$$

As mentioned by Knuth [8], the hyper-sums of powers of integers was introduced for the first time by Faulhaber as a generalization of sums of powers of integers

$$S_p^{(0)}(n) = 1^p + 2^p + \cdots + n^p.$$

The sums $S_p^{(0)}(n)$ have been of interest for a long time and different methods have been proposed to obtain them, see [1–3, 5, 6, 9, 10].

In 2005, Inaba [7] extended the well-known formulas for $S_p^{(0)}(n)$ which is based on Stirling numbers of the second kind to obtain the explicit formulas of the hyper-sums of powers of integers

$$S_p^{(r)}(n) = \sum_{k=1}^{p} k! \binom{n+r+1}{r+k+1} \left\{ {p \atop k} \right\}, \quad n, p \geq 1. \tag{1.1}$$

Note that when $p \geq 0$, we can write $S_p^{(r)}(n)$ as

$$S_p^{(r)}(n) = \sum_{k=0}^{p} \left\{ {p \atop k} \right\} \left(k! \binom{n+r+1}{k+r+1} - \binom{n+r}{r} \delta_{0,k} \right), \tag{1.2}$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

The purpose of the present paper is to give some results concerning the hyper-sums of powers of integers $S_p^{(r)}(n)$.

2. Main results

We begin by the exponential generating function of the hyper-sums of powers of integers.

Theorem 1. *The exponential generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$ is*

$$\sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} = \binom{n+r+1}{r+1} {}_2F_1\left({1, -n \atop r+2} ; 1 - e^z \right) - \binom{n+r}{r},$$

where ${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} ;z\right)$ *denotes the Gaussian hypergeometric function defined by*

$$\sum_{n\geq 0}\frac{(a)^{\overline{n}}\,(b)^{\overline{n}}}{(c)^{\overline{n}}}\frac{z^n}{n!}.$$

Proof. First, we invert the formula (1.2) to obtain

$$\sum_{k=0}^{p} s\left(p,k\right)S_k^{(r)}\left(n\right)=p!\binom{n+r+1}{p+r+1}-\binom{n+r}{r}\delta_{0,p}.$$

Further, we have

$$\begin{aligned} A_0\left(z\right) &= \sum_{p\geq 0}\left(p!\binom{n+r+1}{p+r+1}-\binom{n+r}{r}\delta_{0,p}\right)\frac{z^p}{p!} \\ &= \sum_{p\geq 0}\binom{n+r+1}{p+r+1}z^p-\binom{n+r}{r} \\ &= \binom{n+r+1}{r+1}\sum_{p\geq 0}\frac{(1)^{\overline{p}}\,(-n)^{\overline{p}}}{(r+2)^{\overline{p}}}\frac{(-z)^p}{p!}-\binom{n+r}{r}. \end{aligned}$$

Finally, the result follows from [11, Theorem 3]

$$\sum_{p\geq 0}S_p^{(r)}\left(n\right)\frac{z^p}{p!}=A_0\left(e^z-1\right).$$

This completes the proof. □

We will now derive a few further consequences of Theorem 1.

Corollary 1. *The exponential generating function of the hyper-sums of powers of integers* $S_p^{(r)}(n)$ *is given by*

$$\sum_{p\geq 0}S_p^{(r)}\left(n\right)\frac{z^p}{p!}=\sum_{i=1}^{n}\binom{n+r-i}{r}e^{iz}. \tag{2.1}$$

Proof. It follows from the theory of hypergeometric functions that the Gaussian hypergeometric function ${}_2F_1\left(\begin{matrix} 1,-n \\ r+2 \end{matrix} ;1-e^z\right)$ has an integral representation given by

$${}_2F_1\left(\begin{matrix} 1,-n \\ r+2 \end{matrix} ;1-e^z\right)=(r+1)\int_0^1 (1-x)^r\left(1-x+xe^z\right)^n dx.$$

Hence, we obtain

$$\begin{aligned}\sum_{p\geq 0} S_p^{(r)}(n)\frac{z^p}{p!} &= \frac{(n+r+1)!}{n!r!}\sum_{k=0}^{n}\binom{n}{k}e^{(n-k)z}\int_0^1 (1-x)^{r+k}x^{n-k}dx - \binom{n+r}{r} \\ &= \frac{(n+r+1)!}{n!r!}\sum_{k=0}^{n}\binom{n}{k}e^{(n-k)z}\frac{(r+k)!(n-k)!}{(n+r+1)!} - \binom{n+r}{r} \\ &= \sum_{k=0}^{n}\binom{k+r}{r}e^{(n-k)z} - \binom{n+r}{r} \\ &= \sum_{k=0}^{n-1}\binom{k+r}{r}e^{(n-k)z},\end{aligned}$$

as claimed. □

An explicit formula for $S_p^{(r)}(n)$ is given in the following corollary.

Corollary 2. *The hyper-sums of powers of integers* $S_p^{(r)}(n)$ $(p \geq 0)$ *is given by*

$$S_p^{(r)}(n) = \sum_{i=1}^{n}\binom{n+r-i}{r}i^p.$$

Proof. Since

$$\begin{aligned}\sum_{p\geq 0} S_p^{(r)}(n)\frac{z^p}{p!} &= \sum_{i=1}^{n}\binom{n+r-i}{r}\sum_{p\geq 0}\frac{(iz)^p}{p!} \\ &= \sum_{p\geq 0}\left(\sum_{i=1}^{n}\binom{n+r-i}{r}i^p\right)\frac{z^p}{p!}.\end{aligned}$$

Equating the coefficient of $\frac{z^p}{p!}$, we get the result. □

In the next theorem we will present some congruence relations for $S_p^{(r)}(n)$.

Theorem 2. *Let n be a prime number and for* $p > 0$, $r \geq 0$, *we have*

(1) *If* $n \mid r+1$, *then* $S_p^{(r)}(n) \equiv 0 \pmod{n}$,
(2) *If* $n \nmid r+1$ *and* $n-1 \mid p$, *then* $S_p^{(r)}(n) \equiv -1 \pmod{n}$.

Proof. If $n \mid r+1$ then by Lucas' congruence we get

$$\begin{aligned}S_p^{(r)}(n) &\equiv \sum_{i=1}^{n-1}\binom{r-i}{r}i^p \pmod{n} \\ &\equiv 0 \pmod{n}.\end{aligned}$$

Since $n-1 \mid p$, the Fermat's little theorem and Lucas' congruence gives

$$\begin{aligned} S_p^{(r)}(n) &\equiv \sum_{i=1}^{n-1} \binom{n+r-i}{r} \pmod{n} \\ &\equiv -1 + \binom{n+r}{r+1} \pmod{n} \\ &\equiv -1 \pmod{n}. \end{aligned}$$

This completes the proof. □

Theorem 3. *The ordinary generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$ is given by*

$$\sum_{r\geq 0} S_p^{(r)}(n) z^r = \frac{1}{(1-z)^{n+1}} \sum_{i=1}^{n} (1-z)^i i^p. \tag{2.2}$$

Proof. Since

$$\sum_{r\geq 0} \binom{n+r-i}{r} z^r = (1-z)^{i-n-1},$$

which implies (2.2). □

Further, from (2.1) and (2.2), we deduce the double generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$

$$\begin{aligned} \sum_{r\geq 0}\sum_{p\geq 0} S_p^{(r)}(n) \frac{z^p}{p!} t^r &= \sum_{i=1}^{n}\sum_{r\geq 0} \binom{n+r-i}{r} t^r e^{iz} \\ &= \frac{1}{(1-t)^{n+1}} \sum_{i=1}^{n} \left(e^z(1-t)\right)^i \\ &= \frac{e^{(n+1)z}(1-t)^n - e^z}{(1-t)^n\left(e^z(1-t)-1\right)}. \end{aligned}$$

Now, according to the well-known formula, for $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$

$${}_2F_1\left(\begin{matrix} -n, 1 \\ m \end{matrix}; z\right) = \frac{n!(z-1)^{m-2}}{(m)^{\overline{n}} z^{m-1}} \left[\sum_{k=0}^{m-2} \frac{(n+1)^{\overline{k}}}{k!} \left(\frac{z}{z-1}\right)^k - (1-z)^{n+1}\right],$$

we can rewrite the exponential generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$ as

$$\sum_{p\geq 0} S_p^{(r)}(n) \frac{z^p}{p!} = \frac{(-1)^r e^{rz}}{(1-e^z)^{r+1}} \left(\sum_{k=0}^{r} \binom{n+k}{k} \left(1-e^{-z}\right)^k - e^{(n+1)z}\right)$$

$$-\binom{n+r}{r}. \tag{2.3}$$

The next result gives an explicit formula for $S_p^{(r)}(n)$ involving the generalized Bernoulli polynomials. Recall that the generalized Bernoulli polynomials $\mathbb{B}_n^{(\alpha)}(x)$ of degree n in x are defined by the exponential generating function [12, 13]

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n\geq 0} \mathbb{B}_n^{(\alpha)}(x)\frac{z^n}{n!} \tag{2.4}$$

for arbitrary parameter α.

In particular, $\mathbb{B}_n^{(1)}(x) := \mathbb{B}_n(x)$ denotes the classical Bernoulli polynomials with $\mathbb{B}_1(0) = -\frac{1}{2}$.

Theorem 4. *For all $n, p, r \geq 0$, we have*

$$S_p^{(r)}(n) = \frac{p!}{(p+r+1)!}\mathbb{B}_{p+r+1}^{(r+1)}(n+r+1) - p!\sum_{j=0}^{r}\binom{n+k}{k}\frac{\mathbb{B}_{p+j+1}^{(j+1)}(j)}{(p+j+1)!} - \binom{n+r}{r}\delta_{0,p}.$$

Proof. By (2.3) and (2.4) we have

$$\begin{aligned}\sum_{p\geq 0} S_p^{(r)}(n)\frac{z^p}{p!} &= -\sum_{k=0}^{r}\binom{n+k}{k}\frac{e^{(r-k)z}}{(e^z-1)^{r-k+1}} + \frac{e^{(r+n+1)z}}{(e^z-1)^{r+1}} - \binom{n+r}{r}\\ &= -\sum_{k=0}^{r}\binom{n+k}{k}\sum_{p\geq 0}\mathbb{B}_p^{(r-k+1)}(r-k)\frac{z^{p-r-1}}{p!}\\ &\quad + \sum_{p\geq 0}\mathbb{B}_p^{(r+1)}(n+r+1)\frac{z^{p-r-1}}{p!} - \binom{n+r}{r}.\end{aligned}$$

After some rearrangement, we find

$$\sum_{p\geq 0} S_p^{(r)}(n)\frac{z^p}{p!} = \sum_{p\geq 0}\frac{z^p}{p!}\left(p!\frac{\mathbb{B}_{p+r+1}^{(r+1)}(n+r+1)}{(p+r+1)!} - \sum_{k=0}^{r} p!\frac{\mathbb{B}_{p+r-k+1}^{(r-k+1)}(r-k)}{(p+r-k+1)!}\binom{n+k}{k} - \binom{n+r}{r}\delta_{0,p}\right).$$

Equating the coefficient of $\frac{z^p}{p!}$, we get the result. □

When $r = 0$, Theorem 4 reduces to the known result

$$S_p^{(0)}(n) = \frac{1}{p+1}\left(\mathbb{B}_{p+1}(n+1) - \mathbb{B}_{p+1}(0)\right) - \delta_{0,p}.$$

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REFERENCES

[1] A. Beardon, "Sums of powers of integers." *Am. Math. Mon.*, vol. 103, no. 3, pp. 201–213, 1996, doi: 10.2307/2975368.
[2] R. Chapman, "Evaluating $\sum_{n=1}^{N}(a+nd)^p$ again," *The Mathematical Gazette*, vol. 92, pp. 92–94, 03 2008, doi: 10.1017/S0025557200182622.
[3] A. Edwards, "Sums of powers of integers: a little of the history." *Math. Gaz.*, vol. 66, pp. 22–28, 1982, doi: 10.2307/3617302.
[4] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics: a foundation for computer science. 2nd ed.*, 2nd ed. Amsterdam: Addison-Wesley Publishing Group, 1994.
[5] M. D. Hirschhorn, "Evaluating $\sum_{n=1}^{N}(a+nd)^p$," *The Mathematical Gazette*, vol. 90, pp. 114–116, 03 2006, doi: 10.1017/S0025557200179197.
[6] F. Howard, "Sums of powers of integers via generating functions." *Fibonacci Q.*, vol. 34, no. 3, pp. 244–256, 1996.
[7] Y. Inaba, "Hyper-sums of powers of integers and the Akiyama-Tanigawa matrix." *J. Integer Seq.*, vol. 8, no. 2, pp. art. 05.2.7, 6, 2005.
[8] D. Knuth, "Johann Faulhaber and sums of powers." *Math. Comput.*, vol. 61, no. 203, pp. 277–294, 1993, doi: 10.2307/2152953.
[9] T. C. Kotiah, "Sums of powers of integers – a review." *Int. J. Math. Educ. Sci. Technol.*, vol. 24, no. 6, pp. 863–874, 1993, doi: 10.1080/0020739930240611.
[10] I. Mező and J. L. Ramírez, "The linear algebra of the r-Whitney matrices." *Integral Transforms Spec. Funct.*, vol. 26, no. 3, pp. 213–225, 2015, doi: 10.1080/10652469.2014.984180.
[11] M. Rahmani, "Generalized Stirling transform." *Miskolc Math. Notes*, vol. 15, no. 2, pp. 677–690, 2014.
[12] H. M. Srivastava and J. Choi, *Zeta and q-zeta functions and associated series and integrals.* Amsterdam: Elsevier, 2012.
[13] H. Srivastava and P. G. Todorov, "An explicit formula for the generalized Bernoulli polynomials." *J. Math. Anal. Appl.*, vol. 130, no. 2, pp. 509–513, 1988, doi: 10.1016/0022-247X(88)90326-5.

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