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THETA FUNCTION IDENTITIES AND REPRESENTATION NUMBERS OF CERTAIN QUADRATIC FORMS IN TWELVE VARIABLES

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Abstract. In this paper using the (p,k)-parametrization of theta functions and Eisenstein Series, developed by Alaca, Alaca and Williams, we obtain some new theta function identities and then use them to derive explicit formulae for the number of representations of a positive integer n by certain quadratic forms

$$\sum_{k=1}^{6} a_k(x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2)$$

in twelve variables where $a_k \in \{1,2,4\}$.

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1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the set of natural numbers, non-negative integers, integers, rational numbers and complex numbers respectively so that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a_1,...,a_6 \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we let $R(a_1,...,a_6;n)$ denote the representation numbers of n by the form $a_1(x_1^2 + x_1x_2 + x_2^2) + a_2(x_3^2 + x_3x_4 + x_4^2) + a_3(x_5^2 + x_5x_6 + x_6^2) + a_4(x_7^2 + x_7x_8 + x_8^2) + a_5(x_9^2 + x_9x_{10} + x_{10}^2) + a_6(x_{11}^2 + x_{11}x_{12} + x_{12}^2)$, that is

$$R(a_1,...,a_6;n) := \\ = \text{card}\left\{ (x_1,...,x_{12}) \in \mathbb{Z}^{12} : n = \sum_{k=1}^{6} a_k(x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2) \right\}. \tag{1.1}$$

If l of $a_1,...,a_6$ are equal, say

$$a_i = a_{i+1} = \cdots = a_{i+l-1} = a \tag{1.2}$$

for convenience we indicate this in $R(a_1,...,a_6;n)$ by writing a^l for $a_i, a_{i+1},...,a_{i+l-1}$.

For $k \in \mathbb{N}$ and $n \in \mathbb{Q}$ the sum of divisor function is defined by

$$\sigma_k(n) = \begin{cases} \sum\limits_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \in \mathbb{Q}, n \notin \mathbb{N}. \end{cases} \tag{1.3}$$

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by the product formula

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}). \tag{1.4}$$

Through the remainder of the paper we take $q = q(z) := e^{2\pi i z}$ with $z \in H$ and so by (1.4) we have

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{1.5}$$

Let N be a positive integer. An eta quotient is defined to be a finite product of the form

$$f(z) = \prod_{0 \le \delta \setminus N} \eta^{r_\delta}(\delta z). \tag{1.6}$$

When all of the exponents r_δ are nonnegative, $f(z)$ is said to be an eta product. We now define the following eta products.

$$B_1(q) := \sum_{n=1}^{\infty} b_1(n) q^n = \eta^{12}(2z), \tag{1.7}$$

$$B_2(q) := \sum_{n=1}^{\infty} b_2(n) q^n = \eta^5(z)\eta^5(2z)\eta(3z)\eta(6z), \tag{1.8}$$

$$B_3(q) := \sum_{n=1}^{\infty} b_3(n) q^n = \eta^6(z)\eta^6(3z), \tag{1.9}$$

$$B_4(q) := \sum_{n=1}^{\infty} b_4(n) q^n = \eta^5(2z)\eta^5(4z)\eta(6z)\eta(12z), \tag{1.10}$$

$$B_5(q) := \sum_{n=1}^{\infty} b_5(n) q^n = \eta(z)\eta^5(3z)\eta^5(4z)\eta(12z), \tag{1.11}$$

$$B_6(q) := \sum_{n=1}^{\infty} b_6(n) q^n = \eta(z)\eta(2z)\eta^5(3z)\eta^5(6z), \tag{1.12}$$

$$B_7(q) := \sum_{n=1}^{\infty} b_7(n) q^n = \eta^{12}(6z). \tag{1.13}$$

Determination of representation number formulae for quadratic forms which are sums of binary quadratic form $x_1^2 + x_1x_2 + x_2^2$ was considered before by many mathematicians. See for example [5, 9, 12]. Among the studies some contains formulae for the forms in twelve variables. In a recent publication Yao and Xia [14] obtained formula for $R(1^6;n)$. The authors proved that

$$R(1^6;n) = \frac{252}{13}\sigma_5(n) - \frac{6804}{13}\sigma_5(n/3) + \frac{216}{13}a(n), \tag{1.14}$$

where

$$\sum_{n=1}^{\infty} a(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^6 (1-q^{3n})^6. \tag{1.15}$$

From (1.15) and (1.9) it is clear that $a(n) = b_3(n)$.

A similar formulae for $R(1^6;n)$ was given before by Lomadze [9]. Recently, Köklüce [7] has derived formulae for the convolution sums $W_{1,6}^{1,3}(n)$, $W_{2,3}^{1,3}(n)$, $W_{3,2}^{1,3}(n)$ and $W_{6,1}^{1,3}(n)$. As an application he used these evaluations to derive formulae for the representation numbers $R(1^5,2^1;n)$, $R(1^4,2^2;n)$ and $R(1^3,2^3;n)$. In another work he and his coauthor Eser [8] have found formulae for the convolutions sums $W_{1,12}^{1,3}(n)$, $W_{3,4}^{1,3}(n)$, $W_{4,3}^{1,3}(n)$ and $W_{12,1}^{1,3}(n)$ and then used them for deriving representation number formulae for some quadratic forms in twelve variables which are sums of squares. Köklüce has used another method developed by Lomadze to obtain explicit formulae for quadratic forms in twelve and sixteen variables which are direct sums of binary quadratic forms with discriminant -23 in [6].

In the present paper, we obtain formulae for 21 quadratic forms in twelve variables by using theta function identities. These formulae are given in terms of $\sigma_5(n)$ and the integers $b_i(n)$, ($i = 1,2,3,4,5,6,7$ and $n \in \mathbb{N}$) given by the equation (1.7)-(1.13). We compare our formula for $R(1^6;n)$ with results of Yao and Xia [14] and formulae for $R(1^5,2^1;n)$, $R(1^4,2^2;n)$ and $R(1^3,2^3;n)$ with previous work of Köklüce [7] and see that they are consistent. Similar methods have been used before for deriving representation numbers formulae for sextenary and octonary quadratic forms. See for example [4] and [13], respectively.

The rest of this paper is organized as follows. In Sec. 2, we state our main theorem. In Sec. 3, we give parametrization of theta functions and Eisenstein series in terms of p and k. In Sec. 4, we give a lemma which contains required theta function identities for the proof of the main theorem. In Sec. 5, we prove the main theorem. Finally, in Sec. 6, we give a concluding remark.

2. Statement of the Theorem

Theorem 1. *Let $n \in \mathbb{N}$ then,*

$$\text{(i)}\quad R(1^6;n) = \frac{252}{13}\sigma_5(n) - \frac{6804}{13}\sigma_5(n/3) + \frac{216}{13}b_3(n), \tag{2.1}$$

$$\text{(ii)}\quad R(1^5,2^1;n) = \frac{66}{7}\sigma_5(n) - \frac{192}{7}\sigma_5(n/2) + \frac{1782}{7}\sigma_5(n/3) - \frac{5184}{7}\sigma_5(n/6) + \frac{144}{7}b_2(n) + \frac{1296}{7}b_6(n), \tag{2.2}$$

$$\text{(iii)}\quad R(1^4,2^2;n) = \frac{60}{13}\sigma_5(n) + \frac{192}{13}\sigma_5(n/2) - \frac{1620}{13}\sigma_5(n/3) - \frac{5184}{13}\sigma_5(n/6) - \frac{288}{13}b_2(n) + \frac{540}{13}b_3(n) + \frac{2592}{13}b_6(n), \tag{2.3}$$

$$\text{(iv)}\quad R(1^3,2^3;n) = \frac{18}{7}\sigma_5(n) - \frac{144}{7}\sigma_5(n/2) + \frac{486}{7}\sigma_5(n/3) - \frac{3888}{7}\sigma_5(n/6) + \frac{108}{7}b_2(n) + \frac{972}{7}b_6(n), \tag{2.4}$$

$$\text{(v)}\quad R(1^2,2^4;n) = \frac{12}{13}\sigma_5(n) + \frac{240}{13}\sigma_5(n/2) - \frac{324}{13}\sigma_5(n/3) - \frac{6480}{13}\sigma_5(n/6) - \frac{126}{13}b_2(n) + \frac{270}{13}b_3(n) + \frac{1134}{13}b_6(n), \tag{2.5}$$

$$\text{(vi)}\quad R(1^1,2^5;n) = \frac{6}{7}\sigma_5(n) - \frac{132}{7}\sigma_5(n/2) + \frac{162}{7}\sigma_5(n/3) - \frac{3564}{7}\sigma_5(n/6) + \frac{36}{7}b_2(n) + \frac{324}{7}b_6(n), \tag{2.6}$$

$$\text{(vii)}\quad R(1^5,4^1;n) = \frac{66}{13}\sigma_5(n) - \frac{198}{13}\sigma_5(n/2) - \frac{1782}{13}\sigma_5(n/3) + \frac{384}{13}\sigma_5(n/4) + \frac{5346}{13}\sigma_5(n/6) - \frac{10368}{13}\sigma_5(n/12) + \frac{774}{13}b_1(n) - \frac{1260}{13}b_2(n) + \frac{810}{13}b_3(n) - \frac{16128}{13}b_4(n) + \frac{24192}{13}b_5(n) - \frac{6804}{13}b_6(n) + \frac{33534}{13}b_7(n), \tag{2.7}$$

$$\text{(viii)}\quad R(1^4,4^2;n) = \frac{15}{13}\sigma_5(n) + \frac{45}{13}\sigma_5(n/2) - \frac{405}{13}\sigma_5(n/3) + \frac{192}{13}\sigma_5(n/4) - \frac{1215}{13}\sigma_5(n/6) - \frac{5184}{13}\sigma_5(n/12) + \frac{387}{13}b_1(n) - \frac{846}{13}b_2(n) + \frac{756}{13}b_3(n) - \frac{8064}{13}b_4(n) + \frac{12096}{13}b_5(n) - \frac{1458}{13}b_6(n) + \frac{16767}{13}b_7(n), \tag{2.8}$$

$$\text{(ix)}\quad R(1^3,4^3;n)=\frac{9}{26}\sigma_5(n)-\frac{81}{26}\sigma_5(n/2)-\frac{243}{26}\sigma_5(n/3)+\frac{288}{13}\sigma_5(n/4)$$
$$+\frac{2187}{26}\sigma_5(n/6)-\frac{7776}{13}\sigma_5(n/12)+\frac{459}{26}b_1(n)-\frac{3591}{104}b_2(n)+\frac{3591}{104}b_3(n)$$
$$-\frac{3672}{13}b_4(n)+\frac{5508}{13}b_5(n)-\frac{729}{104}b_6(n)+\frac{12393}{26}b_7(n), \tag{2.9}$$

$$\text{(x)}\quad R(1^2,4^4;n)=\frac{3}{52}\sigma_5(n)+\frac{45}{52}\sigma_5(n/2)-\frac{81}{52}\sigma_5(n/3)+\frac{240}{13}\sigma_5(n/4)$$
$$-\frac{1215}{52}\sigma_5(n/6)-\frac{6480}{13}\sigma_5(n/12)+\frac{531}{52}b_1(n)-\frac{801}{52}b_2(n)+\frac{891}{52}b_3(n)$$
$$-\frac{1656}{13}b_4(n)+\frac{2484}{13}b_5(n)-\frac{243}{52}b_6(n)+\frac{8019}{52}b_7(n), \tag{2.10}$$

$$\text{(xi)}\quad R(1^1,4^5;n)=\frac{3}{104}\sigma_5(n)-\frac{99}{104}\sigma_5(n/2)-\frac{81}{104}\sigma_5(n/3)+\frac{264}{13}\sigma_5(n/4)$$
$$+\frac{2673}{104}\sigma_5(n/6)-\frac{7128}{13}\sigma_5(n/12)+\frac{99}{26}b_1(n)-\frac{1035}{208}b_2(n)+\frac{1485}{208}b_3(n)$$
$$-\frac{558}{13}b_4(n)+\frac{837}{13}b_5(n)-\frac{729}{208}b_6(n)+\frac{2187}{52}b_7(n) \tag{2.11}$$

$$\text{(xii)}\quad R(1^4,2^1,4^1;n)=\frac{15}{7}\sigma_5(n)+\frac{51}{7}\sigma_5(n/2)+\frac{405}{7}\sigma_5(n/3)-\frac{192}{7}\sigma_5(n/4)$$
$$+\frac{1377}{7}\sigma_5(n/6)-\frac{5184}{7}\sigma_5(n/12)+\frac{333}{7}b_1(n)+\frac{36}{7}b_2(n)-\frac{216}{7}b_3(n)$$
$$+\frac{3456}{7}b_5(n)-\frac{3564}{7}b_6(n)+\frac{8991}{7}b_7(n), \tag{2.12}$$

$$\text{(xiii)}\quad R(1^3,2^2,4^1;n)=\frac{18}{13}\sigma_5(n)-\frac{150}{13}\sigma_5(n/2)-\frac{486}{13}\sigma_5(n/3)+\frac{384}{13}\sigma_5(n/4)$$
$$+\frac{4050}{13}\sigma_5(n/6)-\frac{10368}{13}\sigma_5(n/12)+\frac{306}{13}b_1(n)-\frac{909}{26}b_2(n)+\frac{729}{26}b_3(n)$$
$$-\frac{4896}{13}b_4(n)+\frac{7344}{13}b_5(n)-\frac{2835}{26}b_6(n)+\frac{8262}{13}b_7(n), \tag{2.13}$$

$$\text{(xiv)}\quad R(1^3,2^1,4^2;n)=\frac{9}{14}\sigma_5(n)-\frac{69}{14}\sigma_5(n/2)+\frac{243}{14}\sigma_5(n/3)-\frac{96}{7}\sigma_5(n/4)$$
$$-\frac{1863}{14}\sigma_5(n/6)-\frac{2592}{7}\sigma_5(n/12)+\frac{333}{14}b_1(n)+9b_2(n)-\frac{108}{7}b_3(n)$$
$$+\frac{1728}{7}b_5(n)-\frac{1377}{7}b_6(n)+\frac{8991}{14}b_7(n), \tag{2.14}$$

(xv) $$R(1^2,2^3,4^1;n) = \frac{3}{7}\sigma_5(n) + 9\sigma_5(n/2) + \frac{81}{7}\sigma_5(n/3) - \frac{192}{7}\sigma_5(n/4) + 243\sigma_5(n/6) - \frac{5184}{7}\sigma_5(n/12) + \frac{81}{7}b_1(n) + \frac{27}{7}b_2(n) - \frac{27}{7}b_3(n) + \frac{432}{7}b_5(n) - \frac{243}{7}b_6(n) + \frac{2187}{7}b_7(n), \tag{2.15}$$

(xvi) $$R(1^2,2^2,4^2;n) = \frac{3}{13}\sigma_5(n) + \frac{57}{13}\sigma_5(n/2) - \frac{81}{13}\sigma_5(n/3) + \frac{192}{13}\sigma_5(n/4) - \frac{1539}{13}\sigma_5(n/6) - \frac{5184}{13}\sigma_5(n/12) + \frac{153}{13}b_1(n) - \frac{999}{52}b_2(n) + \frac{999}{52}b_3(n) - \frac{2448}{13}b_4(n) + \frac{3672}{13}b_5(n) - \frac{2025}{52}b_6(n) + \frac{4131}{13}b_7(n), \tag{2.16}$$

(xvii) $$R(1^2,2^1,4^3;n) = \frac{3}{28}\sigma_5(n) + \frac{69}{28}\sigma_5(n/2) + \frac{81}{28}\sigma_5(n/3) - \frac{144}{7}\sigma_5(n/4) + \frac{1863}{28}\sigma_5(n/6) - \frac{3888}{7}\sigma_5(n/12) + \frac{243}{28}b_1(n) + \frac{171}{28}b_2(n) - \frac{81}{28}b_3(n) + \frac{324}{7}b_5(n) + \frac{81}{28}b_6(n) + \frac{6561}{28}b_7(n), \tag{2.17}$$

(xviii) $$R(1^1,2^4,4^1;n) = \frac{6}{13}\sigma_5(n) - \frac{138}{13}\sigma_5(n/2) - \frac{162}{13}\sigma_5(n/3) + \frac{384}{13}\sigma_5(n/4) + \frac{3726}{13}\sigma_5(n/6) - \frac{10368}{13}\sigma_5(n/12) + \frac{72}{13}b_1(n) - \frac{243}{26}b_2(n) + \frac{243}{26}b_3(n) - \frac{1152}{13}b_4(n) + \frac{1728}{13}b_5(n) - \frac{405}{26}b_6(n) + \frac{1944}{13}b_7(n), \tag{2.18}$$

(xix) $$R(1^1,2^3,4^2;n) = \frac{3}{14}\sigma_5(n) - \frac{9}{2}\sigma_5(n/2) + \frac{81}{14}\sigma_5(n/3) - \frac{96}{7}\sigma_5(n/4) - \frac{243}{2}\sigma_5(n/6) - \frac{2592}{7}\sigma_5(n/12) + \frac{81}{14}b_1(n) + \frac{27}{14}b_2(n) - \frac{27}{14}b_3(n) + \frac{216}{7}b_5(n) - \frac{243}{14}b_6(n) + \frac{2187}{14}b_7(n), \tag{2.19}$$

(xx) $$R(1^1,2^2,4^3;n) = \frac{3}{26}\sigma_5(n) - \frac{75}{26}\sigma_5(n/2) - \frac{81}{26}\sigma_5(n/3) + \frac{288}{13}\sigma_5(n/4) + \frac{2025}{26}\sigma_5(n/6) - \frac{7776}{13}\sigma_5(n/12) + \frac{54}{13}b_1(n) - \frac{63}{8}b_2(n) + \frac{999}{104}b_3(n) - \frac{864}{13}b_4(n) + \frac{1296}{13}b_5(n) - \frac{405}{104}b_6(n) + \frac{1458}{13}b_7(n), \tag{2.20}$$

(xxi) $R(1^1,2^1,4^4;n) = \frac{3}{56}\sigma_5(n) - \frac{51}{56}\sigma_5(n/2) + \frac{81}{56}\sigma_5(n/3) - \frac{120}{7}\sigma_5(n/4)$
$- \frac{1377}{56}\sigma_5(n/6) - \frac{3240}{7}\sigma_5(n/12) + \frac{153}{56}b_1(n) + \frac{9}{4}b_2(n) + \frac{27}{28}b_3(n)$
$- \frac{108}{7}b_5(n) + \frac{1053}{28}b_6(n) + \frac{4131}{56}b_7(n).$ (2.21)

3. THE (p,k)-PARAMETRIZATION OF THETA FUNCTIONS AND EISENSTEIN SERIES

In his second notebook [11] Ramanujan gives the definitions of Eisenstein series $L(q)$, $M(q)$ and $N(q)$ by

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{3.1}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} \tag{3.2}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{3.3}$$

It can be easily seen that

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \tag{3.4}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \tag{3.5}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n. \tag{3.6}$$

For $q \in \mathbb{C}$, $|q| < 1$ the Jacobi one-dimensional theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{3.7}$$

The Borweins' two-dimensional theta function $a(q)$ is defined by

$$a(q) := \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2+mn+n^2}, \tag{3.8}$$

so that for $a_1,\ldots,a_6 \in \mathbb{N}$ we have

$$\sum_{n=0}^{\infty} N(a_1,...,a_6;n)q^n = a(q^{a_1})...a(q^{a_6}). \tag{3.9}$$

Alaca, Alaca and Williams [2] defined p and k respectively by

$$p = p(q) := \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)} \tag{3.10}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}\cdot \tag{3.11}$$

Alaca, Alaca and Williams [2] derived formulae for $a(q), a(q^2)$ and $a(q^4)$ in terms of p and k. They have proved that

$$a(q) = (1+4p+p^2)k, \tag{3.12}$$

$$a(q^2) = (1+p+p^2)k \tag{3.13}$$

and

$$a(q^4) = (1+p-\frac{1}{2}p^2)k. \tag{3.14}$$

Formulae for the series $N(q), N(q^2), N(q^3), N(q^4), N(q^6)$ and $N(q^{12})$ in terms of p and k are determined by the same authors [1]. Equations (3.22)-(3.27) are respectively as follows

$$N(q) = (1-246p-5532p^2-38614p^3-135369p^4-276084p^5-348024p^6 \\ -276084p^7-135369p^8-38614p^9-5532p^{10}-246p^{11}+p^{12})k^6, \tag{3.15}$$

$$N(q^2) = (1+6p-114p^2-625p^3-\frac{4059}{2}p^4-4302p^5-5556p^6-4302p^7 \\ -\frac{4059}{2}p^8-625p^9-114p^{10}+6p^{11}+p^{12})k^6, \tag{3.16}$$

$$N(q^3) =(1+6p+12p^2-58p^3-297p^4-396p^5-264p^6-396p^7-297p^8 \\ -58p^9+12p^{10}+6p^{11}+p^{12})k^6, \tag{3.17}$$

$$N(q^4) =(1+6p+12p^2+5p^3-45p^4-144p^5-\frac{1167}{8}p^6+\frac{171}{8}p^7+\frac{2151}{32}p^8 -\frac{739}{16}p^9-\frac{345}{8}p^{10}+\frac{129}{32}p^{11}+\frac{1}{64}p^{12})k^6, \tag{3.18}$$

$$N(q^6) =(1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5-12p^6-18p^7-\frac{27}{2}p^8+5p^9 +12p^{10}+6p^{11}+p^{12})k^6, \tag{3.19}$$

and

$$N(q^{12}) =(1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5-\frac{33}{8}p^6+\frac{45}{8}p^7+\frac{135}{32}p^8 +\frac{17}{16}p^9+\frac{3}{16}p^{10}+\frac{3}{32}p^{11}+\frac{1}{64}p^{12})k^6. \tag{3.20}$$

The Ramanujan discriminant function $\triangle(q)$ [10] is defined by

$$\triangle(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Alaca, Alaca and Williams [1] expressed $\triangle(q^i)$ ($i = 1,2,3,4,6$ and 12) in terms of p and k. They solved these equations for $\prod_{n=1}^{\infty}(1-q^{in})$, ($i = 1,2,3,4,6$ and 12) and obtained the following equations in [3].

$$\prod_{n=1}^{\infty}(1-q^n) = q^{-\frac{1}{24}}2^{-\frac{1}{6}}p^{\frac{1}{24}}(1-p)^{\frac{1}{2}}(1+p)^{\frac{1}{6}}(1+2p)^{\frac{1}{8}}(2+p)^{\frac{1}{8}}k^{\frac{1}{2}}, \tag{3.21}$$

$$\prod_{n=1}^{\infty}(1-q^{2n}) = q^{-\frac{1}{12}}2^{-\frac{1}{3}}p^{\frac{1}{12}}(1-p)^{\frac{1}{4}}(1+p)^{\frac{1}{12}}(1+2p)^{\frac{1}{4}}(2+p)^{\frac{1}{4}}k^{\frac{1}{2}}, \tag{3.22}$$

$$\prod_{n=1}^{\infty}(1-q^{3n}) = q^{-\frac{1}{8}}2^{-\frac{1}{6}}p^{\frac{1}{8}}(1-p)^{\frac{1}{6}}(1+p)^{\frac{1}{2}}(1+2p)^{\frac{1}{24}}(2+p)^{\frac{1}{24}}k^{\frac{1}{2}}, \tag{3.23}$$

$$\prod_{n=1}^{\infty}(1-q^{4n}) = q^{-\frac{1}{6}}2^{-\frac{2}{3}}p^{\frac{1}{6}}(1-p)^{\frac{1}{8}}(1+p)^{\frac{1}{24}}(1+2p)^{\frac{1}{8}}(2+p)^{\frac{1}{2}}k^{\frac{1}{2}}, \tag{3.24}$$

$$\prod_{n=1}^{\infty}(1-q^{6n}) = q^{-\frac{1}{4}}2^{-\frac{1}{3}}p^{\frac{1}{4}}(1-p)^{\frac{1}{12}}(1+p)^{\frac{1}{4}}(1+2p)^{\frac{1}{12}}(2+p)^{\frac{1}{12}}k^{\frac{1}{2}} \tag{3.25}$$

and

$$\prod_{n=1}^{\infty}(1-q^{12n}) = q^{-\frac{1}{2}}2^{-\frac{2}{3}}p^{\frac{1}{2}}(1-p)^{\frac{1}{24}}(1+p)^{\frac{1}{8}}(1+2p)^{\frac{1}{24}}(2+p)^{\frac{1}{6}}k^{\frac{1}{2}}. \quad (3.26)$$

Using (3.21)-(3.26) we obtained the following identities.

$$\eta^{12}(2z) = \frac{1}{16}p(1-p)^3(1+p)(1+2p)^3(2+p)^3k^6, \quad (3.27)$$

$$\eta^5(z)\eta^5(2z)\eta(3z)\eta(6z) = \frac{1}{8}p(1-p)^4(1+p)^2(1+2p)^2(2+p)^2k^6, \quad (3.28)$$

$$\eta^6(z)\eta^6(3z) = \frac{1}{4}p(1-p)^4(1+p)^4(1+2p)(2+p)k^6, \quad (3.29)$$

$$\eta^5(2z)\eta^5(4z)\eta(6z)\eta(12z) = \frac{1}{64}p^2(1-p)^2(1+p)(1+2p)^2(2+p)^4k^6, \quad (3.30)$$

$$\eta(z)\eta^5(3z)\eta^5(4z)\eta(12z) = \frac{1}{32}p^2(1-p)^2(1+p)^3(1+2p)(2+p)^3k^6, \quad (3.31)$$

$$\eta(z)\eta(2z)\eta^5(3z)\eta^5(6z) = \frac{1}{8}p^2(1-p)^2(1+p)^4(1+2p)(2+p)k^6, \quad (3.32)$$

$$\eta^{12}(6z) = \frac{1}{16}p^3(1-p)(1+p)^3(1+2p)(2+p)k^6. \quad (3.33)$$

4. Theta function identities

Theorem 2.

(i) $$a^6(q) = -\frac{1}{26}N(q) + \frac{27}{26}N(q^3) + \frac{216}{13}B_3(q), \quad (4.1)$$

(ii) $$a^5(q)a(q^2) = \frac{-11}{588}N(q) + \frac{8}{147}N(q^2) - \frac{99}{196}N(q^3) + \frac{72}{49}N(q^6) + \frac{144}{7}B_2(q) + \frac{1296}{7}B_6(q), \quad (4.2)$$

(iii) $$a^4(q)a^2(q^2) = -\frac{5}{546}N(q) - \frac{8}{273}N(q^2) + \frac{45}{182}N(q^3) + \frac{72}{91}N(q^6) - \frac{288}{13}B_2(q) + \frac{540}{13}B_3(q) + \frac{2592}{13}B_6(q), \tag{4.3}$$

(iv) $$a^3(q)a^3(q^2) = -\frac{1}{196}N(q) + \frac{2}{49}N(q^2) - \frac{27}{196}N(q^3) + \frac{54}{49}N(q^6) + \frac{108}{7}B_2(q) + \frac{972}{7}B_6(q), \tag{4.4}$$

(v) $$a^2(q)a^4(q^2) = -\frac{1}{546}N(q) - \frac{10}{273}N(q^2) + \frac{9}{182}N(q^3) + \frac{90}{91}N(q^6) - \frac{126}{13}B_2(q) + \frac{270}{13}B_3(q) + \frac{1134}{13}B_6(q), \tag{4.5}$$

(vi) $$a(q)a^5(q^2) = -\frac{1}{588}N(q) + \frac{11}{294}N(q^2) - \frac{9}{196}N(q^3) + \frac{99}{98}N(q^6) + \frac{36}{7}B_2(q) + \frac{324}{7}B_6(q), \tag{4.6}$$

(vii) $$a^5(q)a(q^4) = -\frac{11}{1092}N(q) + \frac{11}{364}N(q^2) + \frac{99}{364}N(q^3) - \frac{16}{273}N(q^4) - \frac{297}{364}N(q^6) + \frac{144}{91}N(q^{12}) + \frac{774}{13}B_1(q) - \frac{1260}{13}B_2(q) + \frac{810}{13}B_3(q) - \frac{16128}{13}B_4(q) + \frac{24192}{13}B_5(q) - \frac{6804}{13}B_6(q) + \frac{33534}{13}B_7(q), \tag{4.7}$$

(viii) $$a^4(q)a^2(q^4) = -\frac{5}{2184}N(q) - \frac{5}{728}N(q^2) + \frac{45}{728}N(q^3) - \frac{8}{273}N(q^4) + \frac{135}{728}N(q^6) + \frac{72}{91}N(q^{12}) + \frac{387}{13}B_1(q) - \frac{846}{13}B_2(q) + \frac{756}{13}B_3(q) - \frac{8064}{13}B_4(q) + \frac{12096}{13}B_5(q) - \frac{1458}{13}B_6(q) + \frac{16767}{13}B_7(q), \tag{4.8}$$

(ix) $$a^3(q)a^3(q^4) = -\frac{1}{1456}N(q) + \frac{9}{1456}N(q^2) + \frac{27}{1456}N(q^3) - \frac{4}{91}N(q^4) - \frac{243}{1456}N(q^6) + \frac{108}{91}N(q^{12}) + \frac{459}{26}B_1(q) - \frac{3591}{104}B_2(q) + \frac{3591}{104}B_3(q) - \frac{3672}{13}B_4(q) + \frac{5508}{13}B_5(q) - \frac{729}{104}B_6(q) + \frac{12393}{26}B_7(q), \tag{4.9}$$

(x) $$a^2(q)a^4(q^4) = -\frac{1}{8736}N(q) - \frac{5}{2912}N(q^2) + \frac{9}{2912}N(q^3) - \frac{10}{273}N(q^4) + \frac{135}{2912}N(q^6) + \frac{90}{91}N(q^{12}) + \frac{531}{52}B_1(q) - \frac{801}{52}B_2(q) + \frac{891}{52}B_3(q)$$

$$-\frac{1656}{13}B_4(q)+\frac{2484}{13}B_5(q)-\frac{243}{52}B_6(q)+\frac{8019}{52}B_7(q), \quad (4.10)$$

(xi) $$a(q)a^5(q^4)=-\frac{1}{17472}N(q)+\frac{11}{5824}N(q^2)+\frac{9}{5824}N(q^3)-\frac{11}{273}N(q^4)$$
$$-\frac{297}{5824}N(q^6)+\frac{99}{91}N(q^{12})+\frac{99}{26}B_1(q)-\frac{1035}{208}B_2(q)+\frac{1485}{208}B_3(q)$$
$$-\frac{558}{13}B_4(q)+\frac{837}{13}B_5(q)-\frac{729}{208}B_6(q)+\frac{2187}{52}B_7(q), \quad (4.11)$$

(xii) $$a^4(q)a(q^2)a(q^4)=-\frac{5}{1176}N(q)-\frac{17}{1176}N(q^2)-\frac{45}{392}N(q^3)$$
$$+\frac{8}{147}N(q^4)-\frac{153}{392}N(q^6)+\frac{72}{49}N(q^{12})+B_1(q)+\frac{36}{7}B_2(q)-\frac{216}{7}B_3(q)$$
$$+\frac{3456}{7}B_5(q)-\frac{3564}{7}B_6(q)+\frac{8991}{7}B_7(q), \quad (4.12)$$

(xiii) $$a^3(q)a^2(q^2)a(q^4)=-\frac{1}{364}N(q)+\frac{25}{1092}N(q^2)+\frac{27}{364}N(q^3)-\frac{16}{273}N(q^4)$$
$$-\frac{225}{364}N(q^6)+\frac{144}{91}N(q^{12})+\frac{306}{13}B_1(q)-\frac{909}{26}B_2(q)+\frac{729}{26}B_3(q)$$
$$-\frac{4896}{13}B_4(q)+\frac{7344}{13}B_5(q)-\frac{2835}{26}B_6(q)+\frac{8262}{13}B_7(q), \quad (4.13)$$

(xiv) $$a^3(q)a(q^2)a^2(q^4)=-\frac{1}{784}N(q)+\frac{23}{2352}N(q^2)-\frac{27}{784}N(q^3)+\frac{4}{147}N(q^4)$$
$$+\frac{207}{784}N(q^6)+\frac{36}{49}N(q^{12})+\frac{333}{14}B_1(q)+9B_2(q)-\frac{108}{7}B_3(q)$$
$$+\frac{1728}{7}B_5(q)-\frac{1377}{7}B_6(q)+\frac{8991}{14}B_7(q), \quad (4.14)$$

(xv) $$a^2(q)a^3(q^2)a(q^4)=-\frac{1}{1176}N(q)-\frac{1}{56}N(q^2)-\frac{9}{392}N(q^3)+\frac{8}{147}N(q^4)$$
$$-\frac{27}{56}N(q^6)+\frac{72}{49}N(q^{12})+B_1(q)+\frac{27}{7}B_2(q)-\frac{27}{7}B_3(q)$$
$$+\frac{432}{7}B_5(q)-\frac{243}{7}B_6(q)+\frac{2187}{7}B_7(q), \quad (4.15)$$

(xvi) $$a^2(q)a^2(q^2)a^2(q^4)=-\frac{1}{2184}N(q)-\frac{19}{2184}N(q^2)+\frac{9}{728}N(q^3)-\frac{8}{273}N(q^4)$$
$$+\frac{171}{728}N(q^6)+\frac{72}{91}N(q^{12})+\frac{153}{13}B_1(q)-\frac{999}{52}B_2(q)+\frac{999}{52}B_3(q)$$
$$-\frac{2448}{13}B_4(q)+\frac{3672}{13}B_5(q)-\frac{2025}{52}B_6(q)+\frac{4131}{13}B_7(q), \quad (4.16)$$

$$\text{(xvii)}\ a^2(q)a(q^2)a^3(q^4) = -\frac{1}{4704}N(q) - \frac{23}{4704}N(q^2) - \frac{9}{1568}N(q^3) + \frac{2}{49}N(q^4)$$
$$-\frac{207}{1568}N(q^6) + \frac{54}{49}N(q^{12}) + \frac{243}{28}B_1(q) + \frac{171}{28}B_2(q) - \frac{81}{28}B_3(q)$$
$$+\frac{324}{7}B_5(q) + \frac{81}{28}B_6(q) + \frac{6561}{28}B_7(q), \tag{4.17}$$

$$\text{(xviii)}\ a(q)a^4(q^2)a(q^4) = -\frac{1}{1092}N(q) + \frac{23}{1092}N(q^2) + \frac{9}{364}N(q^3) - \frac{16}{273}N(q^4)$$
$$-\frac{207}{364}N(q^6) + \frac{144}{91}N(q^{12}) + \frac{72}{13}B_1(q) - \frac{243}{26}B_2(q) + \frac{243}{26}B_3(q)$$
$$-\frac{1152}{13}B_4(q) + \frac{1728}{13}B_5(q) - \frac{405}{26}B_6(q) + \frac{1944}{13}B_7(q), \tag{4.18}$$

$$\text{(xix)}\ a(q)a^3(q^2)a^2(q^4) = -\frac{1}{2352}N(q) + \frac{1}{112}N(q^2) - \frac{9}{784}N(q^3) + \frac{4}{147}N(q^4)$$
$$+\frac{27}{112}N(q^6) + \frac{36}{49}N(q^{12}) + \frac{81}{14}B_1(q) + \frac{27}{14}B_2(q) - \frac{27}{14}B_3(q)$$
$$+\frac{216}{7}B_5(q) - \frac{243}{14}B_6(q) + \frac{2187}{14}B_7(q), \tag{4.19}$$

$$\text{(xx)}\ a(q)a^2(q^2)a^3(q^4) = -\frac{1}{4368}N(q) + \frac{25}{4368}N(q^2) + \frac{9}{1456}N(q^3) - \frac{4}{91}N(q^4)$$
$$-\frac{225}{1456}N(q^6) + \frac{108}{91}N(q^{12}) + \frac{54}{13}B_1(q) - \frac{63}{8}B_2(q) + \frac{999}{104}B_3(q)$$
$$-\frac{864}{13}B_4(q) + \frac{1296}{13}B_5(q) - \frac{405}{104}B_6(q) + \frac{1458}{13}B_7(q), \tag{4.20}$$

$$\text{(xxi)}\ a(q)a(q^2)a^4(q^4) = -\frac{1}{9408}N(q) + \frac{17}{9408}N(q^2) - \frac{9}{3136}N(q^3) + \frac{5}{147}N(q^4)$$
$$+\frac{153}{3136}N(q^6) + \frac{45}{49}N(q^{12}) + \frac{153}{56}B_1(q) + \frac{9}{4}B_2(q) + \frac{27}{28}B_3(q)$$
$$-\frac{108}{7}B_5(q) + \frac{1053}{28}B_6(q) + \frac{4131}{56}B_7(q). \tag{4.21}$$

Proof. We just prove part (ii). The rest can be proved similarly. From (3.12), (3.13), (3.15), (3.16), (3.17), (3.19), (1.8) and (1.12) we see that

$$\frac{-11}{588}N(q) + \frac{8}{147}N(q^2) - \frac{99}{196}N(q^3) + \frac{72}{49}N(q^6)$$
$$+\frac{144}{7}\eta^5(z)\eta^5(2z)\eta(3z)\eta(6z) + \frac{1296}{7}\eta(z)\eta(2z)\eta^5(3z)\eta^5(6z)$$
$$= (1+21p+186p^2+905p^3+2655p^4+4914p^5+5964p^6+4914p^7$$
$$+2655p^8+905p^9+186p^{10}+21p^{11}+p^{12})k^6$$

$$= (1+4p+p^2)^5(1+p+p^2)k^6$$
$$= a^5(q)a(q^2).$$

□

5. Proof of Theorem 1

We just prove part (ii) as the remaining parts can be proved similarly.

Proof. From (3.9), (3.12), (3.13) and Theorem 2 we have

$$\sum_{n=0}^{\infty} R(1^5,2^1;n)q^n = a^5(q)a(q^2) = (1+4p+p^2)^5(1+p+p^2)k^6$$
$$= \frac{-11}{588}N(q) + \frac{8}{147}N\left(q^2\right) - \frac{99}{196}N\left(q^3\right) + \frac{72}{49}N\left(q^6\right)$$
$$+ \frac{144}{7}\eta^5(z)\eta^5(2z)\eta(3z)\eta(6z) + \frac{1296}{7}\eta(z)\eta(2z)\eta^5(3z)\eta^5(6z)$$
$$= \frac{-11}{588}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right) + \frac{8}{147}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)$$
$$-\frac{99}{196}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\right) + \frac{72}{49}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}\right)$$
$$+\frac{144}{7}\sum_{n=1}^{\infty}b_2(n)q^n + \frac{1296}{7}\sum_{n=1}^{\infty}b_6(n)q^n$$
$$= 1 + \sum_{n=1}^{\infty}\left(\frac{66}{7}\sigma_5(n) - \frac{192}{7}\sigma_5\left(\frac{n}{2}\right) + \frac{1782}{7}\sigma_5\left(\frac{n}{3}\right) - \frac{5184}{7}\sigma_5\left(\frac{n}{6}\right)\right.$$
$$\left.+\frac{144}{7}b_2(n) + \frac{1296}{7}b_6(n)\right)q^n.$$

For $n \in \mathbb{N}$, equating the coefficients of q^n on both sides in the above equations, we obtain (2.2). □

Denoting the right hand side of (2.2) by $S(1^5,2^1;n)$ we give the first ten values of $R(1^5,2^1;n)$ and $S(1^5,2^1;n)$ in Table 1 to illustrate the equation.

6. Concluding remark

We note that there is no non-trivial linear relationships between the series $N(q^i)$($i = 1,2,3,4,6$ and 12) and $B_j(q)$,($j = 1,2,...7$). However, it is clear from (1.7), (1.8), (1.10) and (1.13) that $B_7(q) = B_1(q^3)$ and $B_4(q) = B_2(q^2)$. Thus

$$b_7(n) = b_1(\frac{n}{3}), b_4(n) = b_2(\frac{n}{2}) \tag{6.1}$$

and hence the number of cusp forms used in formulae given in Theorem 1 could be reduced to 5.

TABLE 1. The first ten values of $R(1^5,2^1;n)$ and $S(1^5,2^1;n)$.

n	$R(1^5,2^1;n)$	$\sigma_5(n)$	$\sigma_5(\frac{n}{2})$	$\sigma_5(\frac{n}{3})$	$\sigma_5(\frac{n}{6})$	$b_2(n)$	$b_6(n)$	$S(1^5,2^1;n)$
1	30	1	0	0	0	1	0	30
2	366	33	1	0	0	-5	1	366
3	2370	244	0	1	0	0	-1	2370
4	9390	1057	33	0	0	34	-2	9390
5	28116	3126	0	0	0	-30	-4	28116
6	76146	8052	244	33	1	-81	5	76146
7	162096	16808	0	0	0	68	12	162096
8	291246	33825	1057	0	0	100	-4	291246
9	622830	59293	0	244	0	81	0	622830
10	881460	103158	3126	0	0	-174	-10	881460

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