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A SIMULATION FUNCTION APPROACH FOR BEST PROXIMITY POINT AND VARIATIONAL INEQUALITY PROBLEMS

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Abstract. We study sufficient conditions for existence of solutions to the global optimization problem $\min_{x\in A} d(x, fx)$, where A, B are nonempty subsets of a metric space (X,d) and $f : A \to B$ belongs to the class of proximal simulative contraction mappings. Our results unify, improve and generalize various comparable results in the existing literature on this topic. As an application of the obtained theorems, we give some solvability theorems of a variational inequality problem.

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1. INTRODUCTION

Let A and B be nonempty subsets of a metric space (X,d) and $f : A \to B$ an arbitrary mapping on A. A fixed point problem defined by a pair (A,B) of sets and a mapping f is to find a point x^* in A such that $d(x^*, fx^*) = 0$. It is clear that if $A \cap B = \varnothing$, the fixed point problem defined by (A,B) and f has no solution. In this case, a point x^* in A satisfying

$$d(x^*, fx^*) \le d(a, fx^*)$$

for every a in A is the nearest point to $fx^* \in B$ in A. Such a point is called an approximate fixed point of f.

It is a reasonable demand to settle down with such a point when an operator equation $fx = x$ does not admit a solution.

The study of conditions that assure existence and uniqueness of approximate fixed point of a mapping f is an important area of research.

Suppose that $\triangle_{AB} = d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$ is the measure of a distance between two sets A and B. A point x^* in A is called a best proximity point of f if

$$d(x^*, fx^*) = \triangle_{AB}.$$

Best proximity point results deal with sufficient conditions under which the nonlinear minimization problem

$$\min_{x\in A} d(x, fx)$$

has at least one solution. Thus a best proximity point problem defined by a mapping f and a pair (A, B) of sets is to find a point x^* in A such that

$$d(x^*, fx^*) = \triangle_{AB}.$$

If $A = B$, the best proximity point problem reduces to a fixed point problem.

In light of this, a best proximity point problem can be viewed as a natural generalization of the fixed point problem. Also, results dealing with existence and uniqueness of best proximity point of certain mappings are more general than the ones dealing with approximate fixed point problem of those mappings.

The theory of best proximity point has proved to be simple and applicable in solving real world problems in nonlinear analysis, optimization, economics, game theory, and so forth. See, for example, [10] and [13].

The purpose of this paper is to study best proximity point results of proximal simulative contraction mappings. These results extend, unify and strengthen various known results in [6, 7, 12, 17] among others. As an application of the obtained theorems, we give some solvability theorems of a variational inequality problem, see [18].

2. Preliminaries

Throughout this paper, we assume that (X, d) is a metric space and (A, B) a pair of nonempty subsets of X.

Consistent with [6], the following definitions and results will be needed in the sequel. Let

$$\begin{aligned} A_0 &= \{x \in A \ : \ d(x, y) = \triangle_{AB} \quad \text{for some} \quad y \in B\}, \\ B_0 &= \{y \in B \ : \ d(x, y) = \triangle_{AB} \quad \text{for some} \quad x \in A\}. \end{aligned}$$

If $A \cap B \neq \varnothing$, then A_0 and B_0 are nonempty. Moreover, A_0 and B_0 are contained in the boundaries of A and B, respectively whenever A and B are closed subsets of a normed linear space such that $\triangle_{AB} > 0$.

Definition 1. A set B is said to be *approximatively compact* with respect to A if every sequence $\{y_n\}$ in B, satisfying $d(x, y_n) \to d(x, B)$ for some $x \in A$, has a convergent subsequence.

Sankar Raj [16] introduced the concept of *P-property* and obtained a best proximity point result for a class of weakly contractive mappings as an interesting generalization of Banach contraction principle.

Definition 2. Let A_0 be nonempty. The pair (A, B) is said to have the *P-property* if and only if

$$d(x_1, y_1) = d(x_2, y_2) = \triangle_{AB} \text{ implies that } d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 1 ([1]). Let $X = \mathbb{R}^2$. Define the metric d on X by

$$d((x_1, y_1), (x_2, y_2)) = \max\{| x_1 - y_1 |, | x_2 - y_2 |\}.$$

If $A := \{(x, 0) : -1 \leq x \leq 1\}$ and $B := \{(0, y) : -1 \leq y \leq 1\}$, then the pair (A, B) has the *P-property*.

Several authors studied best proximity point results for different contractive mappings employing the notion of P-property, see [8] and the references mentioned therein.

Abkar and Gabeleh [2] proved that most of best proximity point results based on P-property can be deduced from existing comparable fixed point results in the literature.

Sadiq Basha [6] modified the concept of P-property and introduced the concept of proximal contractions of first and second kinds as follows:

Definition 3. A mapping $f : A \to B$ is said to be a *proximal Banach contraction of first kind* if there exists a non-negative number $\alpha < 1$ such that for all u_1, u_2, x_1, x_2 in A,

$$d(u_1, fx_1) = d(u_2, fx_2) = \triangle_{AB} \text{ implies that } d(u_1, u_2) \leq \alpha d(x_1, x_2).$$

Definition 4. A mapping $f : A \to B$ is said to be a *proximal Banach contraction of second kind* if there exists a non-negative number $\alpha < 1$ such that for all u_1, u_2, x_1, x_2 in A,

$$d(u_1, fx_1) = d(u_2, fx_2) = \triangle_{AB} \text{ implies that } d(fu_1, fu_2) \leq \alpha d(fx_1, fx_2).$$

Then, Sanhan et al. [17] introduced the concept of proximal φ-contraction of the first and second kinds.

Definition 5. A mapping $f : A \to B$ is said to be a *proximal φ-contraction of first kind* if there exists an upper semi-continuous function from the right, with $\varphi(t) < t$ for all $t > 0$, such that for all u_1, u_2, x_1, x_2 in A,

$$d(u_1, fx_1) = d(u_2, fx_2) = \triangle_{AB} \text{ implies that } d(u_1, u_2) \leq \varphi(d(x_1, x_2)).$$

Definition 6. A mapping $f : A \to B$ is said to be a *proximal φ-contraction of second kind* if there exists an upper semi-continuous function from the right, with $\varphi(t) < t$ for all $t > 0$, such that for all u_1, u_2, x_1, x_2 in A,

$$d(u_1, fx_1) = d(u_2, fx_2) = \triangle_{AB} \text{ implies that } d(fu_1, fu_2) \leq \varphi(d(fx_1, fx_2)).$$

Note that if $\varphi(t) = \alpha t$, $\alpha \in [0,1)$ then proximal φ-contraction of first and second kinds reduce to proximal Banach contraction of first and second kinds, respectively.

Another interesting generalization of proximal Banach contractions was obtained by Sadiq Basha and Shahzad [7].

Definition 7. A mapping $f : A \to B$ is said to be a *generalized proximal contraction of first kind* if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that for all $u_1, u_2, x_1, x_2 \in A$,

$$d(u_1, fx_1) = d(u_2, fx_2) = \triangle_{AB}$$

implies that

$$d(u_1,u_2) \leq \alpha d(x_1,x_2) + \beta d(x_1,u_1) + \gamma d(x_2,u_2) + \delta[d(x_1,u_2) + d(x_2,u_1)].$$

Definition 8. A mapping $f : A \to B$ is said to be a *generalized proximal contraction of second kind* if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that for all $u_1, u_2, x_1, x_2 \in A$,

$$d(u_1, fx_1) = d(u_2, fx_2) = \triangle_{AB}$$

implies that

$$\begin{aligned} d(fu_1, fu_2) \leq{} & \alpha d(fx_1, fx_2) + \beta d(fx_1, fu_1) + \gamma d(fx_2, fu_2) \\ & + \delta[d(fx_1, fu_2) + d(fx_2, fu_1)]. \end{aligned}$$

For more results in this direction, we refer to [3, 11, 14] and the references mentioned therein.

Very recently, Khojasteh et al. [12] introduced the notion of *simulation function* and Argoubi et al. [5] (see also [15]) modified the definition as follows:

Definition 9. A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a *simulation function* if it satisfies the following conditions: (ζ_1) $\zeta(t,s) < s - t$ for all $t,s > 0$; (ζ_2) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n \in (0,\infty)$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Example 2 ([12]). If $\varphi : [0,\infty) \to [0,\infty)$ is an upper semi-continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$, then $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by $\zeta(t,s) = \varphi(s) - t$ for all $s,t \in [0,\infty)$ is a simulation function.

Let $\Im$ be the family of all functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the conditions (ζ_1) and (ζ_2).

Definition 10 ([12])**.** Let X be a metric space. A mapping $f : X \to X$ is said to be $\Im$-contraction (or simulative contraction) if there exists $\zeta \in \Im$ such that

$$\zeta(d(fx, fy), d(x,y)) \geq 0, \quad \text{for all } x,y \in X.$$

Note that Banach contractions are $\Im$-contractions but the converse does not hold in general; see [4, 12].

We introduce the following notions of *proximal simulative contractions.*

Definition 11. A mapping $f : A \to B$ is said to be *proximal simulative contraction of first kind* if there exists a mapping $\zeta \in \Im$ such that for all u_1, u_2, x_1, x_2 in A, $d(u_1, fx_1) = d(u_2, fx_2) = \triangle_{AB}$ implies that

$$\zeta(d(u_1, u_2), d(x_1, x_2)) \geq 0.$$

Definition 12. A mapping $f : A \to B$ is said to be *proximal simulative contraction of second kind* if there exists a mapping $\zeta \in \Im$ such that for all u_1, u_2, x_1, x_2 in A, $d(u_1, fx_1) = d(u_2, fx_2) = \triangle_{AB}$ implies that

$$\zeta(d(fu_1, fu_2), d(fx_1, fx_2)) \geq 0.$$

We note that

(i) if $A = B$, then proximal simulative contractions of the first kind are $\Im$-contractions.

(ii) If $\zeta(t,s) = \varphi(s) - t$ for all $s, t \in [0, \infty)$, where $\varphi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.Then proximal simulative contractions of first and second kinds reduce to proximal φ-contractions of first and second kinds, respectively.

3. Main Results

We start with the following result dealing with a continuous proximal simulative contraction of first kind.

Theorem 1. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that A_0 and B_0 are nonempty, $f : A \to B$ is a continuous proximal simulative contraction of first kind and $f(A_0) \subseteq B_0$. Then there exists a unique element x in A such that $d(x, fx) = \triangle_{AB}$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(x_{n+1}, fx_n) = \triangle_{AB}$ for all $n \in \mathbb{N} \cup \{0\}$ converges to the best proximity point x of f.*

Proof. Let x_0 be a given point in A_0. Since $f(A_0) \subseteq B_0$, we can choose $x_1 \in A_0$ such that $d(x_1, fx_0) = \triangle_{AB}$. As $fx_1 \in B_0$, there exists a point $x_2 \in A_0$ such that $d(x_2, fx_1) = \triangle_{AB}$. Continuing in this manner, we can obtain a sequence $\{x_n\}$ in A_0 such that

$$d(x_n, fx_{n-1}) = \triangle_{AB} \quad \text{and} \quad d(x_{n+1}, fx_n) = \triangle_{AB} \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.1}$$

Since f is proximal simulative contraction of first kind, we have

$$\zeta(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) \geq 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

If for some $m \in \mathbb{N}$, we have $x_{m-1} = x_m$, then

$$d(x_m, fx_{m-1}) = d(x_m, fx_m) = \triangle_{AB},$$

that is, x_m is a best proximity point of f. Thus, we can assume that $d(x_{n-1},x_n) > 0$ for all $n \in \mathbb{N}$. Now, the property (ζ_1) of a simulation function assures that

$$0 \leq \zeta(d(x_n,x_{n+1}), d(x_{n-1},x_n)) < d(x_{n-1},x_n) - d(x_n,x_{n+1})$$

and hence

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n) \quad \text{for all} \quad n \in \mathbb{N}.$$

It follows that $\{d(x_n,x_{n+1})\}$ is a monotonically decreasing sequence of non-negative real numbers which is bounded below, then there exists $r \geq 0$ such that $\{d(x_n,x_{n+1})\}$ converges to r. We claim that $r = 0$. Assume on the contrary that $r > 0$. Obviously,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \lim_{n\to\infty} d(x_{n-1},x_n) = r. \tag{3.2}$$

From (3.2) and the property (ζ_2) of a simulation function, we get

$$0 \leq \limsup_{n\to\infty} \zeta(d(x_n,x_{n+1}), d(x_{n-1},x_n)) < 0$$

and hence $r = 0$, that is

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3.3}$$

Next, to prove that $\{x_n\}$ is a Cauchy sequence in X, it suffices to show that $\{x_{2n}\}$ is a Cauchy sequence in X. If not, there exists $\epsilon > 0$ and two subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_{2n}\}$ with $n_k > m_k \geq k$ such that

$$d(x_{2m_k},x_{2n_k}) \geq \epsilon \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.4}$$

Without any loss of generality, we assume that for all $k \in \mathbb{N}$, n_k is the smallest positive integer greater than m_k for which this inequality holds, then

$$d(x_{2m_k},x_{2n_k-2}) < \epsilon \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.5}$$

By (3.4) and (3.5), we have

$$\begin{aligned}\epsilon &\leq d(x_{2m_k},x_{2n_k}) \\ &\leq d(x_{2m_k},x_{2n_k-2}) + d(x_{2n_k-2},x_{2n_k-1}) + d(x_{2n_k-1},x_{2n_k}) \\ &< \epsilon + d(x_{2n_k-2},x_{2n_k-1}) + d(x_{2n_k-1},x_{2n_k}).\end{aligned} \tag{3.6}$$

On taking the limit as $k \to \infty$ in the previous inequality (3.6) and using (3.3), we obtain that

$$\lim_{k\to\infty} d(x_{2m_k},x_{2n_k}) = \epsilon. \tag{3.7}$$

Similarly, we have

$$\begin{aligned}\epsilon &\leq d(x_{2m_k},x_{2n_k}) \\ &\leq d(x_{2m_k},x_{2m_k+1}) + d(x_{2m_k+1},x_{2n_k+1}) + d(x_{2n_k+1},x_{2n_k}).\end{aligned} \tag{3.8}$$

Also,

$$d(x_{2m_k+1},x_{2n_k+1}) \leq d(x_{2m_k+1},x_{2m_k}) + d(x_{2m_k},x_{2n_k}) + d(x_{2n_k},x_{2n_k+1}). \tag{3.9}$$

On taking the limit as $k \to \infty$ on both sides of (3.8) and (3.9), using (3.3) and (3.7), we have

$$\lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k+1}) = \epsilon. \tag{3.10}$$

Since $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ are subsequences of $\{x_{2n}\}$ with $n_k > m_k \geq k$, then by (3.1) we get

$$d(x_{2m_k+1}, fx_{2m_k}) = \triangle_{AB} = d(x_{2n_k+1}, fx_{2n_k}) \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.11}$$

By using (3.7) and (3.10), it follows from (3.11) and the property (ζ_2) of a simulation function that

$$0 \leq \limsup_{k\to\infty} \zeta(d(x_{2m_k+1}, x_{2n_k+1}), d(x_{2m_k}, x_{2n_k})) < 0,$$

a contradiction and hence $\{x_n\}$ is a Cauchy sequence in X. Since A is a closed subset of a complete metric space (X,d), then there exists $x \in A$ such that

$$\lim_{n\to\infty} x_n = x. \tag{3.12}$$

On taking the limit as $n \to \infty$ on one of equalities in (3.1), by (3.12) and the continuity of f, we obtain that

$$d(x, fx) = \triangle_{AB}.$$

Therefore, x is a best proximity point of f in A. To prove uniqueness, suppose there exists another best proximity point x^* of f in A, other than x. That is

$$d(x^*, x) > 0, \quad d(x^*, fx^*) = \triangle_{AB} \quad \text{and} \quad d(x, fx) = \triangle_{AB}.$$

By the property (ζ_1) of a simulation function, since f is proximal simulative contraction of first kind, we get $d(x^*, x) < d(x^*, x)$, a contradiction, and hence the best proximity point of f in A is unique. This completes the proof. □

Example 3. Let $X = \mathbb{R}^2$ be endowed with the Euclidean metric

$$d((x_1, y_1), (x_2, y_2)) = \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid \qquad \text{for all} \quad x_1, y_1, x_2, y_2 \in X.$$

Let $A = \{(0, y) : 0 \leq y \leq 1\}$ and $B = \{(1, y) : 0 \leq y \leq 1\}$. Then, define $f : A \to B$ and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$f((0, y)) = (1, y - \frac{y^2}{2}) \quad \text{and} \quad \zeta(t,s) = \begin{cases} \frac{1}{2}s - t & \text{if } t < s, \\ 0 & \text{if } t \geq s. \end{cases}$$

Indeed the pair (X,d) is a complete metric space, $A = A_0$, $B = B_0$, $\triangle_{AB} = 1$, and f is a continuous proximal simulative contraction of first kind which is not a proximal contraction of first kind. Moreover f has no fixed point, but the point $z = (0,0) \in A$ is a unique best proximity point of f.

In the next result, we consider a continuous proximal simulative contraction mapping of second kind.

Theorem 2. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A is approximatively compact with respect to B. Suppose that A_0 and B_0 are nonempty, $f : A \to B$ is a continuous proximal simulative contraction of second kind and $f(A_0) \subseteq B_0$. Then there exists an element x in A such that $d(x, fx) = \triangle_{AB}$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(x_{n+1}, fx_n) = \triangle_{AB}$ for all $n \in \mathbb{N} \cup \{0\}$ converges to a best proximity point x of f. Further, if x, x^* are best proximity points of f, then $fx = fx^*$ and f has a unique best proximity point if it is an injective function on A_0.*

Proof. Let x_0 be a given point in A_0. Following arguments similar to those in the proof of Theorem 1, we obtain that $\{x_n\}$ is a sequence in A_0 satisfying

$$d(x_n, fx_{n-1}) = \triangle_{AB} \quad \text{and} \quad d(x_{n+1}, fx_n) = \triangle_{AB} \qquad \text{for all} \quad n \in \mathbb{N}.$$

We assume that $fx_{n-1} \neq fx_n$ for all $n \in \mathbb{N}$; in fact, if $fx_{m-1} = fx_m$ for some $m \in \mathbb{N}$, from

$$d(x_m, fx_m) = d(x_m, fx_{m-1}) = \triangle_{AB},$$

we deduce that x_m is a proximity point of f. Since f is proximal simulative contraction of second kind, we have

$$\zeta(d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n)) \geq 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

It follows from the property (ζ_1) of a simulation function that

$$0 \leq \zeta(d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n)) < d(fx_{n-1}, fx_n) - d(fx_n, fx_{n+1}),$$

which implies that

$$d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n) \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Therefore $\{d(fx_n, fx_{n+1})\}$ is a monotonically decreasing sequence of non-negative real numbers which is bounded below, then there exists $r \geq 0$ such that $\{d(fx_n, fx_{n+1})\}$ converges to r. Following arguments similar to those in the proof of Theorem 1 we obtain that $r = 0$. Thus

$$\lim_{n\to\infty} d(fx_{n-1}, fx_n) = 0 \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Also, $\{fx_n\}$ is a Cauchy sequence in X. Since B is a closed subset of a complete metric space (X,d), then there exists $y \in B$ such that

$$\lim_{n\to\infty} fx_n = y. \tag{3.13}$$

Now, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y,A) &\leq d(y, x_{n+1}) \leq d(y, fx_n) + d(fx_n, x_{n+1}) \\ &= d(y, fx_n) + \triangle_{AB} \\ &\leq d(y, fx_n) + d(y, A), \end{aligned}$$

which implies that

$$d(y,A) \le d(y,x_{n+1}) \le d(y,fx_n) + d(y,A). \tag{3.14}$$

On taking the limit as $n \to \infty$ in (3.14) and using (3.13), we have

$$\lim_{n\to\infty} d(y,x_{n+1}) = d(y,A).$$

Since A is approximatively compact with respect to B, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ convergent to some element $x \in A$. It follows, from the continuity of f, that

$$d(x,fx) = \lim_{k\to\infty} d(x_{n_k}, fx_{n_k-1}) = \triangle_{AB}.$$

Thus $d(x,fx) = \triangle_{AB}$ and hence x is a best proximity point of f in A. If there is another best proximity point x^* of f in A other than x with $fx^* \neq fx$, then we have

$$d(fx^*,fx) > 0 \quad \text{and} \quad d(x^*,fx^*) = d(x,fx) = \triangle_{AB}.$$

By the property (ζ_1) of a simulation function, since f is proximal simulative contraction of the second kind, we have $d(fx^*,fx) < d(fx^*,fx)$, a contradiction, and hence $fx = fx^*$. Thus, f has a unique best proximity point if it is an injective function on A_0. □

Finally, we relax the requirement of continuity for the mapping f in the following result.

Theorem 3. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that B is approximatively compact with respect to A. Suppose that A_0 and B_0 are nonempty, $f : A \to B$ is a proximal simulative contraction of first kind and $f(A_0) \subseteq B_0$. Then there exists a unique element x in A such that $d(x,fx) = \triangle_{AB}$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(x_{n+1},fx_n) = \triangle_{AB}$ for all $n \in \mathbb{N}\cup\{0\}$ converges to the best proximity point x of f.*

Proof. Let x_0 be a given point in A_0. Following arguments similar to those in the proof of Theorem 1 we build a Cauchy sequence $\{x_n\}$ of points of A_0 satisfying

$$d(x_{n+1},x_n) > 0 \text{ and } d(x_{n+1},fx_n) = \triangle_{AB} \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \tag{3.15}$$

Since A is a closed subset of a complete metric space (X,d), then there exists $x \in A$ such that

$$\lim_{n\to\infty} x_n = x. \tag{3.16}$$

Now, for all $n \in \mathbb{N}\cup\{0\}$, we have

$$\begin{aligned} d(x,B) &\le d(x,fx_n) \le d(x,x_{n+1}) + d(x_{n+1},fx_n) \\ &= d(x,x_{n+1}) + \triangle_{AB} \\ &\le d(x,x_{n+1}) + d(x,B). \end{aligned}$$

Thus

$$d(x,B) \leq d(x, fx_n) \leq d(x, x_{n+1}) + d(x,B). \tag{3.17}$$

On taking the limit as $n \to \infty$ in (3.17) and using (3.16), we obtain that

$$\lim_{n\to\infty} d(x, fx_n) = d(x,B).$$

Since B is approximatively compact with respect to A, $\{fx_n\}$ has a subsequence $\{fx_{n_k}\}$ convergent to some element $y \in B$. Thus,

$$d(x,y) = \lim_{k\to\infty} d(x_{n_k+1}, fx_{n_k}) = \triangle_{AB}.$$

This implies that $x \in A_0$. Since $f(A_0) \subseteq B_0$, we can choose $u \in A$ such that

$$d(u, fx) = \triangle_{AB}. \tag{3.18}$$

We shall show that $u = x$. If $\{x_n\}$ has a subsequence convergent to u, then $u = x$, so we can assume that $x_n \neq u$ for all $n \in \mathbb{N} \cup \{0\}$. By (3.15), we deduce that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \neq x$ for all $k \in \mathbb{N}$. Then by (3.15), (3.18) and the property (ζ_1) of a simulation function, we obtain

$$0 \leq \zeta(d(x_{n_k+1}, u), d(x_{n_k}, x)) < d(x_{n_k}, x) - d(x_{n_k+1}, u),$$

that is,

$$d(x_{n_k+1}, u) < d(x_{n_k}, x) \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{3.19}$$

On taking the limit as $n \to \infty$ on both sides of (3.19) and using (3.16), we get that

$$0 \leq d(x,u) \leq 0.$$

Therefore $u = x$ and hence $d(x, fx) = \triangle_{AB}$. Suppose that there exists another best proximity point x^* in A of f, other than x. Then we have

$$d(x^*, x) > 0 \quad \text{and} \quad d(x^*, fx^*) = d(x, fx) = \triangle_{AB}.$$

By the property (ζ_1) of a simulation function, since f is proximal simulative contraction of first kind, we get

$$d(x^*, x) < d(x^*, x),$$

a contradiction. Therefore the best proximity point of f in A is unique. □

Example 4. Let $X = \mathbb{R}^2$ be endowed with the Euclidean metric

$$d((x_1,y_1),(x_2,y_2)) = \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid \quad \text{for all} \quad x_1, y_1, x_2, y_2 \in X.$$

Let $A = \{(0,y) : 0 \leq y < \infty\}$ and $B = \{(1,y) : 0 \leq y < \infty\}$. Then, define $f : A \to B$ and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$f((0,y)) = \begin{cases} (1, \frac{\ln(1+y)}{y}) & \text{if } y > 0, \\ (1,0) & \text{otherwise,} \end{cases} \quad \text{and} \quad \zeta(t,s) = \begin{cases} \frac{1}{2}s - t & \text{if } t < s, \\ 0 & \text{if } t \geq s. \end{cases}$$

Indeed $A = A_0$, $B = B_0$, $\triangle_{AB} = 1$, and f is a proximal simulative contraction of first kind which is not continuous. Further, B is approximatively compact with respect to A and $z = (0,0)$ is a unique point in A satisfying the conclusion of Theorem 3.

We conclude with the following remarks:

(i) If we put $A = B$ in Theorem 3, we obtain the main result in [12].
(ii) If we put $A = B$ and $\zeta(s,t) = \alpha s - t$ for all $s,t \in [0,\infty)$ where $\alpha \in [0,1)$ in Theorem 3, we obtain a well known Banach fixed point theorem.
(iii) If we choose $\zeta(t,s) = \varphi(s) - t$ for all $s,t \in [0,\infty)$ where $\varphi : [0,\infty) \to [0,\infty)$ is an upper semi-continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$, we obtain the result in [17].

4. Variational Inequalities

Let K be a nonempty closed and convex subset of a real Hilbert space H, with inner product $\langle \cdot,\cdot \rangle$ and corresponding norm $\|\cdot\|$. A variational inequality problem can be stated as follows:

Find $u \in K$ such that $\langle gu, v-u \rangle \geq 0$ for all $v \in K$, where $g : H \to H$ is a given operator.

The interest for such a kind of mathematical tool is due to the fact that a wide class of equilibrium and economic problems, arising in the applied sciences, can be described by variational inequalities. Here, we consider the metric projection operator, say $P_K : H \to K$, for solving the variational inequality problem. Indeed, it is well-known that, for each $u \in H$, there exists a unique nearest point $P_K u \in K$ such that $\|u - P_K u\| \leq \|u - v\|$ for all $v \in K$, see [9].

The following Lemmas hold.

Lemma 1. *Let $z \in H$. Then $u \in K$ satisfies the inequality $\langle u - z, y - u \rangle \geq 0$, for all $y \in K$ if and only if $u = P_K z$.*

Lemma 2. *Let $g : H \to H$. Then $u \in K$ is a solution of $\langle gu, v - u \rangle \geq 0$, for all $v \in K$, if and only if $u = P_K(u - \lambda gu)$, with $\lambda > 0$.*

It is obvious that any set is approximatively compact with respect to itself. Therefore, we prove our first result.

Theorem 4. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $g : H \to H$ is such that $P_K(I - \lambda g) : K \to K$ is a proximal simulative contraction of first kind. Then there exists a unique element $u \in K$ such that $\langle gu, v - u \rangle \geq 0$ for all $v \in K$. Moreover, for any fixed element $u_0 \in K$, the sequence $\{u_n\}$, with $u_{n+1} = P_K(u_n - \lambda g u_n)$ where $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, converges to u.*

Proof. Define $f : K \to K$ by $fx = P_K(x - \lambda gx)$ so that, by Lemma 2, $u \in K$ is a solution of $\langle gu, v - u \rangle \geq 0$ for all $v \in K$, if and only if $u = fu$. Now, f satisfies all the hypotheses of Theorem 1 with $A = B = K$, then the conclusions of Theorem 4 hold true as a particular case of Theorem 3. □

Similarly, the following result is related to Theorem 2.

Theorem 5. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $g : H \to H$ is such that $P_K(I - \lambda g) : K \to K$ is a continuous proximal simulative contraction of second kind. Then there exists a unique element $u \in K$ such that $\langle gu, v-u \rangle \geq 0$ for all $v \in K$. Moreover, for any fixed element $u_0 \in K$, the sequence $\{u_n\}$, with $u_{n+1} = P_K(u_n - \lambda g u_n)$ where $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, converges to u.*

Of course, these type-theorems can be extended and particularized for different variational inequality problems. For instance, we think to the case of a variational inequality of the second kind, that is, find $u \in K$ such that $\langle gu, v-u \rangle + \psi_K(v) - \psi_K(u) \geq 0$ for all $v \in H$, where $g : H \to H$ is a given operator and $\psi_K : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and continuous function. Clearly, if we assume

$$\psi_K(v) = \begin{cases} +\infty & \text{if } v \notin K, \\ 0 & \text{if } v \in K, \end{cases}$$

that is the indicator function of nonempty closed and convex set K, then we retrieve the classical variational inequality problem at the beginning of this section.

5. Conclusions

In this paper, we present an abstract approach to the solution of best proximity point and variational inequality problems, via classical arguments of fixed point theory. The motivation in the use of known fixed point techniques is their usefulness in covering a wide range of situations, without artificial expedients. On the other hand, we give sufficient flexibility to our theory, by using classes of simulation functions. As shown in dealing with variational inequality problems, the approach give us the possibility to design a convergent algorithm for the solution of the problem under investigation. Precisely, by using the metric projection, we are able to construct a sequence of approximations, say $\{u_n\}$, which is convergent to the solution of the problem.

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