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DILATIONS, MODELS AND SPECTRAL PROBLEMS OF NON-SELF-ADJOINT SRURM-LIUVILLE OPERATORS

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Abstract. In this study, we investigate the maximal dissipative singular Sturm-Liouville operators acting in the Hilbert space $L_r^2(a,b)$ $(-\infty \leq a < b \leq \infty)$, that the extensions of a minimal symmetric operator with defect index $(2,2)$ (in limit-circle case at singular end points a and b). We examine two classes of dissipative operators with separated boundary conditions and we establish, for each case, a self-adjoint dilation of the dissipative operator as well as its incoming and outgoing spectral representations, which enables us to define the scattering matrix of the dilation. Moreover, we construct a functional model of the dissipative operator and identify its characteristic function in terms of the Weyl function of a self-adjoint operator. We present several theorems on completeness of the system of root functions of the dissipative operators and verify them.

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1. INTRODUCTION

Dissipative operators are one of the important classes of non-self-adjoint operators. It is well recognized ([1–3, 9, 13–16]), that the theory of dilations with application of functional models gives an ample approach to the spectral theory of dissipative (contractive) operators. By carrying the complete information on the spectral properties of the dissipative operator, we can say that characteristic function plays the primary role in this theory. Hence, in the incoming spectral representation of the dilation, the dissipative operator becomes the model. Completeness problem of the system of eigenvectors and associated (or root) vectors is solved through the factorization of the characteristic function. The computation of the characteristic functions of dissipative operators is preceded by the construction and investigation of the self-adjoint dilation and the corresponding scattering problem, in which the characteristic function is considered as the scattering matrix. According to the Lax-Phillips scattering theory

[10], the unitary group $\{U(s)\}$ $(s \in \mathbb{R} := (-\infty, \infty))$ has typical properties in the subspaces D^- and D^+ of the Hilbert space H, which are called respectively the incoming and outgoing subspaces. One can find the adequacy of this approach to dissipative Schrödinger and Sturm-Liouville operators, for example, in [1–3, 9, 13–15].

In this paper, we take the minimal symmetric singular Sturm-Liouville operator acting in the Hilbert space $\mathcal{L}_r^2(a,b)$ $(-\infty \le a < b \le \infty)$ with maximal defect index $(2,2)$ (in Weyl's limit-circle cases at singular end points a and b) into consideration. We define all maximal dissipative, maximal accumulative and self-adjoint extensions of such a symmetric operator using the boundary conditions at a and b. We investigate two classes of non-self-adjoint operators with separated boundary conditions, called 'dissipative at a' and 'dissipative at b'. In each of these two cases, we construct a self-adjoint dilation of the maximal dissipative operator together with its incoming and outgoing spectral representations so that we can determine the scattering matrix (function) of the dilation as stated in the scheme of Lax and Phillips [10]. Then, we create a functional model of the maximal dissipative operator via the incoming spectral representation and define its characteristic function in terms of the Weyl function (or scattering matrix of the dilation) of a self-adjoint operator. Finally, using the results found for characteristic functions, we prove the theorems on completeness of the system of eigenfunctions and associated functions (or root functions) of the maximal dissipative Sturm-Liouville operators. Results of the present paper are new even in the case $p = r = 1$ (in the case of the one-dimensional Schrödinger operator).

2. Extensions of a symmetric operator and self-adjoint dilations of the dissipative operators

We address the following Sturm-Liouville differential expression with two singular end points a and b:

$$\tau(x) := \frac{1}{r(t)}[-(p(t)x'(t))' + q(t)x(t)] \ (t \in J := (a,b), \ -\infty \le a < b \le +\infty), \quad (2.1)$$

where p, q and r are real-valued, Lebesgue measurable functions on J, and $p^{-1}, q, r \in \mathcal{L}_{loc}^1(J)$, $p \neq 0$ and $r > 0$ almost everywhere on J.

In order to pass from the differential expression to operators, we shall take the Hilbert space $\mathcal{L}_r^2(J)$ consisting of all complex-valued functions f satisfying $\int_a^b r(t)\,|f(t)|^2\,dt < \infty$, with the inner product $(f,g) = \int_a^b r(t) f(t) \overline{g(t)} dt$.

Let $\mathcal{D}_{\max}$ represent the linear set of all functions $f \in \mathcal{L}_r^2(J)$ such that f and pf' are locally absolutely continuous functions on J, and $\tau(f) \in \mathcal{L}_r^2(J)$. Let us define the *maximal operator* $T_{\max}$ on $\mathcal{D}_{\max}$ as $T_{\max} f = \tau(f)$.

For any two functions $f, g \in \mathcal{D}_{\max}$, Green's formula is given by

$$(T_{\max} f, g) - (f, T_{\max} g) = [f,g](b) - [f,g](a), \quad (2.2)$$

where

$$[f,g](t) := W_t(f, \overline{g}) := f(t)\,(p\overline{g}')\,(t) - (pf')\,(t)\,\overline{g}\,(t) \ \ (t \in J),$$

$$[f,g](a) := \lim_{t\to a^+}[f,g](t),\ [f,g](b) := \lim_{t\to b^-}[f,g](t).$$

In $\mathcal{L}_r^2(J)$, we consider the dense linear set $\mathcal{D}_{\min}$ consisting of smooth, compactly supported functions on J. Let us indicate the restriction of the operator $T_{\max}$ to $\mathcal{D}_{\min}$ by $T_{\min}$. We can conclude from (2.2) that $T_{\min}$ is symmetric. Thus, it admits closure denoted by $T_{\min}$. The *minimal operator* $T_{\min}$ is a symmetric operator with defect index $(0,0)$, $(1,1)$ or $(2,2)$, and $T_{\max} = T_{\min}^*$ ([4, 5, 12, 18, 19]). Note that the operator $T_{\min}$ is self-adjoint for defect index $(0,0)$, that is, $T_{\min}^* = T_{\min} = T_{\max}$.

Moreover, we assume that $T_{\min}$ has defect index $(2,2)$. Under this assumption, Weyl's limit-circle cases are obtained for the differential expression τ at a and b (see [4–6, 8, 11, 12, 17–19]). The domain of the operator $T_{\min}$ consists of precisely the functions $f \in \mathcal{D}_{\max}$, which satisfy the following condition

$$[f,g](b) - [f,g](a) = 0,\ \forall g \in \mathcal{D}_{\max}. \tag{2.3}$$

Let $T_{\min}^-$ and $T_{\min}^+$ denote respectively the minimal symmetric operators generated by the expression τ on the intervals $(a,c]$ and $[c,b)$ for some $c \in J$, and $\mathcal{D}_{\min}^{\mp}$ represents the domain of $T_{\min}^{\mp}$. It is known ([5, 12, 18]), that the defect number $def T_{\min}$ of $T_{\min}$ can be computed using the formula $def T_{\min} = def T_{\min}^+ + def T_{\min}^- - 2$. Thus, we obtain that $def T_{\min}^+ + def T_{\min}^- = 4$, $def T_{\min}^+ = 2$ and $def T_{\min}^- = 2$.

We denote by $\theta(t)$ and $\chi(t)$ the solutions of the equation

$$\tau(y) = 0\ (t \in J) \tag{2.4}$$

satisfying the conditions

$$\theta(c) = 1,\ (p\theta')(c) = 0,\ \chi(c) = 0,\ (p\chi')(c) = 1,\ c \in J. \tag{2.5}$$

The Wronskian of the two solutions of (2.4) does not depend on t, and the two solutions of this equation are linearly independent if and only if their Wronskian is nonzero. Conditions (2.5) and the constancy of the Wronskian imply that

$$W_t(\theta,\chi) = W_c(\theta,\chi) = 1\ (a \le t \le b). \tag{2.6}$$

Hence, θ and χ form a fundamental set of solutions of (2.4). Since $T_{\min}$ has defect index $(2,2)$, we have $\theta,\chi \in \mathcal{L}_r^2(J)$, and $\theta,\chi \in \mathcal{D}_{\max}$ as well.

The following equality holds for arbitrary functions $f,g \in \mathcal{D}_{\max}$ ([2])

$$[f,g](t) = [f,\theta](t)[\overline{g},\chi](t) - [f,\chi](t)[\overline{g},\theta](t)\ (a \le t \le b). \tag{2.7}$$

The domain $\mathcal{D}_{\min}$ of the operator $T_{\min}$ is composed of precisely the functions $f \in \mathcal{D}_{\max}$ satisfying the boundary conditions given as follows ([1])

$$[f,\theta](a) = [f,\chi](a) = [f,\theta](b) = [f,\chi](b) = 0. \tag{2.8}$$

Recall that a linear operator $\mathbf{A}$ (with dense domain $\mathcal{D}(\mathbf{A})$) acting on some Hilbert space $\mathbf{H}$ is called *dissipative* (*accumulative*) if $\Im(\mathbf{A}y,y) \ge 0$ ($\Im(\mathbf{A}y,y) \le 0$) for all $y \in \mathcal{D}(\mathbf{A})$ and *maximal dissipative* (*maximal accumulative*) if it does not have a proper dissipative (accumulative) extension ([7], p.149).

Now, consider the linear maps of $\mathcal{D}_{\max}$ into $\mathbb{C}^2$ given by

$$\Psi_1 f = \begin{pmatrix} [f,\chi](a) \\ [f,\theta](b) \end{pmatrix}, \ \Psi_2 f = \begin{pmatrix} [f,\theta](a) \\ [f,\chi](b) \end{pmatrix}. \tag{2.9}$$

Then we get the following statement ([1]).

Theorem 1. *For any contraction $S \in \mathbb{C}^2$ the restriction of the operator $T_{\max}$ to the set of vectors $f \in \mathcal{D}_{\max}$ satisfying the boundary condition*

$$(S-I)\Psi_1 f + i(S+I)\Psi_2 f = 0 \tag{2.10}$$

or

$$(S-I)\Psi_1 f - i(S+I)\Psi_2 f = 0 \tag{2.11}$$

is, respectively, a maximal dissipative or a maximal accumulative extension of the operator $T_{\min}$. Conversely, every maximally dissipative (accumulative) extension of $T_{\min}$ is the restriction of $T_{\max}$ to the set consisting of vectors $f \in \mathcal{D}_{\max}$ satisfying (2.10) ((2.11)), and the contraction S is uniquely determined by the extension. These conditions describe a self-adjoint extension if and only if S is unitary. In the latter case, (2.10) and (2.11) are equivalent to the condition $(\cos B)\Psi_1 f - (\sin B)\Psi_2 f = 0$, where B is a self-adjoint operator (Hermitian matrix) in $\mathbb{C}^2$. The general forms of dissipative and accumulative extensions of the operator $T_{\min}$ are respectively given by the conditions

$$S(\Psi_1 f + i\Psi_2 f) = \Psi_1 f - i\Psi_2 f, \ \Psi_1 f + i\Psi_2 f \in \mathcal{D}(S), \tag{2.12}$$

$$S(\Psi_1 f - i\Psi_2 f) = \Psi_1 f + i\Psi_2 f, \ \Psi_1 f - i\Psi_2 f \in \mathcal{D}(S), \tag{2.13}$$

where S is a linear operator with $\|Sf\| \leq \|f\|, f \in \mathcal{D}(S)$. For an isometric operator S in (2.12) and (2.13) we have the general forms of symmetric extensions.

Particularly, the boundary conditions ($f \in \mathcal{D}_{\max}$)

$$[f,\chi](a) - \alpha_1[f,\theta](a) = 0, \tag{2.14}$$

$$[f,\theta](b) - \alpha_2[f,\chi](b) = 0 \tag{2.15}$$

with $\Im\alpha_1 \geq 0$ or $\alpha_1 = \infty$, and $\Im\alpha_2 \geq 0$ or $\alpha_2 = \infty$ $\Im\alpha_1 \leq 0$ or $\alpha_1 = \infty$, and $\Im\alpha_2 \leq 0$ or $\alpha_2 = \infty$) characterize all maximal dissipative (maximal accumulative) extensions of $T_{\min}$ with separated boundary conditions. If $\Im\alpha_1 = 0$ or $\alpha_1 = \infty$, and $\Im\alpha_2 = 0$ or $\alpha_2 = \infty$ hold true, then self-adjoint extensions of $T_{\min}$ are obtained. Here for $\alpha_1 = \infty$ ($\alpha_2 = \infty$), condition (2.14) ((2.15)) should be replaced by $[f,\theta](a) = 0$ ($[f,\chi](b) = 0$).

Next, we shall consider the maximal dissipative operators $T^{\mp}_{\alpha_1\alpha_2}$ generated by (2.1) and the boundary conditions given by (2.14) and (2.15) of two different types: 'dissipative at a', i.e., either $\Im\alpha_1 > 0$ and $\Im\alpha_2 = 0$ or $\alpha_2 = \infty$; and 'dissipative at b', i.e., $\Im\alpha_1 = 0$ or $\alpha_1 = \infty$ and $\Im\alpha_2 > 0$.

In order to establish a self-adjoint dilation of the maximal dissipative operator $T^{-}_{\alpha_1\alpha_2}$ for the case 'dissipative at a' (i.e., $\Im\alpha_1 > 0$ and $\Im\alpha_2 = 0$ or $\alpha_2 = \infty$), we associate with $\mathcal{H} := \mathcal{L}^2_r(J)$ the 'incoming' and 'outgoing' channels $\mathcal{L}^2$ $(\mathbb{R}_-)$ $(\mathbb{R}_- := (-\infty, 0])$ and $\mathcal{L}^2(\mathbb{R}_+)$ $(\mathbb{R}_+ := [0,\infty))$, we form the orthogonal sum $\mathfrak{H} := \mathcal{L}^2(\mathbb{R}_-) \oplus \mathcal{H} \oplus \mathcal{L}^2(\mathbb{R}_+)$. Let us call the space $\mathfrak{H}$ as the *main Hilbert space of the dilation* and consider in this space the operator $\mathfrak{T}^{-}_{\alpha_1\alpha_2}$ generated by the expression

$$\mathfrak{T}\langle u_-, y, u_+\rangle = \langle i\frac{du_-}{d\xi}, \tau(y), i\frac{du_+}{d\varsigma}\rangle \tag{2.16}$$

on the set $\mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2})$ consisting of vectors $\langle u_-, y, u_+\rangle$, where $u_- \in \mathcal{W}^1_2(\mathbb{R}_-)$, $u_+ \in \mathcal{W}^1_2(\mathbb{R}_+)$, $y \in \mathcal{D}_{\max}$ and

$$[y,\chi](a) - \alpha_1[y,\theta](a) = \gamma u_-(0),\ [y,\chi](a) - \overline{\alpha}_1[y,\theta](a) = \gamma u_+(0),$$

$$[y,\theta](b) - \alpha_2[y,\chi](b) = 0. \tag{2.17}$$

Here $\mathcal{W}^1_2(\mathbb{R}_\mp)$ denotes the Sobolev space, and $\gamma^2 := 2\Im\alpha_1$, $\gamma > 0$. Then we obtain the next assertion.

Theorem 2. *The operator $\mathfrak{T}^{-}_{\alpha_1\alpha_2}$ is self-adjoint in the space $\mathfrak{H}$and it is a self-adjoint dilation of the maximal dissipative operator $T^{-}_{\alpha_1\alpha_2}$.*

Proof. We assume that $Y, Z \in \mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2})$, $Y = \langle u_-, y, u_+\rangle$ and $Z = \langle v_-, z, v_+\rangle$. If we use integration by parts and (2.16), we find that

$$\begin{aligned}(\mathfrak{T}^{-}_{\alpha_1\alpha_2}Y, Z)_{\mathfrak{H}} &= \int_{-\infty}^{0} iu'_-\overline{v}_- d\xi + (T_{\max}y, z)_{\mathcal{H}} + \int_0^{\infty} iu'_+\overline{v}_+ d\varsigma \\ &= iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} + [y,z](b) - [y,z](a) + (Y, \mathfrak{T}^{-}_{\alpha_1\alpha_2}Z)_{\mathfrak{H}}.\end{aligned} \tag{2.18}$$

Moreover, if the boundary conditions (2.17) for the components of the vectors Y, Z and (2.7) are used, it can be seen easily that $iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} + [y,z](b) - [y,z](a) = 0$. Hence, we conclude that $\mathfrak{T}^{-}_{\alpha_1\alpha_2}$ is symmetric. Thus, in order to prove that $\mathfrak{T}^{-}_{\alpha_1\alpha_2}$ is self-adjoint, it is sufficient to show that $(\mathfrak{T}^{-}_{\alpha_1\alpha_2})^* \subseteq \mathfrak{T}^{-}_{\alpha_1\alpha_2}$ Take $Z = \langle v_-, z, v_+\rangle \in \mathcal{D}((\mathfrak{T}^{-}_{\alpha_1\alpha_2})^*)$. Let $(\mathfrak{T}^{-}_{\alpha_1\alpha_2})^*Z = Z^* = \langle v^*_-, z^*, v^*_+\rangle \in \mathfrak{H}$, so that

$$(\mathfrak{T}^{-}_{\alpha_1\alpha_2}Y, Z)_{\mathfrak{H}} = (Y, Z^*)_{\mathfrak{H}},\ \forall Y \in \mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2}). \tag{2.19}$$

If we choose suitable components for $Y \in \mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2})$ in (2.19), it can be shown easily that $v_- \in \mathcal{W}^1_2(\mathbb{R}_-)$, $v_+ \in \mathcal{W}^1_2(\mathbb{R}_+)$, $z \in \mathcal{D}_{\max}$ and $Z^* = \mathfrak{T}Z$, where $\mathfrak{T}$ is given by (2.16). Therefore, (2.19) takes the following form $(\mathfrak{T}Y, Z)_{\mathfrak{H}} = (Y, \mathfrak{T}Z)_{\mathfrak{H}}$, $\forall Y \in \mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2})$. Hence, the sum of the integrated terms in the bilinear form $(\mathfrak{T}Y, Z)_{\mathfrak{H}}$ must be zero:

$$iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} + [y,z](b) - [y,z](a) = 0 \tag{2.20}$$

for all $Y=\langle u_-,y,u_+\rangle\in\mathcal{D}(\mathfrak{T}^-_{\alpha_1\alpha_2})$. Additionally, after the boundary conditions (2.17) for $[y,\theta](a)$ and $[y,\chi](a)$ are solved, it is found that

$$[y,\theta](a)=-\frac{i}{\gamma}(u_+(0)-u_-(0)),\ [y,\chi](a)=\gamma u_-(0)-\frac{i\alpha_1}{\gamma}(u_+(0)-u_-(0)). \quad (2.21)$$

Therefore, (2.7) and (2.21) imply that (2.20) is equivalent to the equality given as follows

$$\begin{aligned} &iu_-(0)\overline{v_-(0)}-iu_+(0)\overline{v_+(0)}=[y,z](a)-[y,z](b)\\ &=-\frac{i}{\gamma}(u_+(0)-u_-(0))\,[\bar z,\chi]\,(a)-\gamma[u_-(0)-\frac{i\alpha_1}{\gamma^2}\,(u_+(0)-u_-(0))]\,[\bar z,\theta]\,(a)\\ &\quad-[y,\theta](b)\,[\bar z,\chi]\,(b)+[y,\chi](b)\,[\bar z,\theta]\,(b)\\ &=-\frac{i}{\gamma}(u_+(0)-u_-(0))\,[\bar z,\chi]\,(a)-\gamma[u_-(0)-\frac{i\alpha_1}{\gamma^2}\,(u_+(0)-u_-(0))]\,[\bar z,\theta]\,(a)\\ &\quad+([\bar z,\theta]\,(b)-\alpha_2\,[\bar z,\chi]\,(b))[y,\chi](b). \end{aligned}$$

Note that $u_\pm(0)$ can be arbitrary complex numbers. If we compare the coefficients of $u_\pm(0)$ on the left and right sides of the last equality, we see that the vector $Z=\langle v_-,z,v_+\rangle$ satisfies the boundary conditions $[z,\chi](a)-\alpha_1[z,\theta](a)=\gamma v_-(0)$, $[z,\chi](a)-\overline{\alpha}_1[z,\theta](a)=\gamma v_+(0)$, $[z,\theta](b)-\alpha_2[z,\chi](b)=0$. Consequently, the inclusion $(\mathfrak{T}^-_{\alpha_1\alpha_2})^*\subseteq\mathfrak{T}^-_{\alpha_1\alpha_2}$ is fulfilled. This proves that $\mathfrak{T}^-_{\alpha_1\alpha_2}=(\mathfrak{T}^-_{\alpha_1\alpha_2})^*$.

In the space $\mathfrak{H}$, self-adjoint operator $\mathfrak{T}^-_{\alpha_1\alpha_2}$ generates a unitary group $\mathfrak{U}^-(s):=\exp[i\mathfrak{T}^-_{\alpha_1\alpha_2}s]$ $(s\in\mathbb{R})$. Denote by $\mathcal{P}:\mathfrak{H}\to\mathcal{H}$ and $\mathcal{P}_1:\mathcal{H}\to\mathfrak{H}$ the mappings acting in keeping with the formulas $\mathcal{P}:\langle u_-,y,u_+\rangle\to y$ and $\mathcal{P}_1:y\to\langle 0,y,0\rangle$. Set $\mathcal{V}(s)=\mathcal{P}\mathfrak{U}^-(s)\mathcal{P}_1$ $(s\geq 0)$. The family $\{\mathcal{V}(s)\}$ $(s\geq 0)$ of operators is a strongly continuous semigroup of completely non-unitary contractions on $\mathcal{H}$. Let A represent the generator of this semigroup, i.e, $Az=\lim_{s\to+0}[(is)^{-1}(\mathcal{V}(s)z-z)]$. All vectors for which this limit exists belong to the domain of A. The operator A is maximal dissipative and the operator $\mathfrak{T}^-_{\alpha_1\alpha_2}$ is called the *self-adjoint dilation* of A ([13–15]). We aim to show that $A=T^-_{\alpha_1\alpha_2}$, which implies in turn that $\mathfrak{T}^-_{\alpha_1\alpha_2}$ is a self-adjoint dilation of $T^-_{\alpha_1\alpha_2}$. To achieve this goal, we first verify the following equality ([13–15])

$$\mathcal{P}(\mathfrak{T}^-_{\alpha_1\alpha_2}-\lambda I)^{-1}\mathcal{P}_1 y=(T^-_{\alpha_1\alpha_2}-\lambda I)^{-1}y,\ y\in\mathcal{H},\ \Im\lambda<0. \quad (2.22)$$

Let $(\mathfrak{T}^-_{\alpha_1\alpha_2}-\lambda I)^{-1}\mathcal{P}_1 y=Z=\langle v_-,z,v_+\rangle$. Then $(\mathfrak{T}^-_{\alpha_1\alpha_2}-\lambda I)Z=\mathcal{P}_1 y$, and so, $T_{\max}z-\lambda z=y$, $v_-(\xi)=v_-(0)e^{-i\lambda\xi}$ and $v_+(\varsigma)=v_+(0)e^{-i\lambda\varsigma}$. Since $Z\in\mathcal{D}(\mathfrak{T}^-_{\alpha_1\alpha_2})$ and hence, $v_-\in\mathcal{L}^2(\mathbb{R}_-)$; we have $v_-(0)=0$, and consequently, z satisfies the boundary conditions $[z,\chi](a)-\alpha_1[z,\theta](a)=0$, $[z,\theta](b)-\alpha_2[z,\chi](b)=0$. Therefore, $z\in\mathcal{D}(T^-_{\alpha_1\alpha_2})$, and since a dissipative operator cannot have an eigenvalue λ with $\Im\lambda<0$, we conclude that $z=(T^-_{\alpha_1\alpha_2}-\lambda I)^{-1}y$. Here, we evaluate $v_+(0)$ using the formula $v_+(0)=\gamma^{-1}([z,\chi](a)-\overline{\alpha}_1[z,\theta](a))$. Then

$$(\mathfrak{T}^-_{\alpha_1\alpha_2}-\lambda I)^{-1}\mathcal{P}_1 y=\left\langle 0,(T^-_{\alpha_1\alpha_2}-\lambda I)^{-1}y,\gamma^{-1}([z,\chi](a)-\overline{\alpha}_1[z,\theta](a))\,e^{-i\lambda\varsigma}\right\rangle$$

for $y \in \mathcal{H}$ and $\Im\lambda < 0$. Applying $\mathcal{P}$, we get the desired equality (2.22).

Now, it is not difficult to show that $A = T^{-}_{\alpha_1\alpha_2}$. In fact, it follows from (2.22) that

$$(T^{-}_{\alpha_1\alpha_2} - \lambda I)^{-1} = \mathcal{P}(\mathfrak{T}^{-}_{\alpha_1\alpha_2} - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^{\infty} \mathfrak{U}^{-}(s)e^{-i\lambda s}ds\mathcal{P}_1$$
$$= -i\int_0^{\infty} \mathcal{V}(s)e^{-i\lambda s}ds = (A - \lambda I)^{-1},\ \Im\lambda < 0,$$

and thus we have $T^{-}_{\alpha_1\alpha_2} = A$ proving Theorem 2. □

In order to construct a self-adjoint dilation of the maximal dissipative operator $T^{+}_{\alpha_1\alpha_2}$ in the case 'dissipative at b' (i.e., $\Im\alpha_1 = 0$ or $\alpha_1 = 0$ and $\Im\alpha_2 > 0$) in $\mathfrak{H}$, we consider the operator $\mathfrak{T}^{+}_{\alpha_1\alpha_2}$ generated by the expression (2.16) on the set $\mathcal{D}(\mathfrak{T}^{+}_{\alpha_1\alpha_2})$ of vectors $\langle u_-, y, u_+\rangle$ satisfying the conditions: $u_- \in \mathcal{W}^1_2(\mathbb{R}_-)$, $u_+ \in \mathcal{W}^1_2(\mathbb{R}_+)$, $y \in \mathcal{D}_{\max}$ and

$$[y,\chi](a) - \alpha_1[y,\theta](a) = 0,\ \ [y,\theta](b) - \alpha_2[y,\chi](b) = \beta u_-(0),$$
$$[y,\theta](b) - \overline{\alpha}_2[y,\chi](b) = \beta u_+(0), \tag{2.23}$$

where $\beta^2 := 2\Im\alpha_2$, $\beta > 0$.

Since the proof of the next theorem is similar to that of Theorem 2, we omit it here.

Theorem 3. *The operator $\mathfrak{T}^{+}_{\alpha_1\alpha_2}$ is self-adjoint in $\mathfrak{H}$ and it is a self-adjoint dilation on the maximal dissipative operator $T^{+}_{\alpha_1\alpha_2}$.*

3. Scattering theory of the dilations, functional models and completeness of root functions of the dissipative operators

The unitary group $\mathfrak{U}^{\pm}(s) = \exp[i\mathfrak{T}^{\pm}_{\alpha_1\alpha_2}s]$ $(s \in \mathbb{R})$ possesses a crucial feature through which we can apply to it the Lax-Phillips scheme ([10]). Namely, it has incoming and outgoing subspaces $\mathfrak{D}^- := \langle \mathcal{L}^2(\mathbb{R}_-), 0, 0\rangle$ and $\mathfrak{D}^+ := \langle 0, 0, \mathcal{L}^2(\mathbb{R}_+)\rangle$ satisfying the following properties:

(1) $\mathfrak{U}^{\pm}(s)\mathfrak{D}^- \subset \mathfrak{D}^-$, $s \le 0$ and $\mathfrak{U}^{\pm}(s)\mathfrak{D}^+ \subset \mathfrak{D}^+$, $s \ge 0$;
(2) $\bigcap_{s\le 0} \mathfrak{U}^{\pm}(s)\mathfrak{D}^- = \bigcap_{s\ge 0} \mathfrak{U}^{\pm}(s)\mathfrak{D}^+ = \{0\}$;
(3) $\overline{\bigcup_{s\ge 0} \mathfrak{U}^{\pm}(s)\mathfrak{D}^-} = \overline{\bigcup_{s\le 0} \mathfrak{U}^{\pm}(s)\mathfrak{D}^+} = \mathfrak{H}$;
(4) $\mathfrak{D}^- \perp \mathfrak{D}^+$.

It is evident that property (**4**) holds true. Let us prove property (**1**) for $\mathfrak{D}^+$ (the proof for $\mathfrak{D}^-$ is similar). For this end, we define $\mathcal{R}^{\pm}_{\lambda} = (\mathfrak{T}^{\pm}_{\alpha_1\alpha_2} - \lambda I)^{-1}$ for all λ with $\Im\lambda < 0$. Then, for any $Y = \langle 0, 0, u_+\rangle \in \mathfrak{D}^+$, we get

$$\mathcal{R}^{\pm}_{\lambda}Y = \langle 0, 0, -ie^{-i\lambda\varsigma}\int_0^{\varsigma} e^{-i\lambda\xi}u_+(\xi)d\xi\rangle.$$

Therefore, we see that $\mathcal{R}_{\lambda}Y \in \mathfrak{D}^+$. Further, if $Z \perp \mathfrak{D}^+$, then

$$0 = \left(\mathcal{R}^{\pm}_{\lambda}Y, Z\right)_{\mathfrak{H}} = -i\int_0^{\infty} e^{-i\lambda s}\left(\mathfrak{U}^{\pm}(s)Y, Z\right)_{\mathfrak{H}} ds,\ \Im\lambda < 0,$$

which implies that $(\mathfrak{U}^{\pm}(s)Y,Z)_{\mathfrak{H}}=0$ for all $s\geq 0$. So, we obtain $\mathfrak{U}^{\pm}(s)\mathfrak{D}^{+}\subset\mathfrak{D}^{+}$ for $s\geq 0$, proving property (**1**).

To prove property (**2**) for $\mathfrak{D}^{+}$ (the proof for $\mathfrak{D}^{-}$ is similar), we denote by $P^{+}:\mathfrak{H}\to\mathcal{L}^{2}(\mathbb{R}_{+})$ and $P_{1}^{+}:\mathcal{L}^{2}(\mathbb{R}_{+})\to\mathfrak{D}^{+}$ the mappings acting according to the formulae $P^{+}:\langle u_{-},y,u_{+}\rangle\to u_{+}$ and $P_{1}^{+}:u\to\langle 0,0,u\rangle$, respectively. The semigroup of isometries $\mathcal{X}(s)=P^{+}\mathfrak{U}^{-}(s)P_{1}^{+}$, $s\geq 0$ is a one-sided shift in $\mathcal{L}^{2}(\mathbb{R}_{+})$. In fact, the generator of the semigroup of the one-sided shift $\mathcal{Y}(s)$ in $\mathcal{L}^{2}(\mathbb{R}_{+})$ is the differential operator $i\frac{d}{d\xi}$ satisfying the boundary condition $u(0)=0$. On the other hand, the generator $\mathcal{B}$ of the semigroup of isometries $\mathcal{X}(s)$, $s\geq 0$, is the operator defined by

$$\mathcal{B}u=P^{+}\mathfrak{T}_{\alpha_1\alpha_2}^{-}P_{1}^{+}Y=P^{+}\mathfrak{T}_{\alpha_1\alpha_2}^{-}\langle 0,0,u\rangle=P^{+}\langle 0,0,i\frac{du}{d\xi}\rangle=i\frac{du}{d\xi},$$

where $u\in\mathcal{W}_{2}^{1}(\mathbb{R}_{+})$ and $u(0)=0$. However, since a semigroup is uniquely determined by its generator, we have $\mathcal{X}(s)=\mathcal{Y}(s)$, and thus,

$$\bigcap_{s\geq 0}\mathfrak{U}^{-}(s)\mathfrak{D}^{+}=\langle 0,0,\bigcap_{s\geq 0}\mathcal{Y}(s)\mathcal{L}^{2}(\mathbb{R}_{+})\rangle=\{0\},$$

(the proof for $\mathfrak{U}^{+}(s)$ is similar) verifying that property (**2**) is valid.

As stated in the scheme of the Lax-Phillips scattering theory, the scattering matrix is defined using the spectral representations theory. Now, we shall continue with their construction. During this process, we shall also have proved property (**3**) of the incoming and outgoing subspaces.

Recall that the linear operator **A** (with domain $\mathcal{D}(\mathbf{A})$) acting in the Hilbert space **H** is called *completely non-self-adjoint* (or *pure*) if invariant subspace $\mathbf{M}\subseteq\mathcal{D}(\mathbf{A})$ $(\mathbf{M}\neq\{0\})$ of the operator **A** whose restriction on **M** is self-adjoint, does not exist.

Lemma 1. *The operator $T_{\alpha_1\alpha_2}^{\pm}$ is completely non-self-adjoint (pure).*

Proof. Let $\mathcal{H}'\subset\mathcal{H}$ be a non-trivial subspace in which the operator $T_{\alpha_1\alpha_2}^{-}$ (the proof for $T_{\alpha_1\alpha_2}^{+}$ is similar) induces a self-adjoint operator T' with domain $\mathcal{D}(T')=\mathcal{H}'\cap\mathcal{D}(T_{\alpha_1\alpha_2}^{-})$. If $z\in\mathcal{D}(T')$, then we have $z\in\mathcal{D}(T'^{*})$ and $[z,\chi](a)-\alpha_1[z,\theta](a)=0$, $[z,\chi](a)-\overline{\alpha}_1[z,\theta](a)=0$, $[z,\theta](b)-\alpha_2[z,\chi](b)=0$. Hence, we have $[z,\theta](a)=0$ for the eigenfunctions $z(t,\lambda)$ of the operator $T_{\alpha_1\alpha_2}^{-}$ that lie in $\mathcal{H}'$ and are eigenfunctions of T'. Since $[z,\chi](a)-\alpha_1[z,\theta](a)=0$, we derive that $[z,\chi](a)=0$ and $z(t,\lambda)\equiv 0$. Since all solutions of $\tau(z)=\lambda z$ $(t\in J)$ lie in $\mathcal{L}_{r}^{2}(J)$, we can see that the resolvent $R_{\lambda}(T_{\alpha_1\alpha_2}^{-})$ of the operator $T_{\alpha_1\alpha_2}^{-}$ is a Hilbert-Schmidt operator, and thus the spectrum of $T_{\alpha_1\alpha_2}^{-}$ is purely discrete. Hence, the theorem on the expansion of the self-adjoint operator T' in eigenfunctions implies that $\mathcal{H}'=\{0\}$, that is, $T_{\alpha_1\alpha_2}^{-}$ is pure. This completes the proof. □

In order to prove third property, we set

$$\mathfrak{H}_{-}^{\pm}=\overline{\bigcup_{s\geq 0}\mathfrak{U}^{\pm}(s)\mathfrak{D}^{-}},\ \mathfrak{H}_{+}^{\pm}=\overline{\bigcup_{s\leq 0}\mathfrak{U}^{\pm}(s)\mathfrak{D}^{+}}$$

and first prove the next result.

Lemma 2. *The equality* $\mathfrak{H}_{-}^{\pm} + \mathfrak{H}_{+}^{\pm} = \mathfrak{H}$ *is fulfilled.*

Proof. By means of the property (**1**) of the subspace $\mathfrak{D}_{\pm}$, it can be shown that the subspace $\mathfrak{H}'_{\pm} = \mathfrak{H} \ominus \left(\mathfrak{H}_{-}^{\pm} + \mathfrak{H}_{+}^{\pm}\right)$ is invariant with respect to the group $\{\mathfrak{U}^{\pm}(s)\}$ and it can be described as $\mathfrak{H}'_{\pm} = \left\langle 0, \mathcal{H}'_{\pm}, 0 \right\rangle$, where $\mathcal{H}'_{\pm}$ is a subspace in $\mathcal{H}$. Therefore, if the subspace $\mathfrak{H}'_{\pm}$ (and hence also $\mathcal{H}'_{\pm}$) were non-trivial, then the unitary group $\{\mathfrak{U}^{\pm\prime}(s)\}$, restricted to this subspace, would be a unitary part of the group $\{\mathfrak{U}^{\pm}(s)\}$, and thus the restriction $T^{\pm\prime}_{\alpha_1\alpha_2}$ of $T^{\pm}_{\alpha_1\alpha_2}$ to $\mathcal{H}'_{\pm}$ would be a self-adjoint operator in $\mathcal{H}'_{\pm}$. Since the operator $T^{\pm}_{\alpha_1\alpha_2}$ is pure, we conclude that $\mathcal{H}'_{\pm} = \{0\}$, i.e., $\mathfrak{H}'_{\pm} = \{0\}$. Hence, the lemma is proved. □

Let $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ be the solutions of the equation $\tau(y) = \lambda y$ $(t \in J)$ satisfying the conditions given by

$$[\varphi,\theta](a) = -1,\ [\varphi,\chi](a) = 0,\ [\psi,\theta](a) = 0,\ [\psi,\chi](a) = 1. \tag{3.1}$$

The *Weyl function* $m_{\infty\alpha_2}(\lambda)$ of the self-adjoint operator $T^{-}_{\infty\alpha_2}$ is determined by the condition

$$[\psi + m_{\infty\alpha_2}\varphi, \theta](b) - \alpha_2[\psi + m_{\infty\alpha_2}\varphi, \chi](b) = 0,$$

which implies in turn that

$$m_{\infty\alpha_2}(\lambda) = -\frac{[\psi,\theta](b) - \alpha_2[\psi,\chi](b)}{[\varphi,\theta](b) - \alpha_2[\varphi,\chi](b)}. \tag{3.2}$$

It follows from (3.2) that $m_{\infty\alpha_2}(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. We note that these poles coincide with the eigenvalues of the self-adjoint operator $T^{-}_{\infty\alpha_2}$. Furthermore, we can show that the function $m_{\infty\alpha_2}(\lambda)$ has the following properties: $\Im\lambda\Im m_{\infty\alpha_2}(\lambda) > 0$ for $\Im\lambda \neq 0$ and $m_{\infty\alpha_2}(\bar{\lambda}) = \overline{m_{\infty\alpha_2}(\lambda)}$ for complex λ, except the real poles of $m_{\infty\alpha_2}(\lambda)$.

For convenience, we adopt the following notations:

$$\omega(t,\lambda) = \psi(t,\lambda) + m_{\infty\alpha_2}(\lambda)\varphi(t,\lambda),$$

$$\Theta^{-}_{\alpha_1\alpha_2}(\lambda) = \frac{m_{\infty\alpha_2}(\lambda) - \alpha_1}{m_{\infty\alpha_2}(\lambda) - \overline{\alpha}_1}. \tag{3.3}$$

Set

$$\mathcal{V}^{-}_{\lambda}(t,\xi,\varsigma) = \langle e^{-i\lambda\xi}, (m_{\infty\alpha_2}(\lambda) - \alpha_1)^{-1}\gamma\omega(t,\lambda), \overline{\Theta}^{-}_{\alpha_1\alpha_2}(\lambda)e^{-i\lambda\varsigma}\rangle.$$

By means of the vector $\mathcal{V}^{-}_{\lambda}(t,\xi,\varsigma)$, we consider the transformation $\Phi_{-} : Y \to \tilde{Y}_{-}(\lambda)$ by $(\Phi_{-}Y)(\lambda) := \tilde{Y}_{-}(\lambda) := \frac{1}{\sqrt{2\pi}}(Y, \mathcal{V}^{-}_{\lambda})_{\mathfrak{H}}$ on the vector $Y = \langle u_{-}, y, u_{+}\rangle$, where u_{-}, u_{+}, and y are smooth, compactly supported functions.

Lemma 3. *The transformation Φ_- maps $\mathfrak{H}_-^-$ onto $\mathcal{L}^2(\mathbb{R})$ isometrically. For all vectors $Y,Z\in\mathfrak{H}_-^-$ the Parseval equality and the inversion formula hold:*

$$(Y,Z)_{\mathfrak{H}}=(\tilde{Y}_-,\tilde{Z}_-)_{\mathcal{L}^2}=\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}d\lambda,\ Y=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\mathcal{V}_\lambda^-d\lambda,$$

where $\tilde{Y}_-(\lambda):=(\Phi_-Y)(\lambda)$ and $\tilde{Z}_-(\lambda):=(\Phi_-Z)(\lambda)$.

Proof. For $Y,Z\in\mathfrak{D}^-$, $Y=\langle u_-,0,0\rangle$, $Z=\langle v_-,0,0\rangle$, we get

$$\tilde{Y}_-(\lambda):=\frac{1}{\sqrt{2\pi}}(Y,\mathcal{V}_\lambda^-)_{\mathfrak{H}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{9}u_-(\xi)e^{i\lambda\xi}d\xi\in\mathcal{H}_-^2$$

and

$$(Y,Z)_{\mathfrak{H}}=\int_{-\infty}^{0}u_-(\xi)\overline{v_-(\xi)}d\xi=\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}d\lambda=(\Phi_-Y,\Phi_-Z)_{\mathcal{L}^2}$$

in view of the usual Parseval equality for Fourier integrals. Here and below, $\mathcal{H}_\pm^2$ denote the Hardy classes in $\mathcal{L}^2(\mathbb{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively.

We aim to extend the Parseval equality to the whole of $\mathfrak{H}_-^-$. In this context, we consider in $\mathfrak{H}_-^-$ the dense set $\mathfrak{H}_-'$ of vectors acquired from the smooth, compactly supported functions in $\mathfrak{D}^-$: $Y\in\mathfrak{H}_-'$ if $Y=\mathfrak{U}^-(s)Y_0$, $Y_0=\langle u_-,0,0\rangle$, $u_-\in C_0^\infty(\mathbb{R}_-)$, where $s=s_Y$ is a non-negative number depending on Y. If $Y,Z\in\mathfrak{H}_-'$, then for $s>s_Y$ and $s>s_Z$ we have $\mathfrak{U}^-(-s)Y$, $\mathfrak{U}^-(-s)Z\in\mathfrak{D}^-$ and, moreover, the first components of these vectors lie in $C_0^\infty(\mathbb{R}_-)$. Then, as the operators $\mathfrak{U}^-(s)$ $(s\in\mathbb{R})$ are unitary, it follows from the equality

$$\Phi_-\mathfrak{U}^-(-s)Y=(\mathfrak{U}^-(-s)Y,\mathcal{V}_\lambda^-)_{\mathfrak{H}}=e^{-i\lambda s}(Y,\mathcal{V}_\lambda^-)_{\mathfrak{H}}=e^{-i\lambda s}\Phi_-Y,$$

that

$$\begin{aligned}(Y,Z)_{\mathfrak{H}}&=(\mathfrak{U}^-(-s)Y,\mathfrak{U}^-(-s)Z)_{\mathfrak{H}}=(\Phi_-\mathfrak{U}^-(-s)Y,\Phi_-\mathfrak{U}^-(-s)Z)_{\mathcal{L}^2}\\&=(e^{-i\lambda s}\Phi_-Y,e^{-i\lambda s}\Phi_-Z)_{\mathcal{L}^2}=(\Phi_-Y,\Phi_-Z)_{\mathcal{L}^2}.\end{aligned}\tag{3.4}$$

If we take the closure in (3.4), we find the Parseval equality for the entire space $\mathfrak{H}_-^-$. If all integrals in the Parseval equality are considered as limits in the mean of integrals over finite intervals, we get the inversion formula. In conclusion, we have

$$\Phi_-\mathfrak{H}_-^-=\overline{\bigcup_{s\geq 0}\Phi_-\mathfrak{U}^-(s)\mathfrak{D}^-}=\overline{\bigcup_{s\geq 0}e^{-i\lambda s}\mathcal{H}_-^2}=\mathcal{L}^2(\mathbb{R}),$$

i.e., Φ_- maps $\mathfrak{H}_-^-$ onto whole $\mathcal{L}^2(\mathbb{R})$, proving the lemma. □

Let us set

$$\mathcal{V}_\lambda^+(t,\xi,\varsigma)=\langle\Theta_{\alpha_1\alpha_2}^-(\lambda)e^{-i\lambda\xi},(m_{\infty\alpha_2}(\lambda)-\overline{\alpha}_1)^{-1}\gamma\omega(t,\lambda),e^{-i\lambda\varsigma}\rangle.$$

By using the vectors $\mathcal{V}_\lambda^+(t,\xi,\varsigma)$, we define the map $\Phi_+:Y\to\tilde{Y}_+(\lambda)$ on vectors $Y=\langle u_-,y,u_+\rangle$ in which u_-,u_+, and y are smooth, compactly supported functions by setting $(\Phi_+ Y)(\lambda) := \tilde{Y}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(Y, \mathcal{V}_\lambda^+)_{\mathfrak{H}}$. The next result can be proved by following the procedure used in the proof of Lemma 3.

Lemma 4. *The transformation Φ_+ isometrically maps $\mathfrak{H}_+^-$ onto $\mathcal{L}^2(\mathbb{R})$ and besides, the Parseval equality and the inversion formula hold for all vectors $Y, Z \in \mathfrak{H}_+^-$ as follows:*

$$(Y,Z)_{\mathfrak{H}} = (\tilde{Y}_+, \tilde{Z}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{Y}_+(\lambda)\overline{\tilde{Z}_+(\lambda)}d\lambda, \; Y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{Y}_+(\lambda)\mathcal{V}_\lambda^+ d\lambda,$$

where $\tilde{Y}_+(\lambda) := (\Phi_+ Y)(\lambda)$ and $\tilde{Z}_+(\lambda) := (\Phi_+ Z)(\lambda)$.

Equality given by (3.3) implies that $\Theta_{\alpha_1\alpha_2}^-(\lambda)$ satisfies $\left|\Theta_{\alpha_1\alpha_2}^-(\lambda)\right| = 1$ for all $\lambda \in \mathbb{R}$. Then, we conclude from the explicit formula for the vectors $\mathcal{V}_\lambda^+$ and $\mathcal{V}_\lambda^-$ that

$$\mathcal{V}_\lambda^- = \overline{\Theta_{\alpha_1\alpha_2}^-}(\lambda)\mathcal{V}_\lambda^+ \; (\lambda \in \mathbb{R}). \tag{3.5}$$

Lemmas 3 and 4 imply that $\mathfrak{H}_-^- = \mathfrak{H}_+^-$. This, together with Lemma 2, verifies that $\mathfrak{H} = \mathfrak{H}_-^- = \mathfrak{H}_+^-$ and property (**3**) for $\mathfrak{U}^-(s)$ above has been established for the incoming and outgoing subspaces.

Hence, Φ_- isometrically maps onto $\mathcal{L}^2(\mathbb{R})$ with the subspace $\mathfrak{D}^-$ mapped onto $\mathcal{H}_-^2$, and the operators $\mathfrak{U}^-(s)$ are transformed by the operators of multiplication by $e^{i\lambda s}$. This means that Φ_- (Φ_+) is the incoming (outgoing) spectral representation for the group $\{\mathfrak{U}^-(s)\}$. Using (3.5), we can pass from the Φ_+-representation of a vector $Y \in \mathfrak{H}$ to its Φ_--representation by multiplication of the function $\Theta_{\alpha_1\alpha_2}^-(\lambda)$: $\tilde{Y}_-(\lambda) = \Theta_{\alpha_1\alpha_2}^-(\lambda)\tilde{Y}_+(\lambda)$. Based on [10], the scattering function (matrix) of the group $\{\mathfrak{U}^-(s)\}$ with respect to the subspaces $\mathfrak{D}^-$ and $\mathfrak{D}^+$, is the coefficient by which the Φ_--representation of a vector $Y \in \mathfrak{H}$ must be multiplied in order to get the corresponding Φ_+-representation: $\tilde{Y}_+(\lambda) = \overline{\Theta_{\alpha_1\alpha_2}^-}(\lambda)\tilde{Y}_-(\lambda)$ and thus we have proved the following statement.

Theorem 4. *The function $\overline{\Theta_{\alpha_1\alpha_2}^-}(\lambda)$ is the scattering function (matrix) of the group $\{\mathfrak{U}^-(s)\}$ or of the self-adjoint operator $\mathfrak{T}_{\alpha_1\alpha_2}^-$).*

Let $\mathcal{S}(\lambda)$ be an arbitrary non-constant inner function ([16]) defined on the upper half-plane (we recall that a function $\mathcal{S}(\lambda)$ analytic in the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|\mathcal{S}(\lambda)| \leq 1$ for $\lambda \in \mathbb{C}_+$, and $|\mathcal{S}(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Setting $\mathcal{K} = \mathcal{H}_+^2 \ominus \mathcal{S}\mathcal{H}_+^2$, we can see that $\mathcal{K} \neq \{0\}$ is a subspace of the Hilbert space $\mathcal{H}_+^2$. We deal with the semigroup of the operators $\mathcal{X}(s)$ $(s \geq 0)$ acting in $\mathcal{K}$ according to the formula $\mathcal{X}(s)u = P\left[e^{i\lambda s}u\right]$, $u := u(\lambda) \in \mathcal{K}$, where P is the orthogonal projection from $\mathcal{H}_+^2$ onto $\mathcal{K}$. The generator of the semigroup $\{\mathcal{X}(s)\}$ is represented as $\mathcal{B} : \mathcal{B}u = \lim_{s\to+0}[(is)^{-1}(\mathcal{X}(s)u - u)]$. $\mathcal{B}$ is a maximal dissipative operator acting in $\mathcal{K}$ and its domain $\mathcal{D}(\mathcal{B})$ consisting of all functions $u \in \mathcal{K}$, for which the limit given above exists. The operator $\mathcal{B}$ is called a *model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax and Phillips

[10], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [16]). It is the basic assertion that $\mathcal{S}(\lambda)$ is the *characteristic function* of the operator $\mathcal{B}$.

If we set $\mathfrak{N} = \langle 0, \mathcal{H}, 0\rangle$, then it is obtained that $\mathfrak{H} = \mathfrak{D}^- \oplus \mathfrak{N} \oplus \mathfrak{D}^+$. From the explicit form of the unitary transformation Φ_- that under the mapping Φ_-, we have

$$\begin{gathered}\mathfrak{H} \to \mathcal{L}^2(\mathbb{R}),\ Y \to \tilde{Y}_-(\lambda) = (\Phi_- Y)(\lambda),\ \mathfrak{D}^- \to \mathcal{H}_-^2,\\ \mathfrak{D}^+ \to \Theta_{\alpha_1\alpha_2}^- \mathcal{H}_+^2,\ \mathfrak{N} \to \mathcal{H}_+^2 \ominus \Theta_{\alpha_1\alpha_2}^- \mathcal{H}_+^2,\\ \mathfrak{U}^-(s)Y \to (\Phi_- \mathfrak{U}^-(s)\Phi_-^{-1}\tilde{Y}_-)(\lambda) = e^{i\lambda s}\tilde{Y}_-(\lambda). \qquad (3.6)\end{gathered}$$

The formulas in (3.6) imply that our operator $T_{\alpha_1\alpha_2}^-$ is unitary equivalent to the model dissipative operator with the characteristic function $\Theta_{\alpha_1\alpha_2}^-(\lambda)$. The fact that characteristic functions of unitary equivalent dissipative operators coincide ([13–16]) leads us the following theorem.

Theorem 5. *The characteristic function of the maximal dissipative operator $T_{\alpha_1\alpha_2}^-$ coincides with the function $\Theta_{\alpha_1\alpha_2}^-(\lambda)$given by* (3.3).

Weyl function of the self-adjoint operator $T_{\alpha_1\infty}^+$, denoted by $m_{\alpha_1\infty}(\lambda)$, can be expressed in terms of the Wronskians of the solutions:

$$m_{\alpha_1\infty}(\lambda) = -\frac{[\vartheta,\chi](b)}{[\phi,\chi](b)},$$

where $\phi(t,\lambda)$ and $\vartheta(t,\lambda)$ are solutions of $\tau(y) = \lambda y\,(t \in J)$ and satisfying the conditions

$$\begin{gathered}[\phi,\theta](a) = -\frac{1}{\sqrt{1+\alpha_1^2}},\ [\phi,\chi](a) = -\frac{\alpha_1}{\sqrt{1+\alpha_1^2}},\\ [\vartheta,\theta](a) = \frac{\alpha_1}{\sqrt{1+\alpha_1^2}},\ [\vartheta,\chi](a) = \frac{1}{\sqrt{1+\alpha_1^2}}.\end{gathered}$$

Let us adopt the following notations:

$$\begin{gathered}k(\lambda) := \frac{[\phi,\theta](b)}{[\vartheta,\chi](b)},\ m(\lambda) := m_{\alpha_1\infty}(\lambda),\\ \Theta^+(\lambda) := \Theta_{\alpha_1\alpha_2}^+(\lambda) := \frac{m(\lambda)k(\lambda)-\alpha_2}{m(\lambda)k(\lambda)-\overline{\alpha}_2}. \qquad (3.7)\end{gathered}$$

Let

$$\mathcal{W}_\lambda^-(t,\xi,\varsigma) = \langle e^{-i\lambda\xi}, \beta m(\lambda)[(m(\lambda)k(\lambda)-\alpha_2)[[\vartheta,v](b)]^{-1}\phi(t,\lambda), \overline{\Theta}^+(\lambda)e^{-i\lambda\varsigma}\rangle.$$

By means of the vector $\mathcal{W}_\lambda^-$, we set the transformation $\Upsilon_- : Y \to \tilde{Y}_-(\lambda)$ given by $(\Upsilon_- Y)(\lambda) := \tilde{Y}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(Y, \mathcal{W}_\lambda^-)_{\mathfrak{H}}$ on the vector $Y = \langle u_-, y, u_+\rangle$ in which u_-, u_+, and y are smooth, compactly supported functions. The proof of the next result is similar to that of Lemma 3.

Lemma 5. *The transformation Υ_- isometrically maps $\mathfrak{H}_-^+$ onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $Y, Z \in \mathfrak{H}_-^+$, we obtain the Parseval equality and the inversion formula given by:*

$$(Y,Z)_{\mathfrak{H}} = (\tilde{Y}_-, \tilde{Z}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}d\lambda,\ Y = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\mathcal{W}_\lambda^- d\lambda,$$

where $\tilde{Y}_-(\lambda) = (\Upsilon_- Y)(\lambda)$ and $\tilde{Z}_-(\lambda) = (\Upsilon_- Z)(\lambda)$.

Let

$$\mathcal{W}_\lambda^+(t,\xi,\varsigma) = \langle \Theta^+(\lambda)e^{-i\lambda\xi}, \beta m(\lambda)[(m(\lambda)k(\lambda) - \overline{\alpha}_2)[\vartheta,\chi](b)]^{-1}\phi(t,\lambda), e^{-i\lambda\varsigma}\rangle.$$

With the help of the vector $\mathcal{W}_\lambda^+(t,\xi,\varsigma)$, define the transformation $\Upsilon_+ : Y \to \tilde{Y}_+(\lambda)$ on vectors $Y = \langle u_-, y, u_+\rangle$ by setting $(\Upsilon_+ Y)(\lambda) := \tilde{Y}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(Y, \mathcal{W}_\lambda^+)_{\mathfrak{H}}$. Here, we consider u_-, u_+, and y as smooth, compactly supported functions.

Lemma 6. *The transformation Υ_+ isometrically maps $\mathfrak{H}_+^+$ onto $\mathcal{L}^2(\mathbb{R})$, and for all vectors $Y, Z \in \mathfrak{H}_+^+$, the Parseval equality and the inversion formula hold:*

$$(Y,Z)_{\mathfrak{H}} = (\tilde{Y}_+, \tilde{Z}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}d\lambda,\ Y = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{Y}_+(\lambda)\mathcal{W}_\lambda^+ d\lambda,$$

where $\tilde{Y}_+(\lambda) := (\Upsilon_+ Y)(\lambda)$and $\tilde{Z}_+(\lambda) := (\Upsilon_+ Z)(\lambda)$

It follows from (3.7) that the function $\Theta_{\alpha_1\alpha_2}^+(\lambda)$ satisfies $\left|\Theta_{\alpha_1\alpha_2}^+(\lambda)\right| = 1$ for $\lambda \in \mathbb{R}$. Then, the explicit formula for the vectors $\mathcal{W}_\lambda^+$ and $\mathcal{W}_\lambda^-$ implies that

$$\mathcal{W}_\lambda^- = \overline{\Theta}_{\alpha_1\alpha_2}^+(\lambda)\mathcal{W}_\lambda^+,\ \lambda \in \mathbb{R}. \tag{3.8}$$

Lemmas 5 and 6 result in $\mathfrak{H}_-^+ = \mathfrak{H}_+^+$. By means of Lemma 2, we can conclude that $\mathfrak{H} = \mathfrak{H}_-^+ = \mathfrak{H}_+^+$. According to the formula (3.8), we can see that the passage from the Υ_--representation of a vector $Y \in \mathfrak{H}$ to its Υ_+-representation is achieved as follows: $\tilde{Y}_+(\lambda) = \overline{\Theta}_{\alpha_1\alpha_2}^+(\lambda)\tilde{Y}_-(\lambda)$. Hence, according to [10], the following theorem follows.

Theorem 6. *The function $\overline{\Theta}_{\alpha_1\alpha_2}^+(\lambda)$ is the scattering matrix of the group $\{\mathfrak{U}^+(s)\}$ of the self-adjoint operator $\mathfrak{T}_{\alpha_1\alpha_2}^+$).*

We derive from the explicit form of the unitary transformation Φ_- that

$$\mathfrak{H} \to \mathcal{L}^2(\mathbb{R}),\ \ Y \to \tilde{Y}_-(\lambda) = (\Upsilon_- Y)(\lambda),\ \ \mathfrak{D}^- \to \mathcal{H}_-^2,$$

$$\mathfrak{D}^+ \to \Theta_{\alpha_1\alpha_2}^+\mathcal{H}_+^2,\ \ \mathfrak{N} \to \mathcal{H}_+^2 \ominus \Theta_{\alpha_1\alpha_2}^+\mathcal{H}_+^2,$$

$$\mathfrak{U}^+(s)Y \to (\Upsilon_-\mathfrak{U}^+(s)\Upsilon_-^{-1}\tilde{Y}_-)(\lambda) = e^{i\lambda s}\tilde{Y}_-(\lambda). \tag{3.9}$$

The formulas given by (3.9) state that the operator $T_{\alpha_1\alpha_2}^+$ is a unitary equivalent to the model dissipative operator with characteristic function $\Theta_{\alpha_1\alpha_2}^+(\lambda)$. We have thus proved the next assertion.

Theorem 7. *The characteristic function of the maximal dissipative operator $T^{+}_{\alpha_1\alpha_2}$ coincides with the function $\Theta^{+}_{\alpha_1\alpha_2}(\lambda)$defined by (3.7).*

Let $\mathbf{S}$ represent the linear operator acting in the Hilbert space $\mathbf{H}$ with the domain $\mathcal{D}(\mathbf{S})$. We know that a complex number λ_0 is called an *eigenvalue* of an operator $\mathbf{S}$ if there exists a non-zero vector $z_0 \in \mathcal{D}(\mathbf{S})$ satisfying the equation $\mathbf{S}z_0 = \lambda_0 z_0$; here, z_0 is called an *eigenvector* of $\mathbf{S}$ for λ_0. The eigenvector for λ_0 spans a subspace of $\mathcal{D}(\mathbf{S})$, called the *eigenspace* for λ_0 and the *geometric multiplicity* of λ_0 is the dimension of its eigenspace. The vectors $z_1, z_2, ..., z_k$ are called the *associated vectors* of the eigenvector z_0 if they belong to $\mathcal{D}(\mathbf{S})$ and $\mathbf{S}z_j = \lambda_0 z_j + z_{j-1}$, $j = 1, 2, ..., k$. The non-zero vector $z \in \mathcal{D}(\mathbf{S})$ is called a *root vector* of the operator $\mathbf{S}$ corresponding to the eigenvalue λ_0, if all powers of $\mathbf{S}$ are defined on this element and $(\mathbf{S} - \lambda_0 I)^m z = 0$ for some integer m. The set of all root vectors of $\mathbf{S}$ corresponding to the same eigenvalue λ_0 with the vector $z = 0$ forms a linear set $\mathbf{M}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathbf{M}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathbf{M}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathbf{S}$ corresponding to the eigenvalue λ_0. As a result, the completeness of the system of all eigenvectors and associated vectors of $\mathbf{S}$ is equivalent to the completeness of the system of all root vectors of this operator.

Characteristic function of a maximal dissipative operator $T^{\pm}_{\alpha_1\alpha_2}$ carries complete information about the spectral properties of this operator ([9, 13–16]). For example, when a singular factor $\theta^{\pm}(\lambda)$ of the characteristic function $\Theta^{\pm}_{\alpha_1\alpha_2}(\lambda)$ in the factorization $\Theta^{\pm}_{\alpha_1\alpha_2}(\lambda) = \theta^{\pm}(\lambda)B^{\pm}(\lambda)$ (where $B^{\pm}(\lambda)$ is a Blaschke product) is absent, we are sure that system of eigenfunctions and associated functions (or root functions) of the maximal dissipative Sturm-Liouville operator $T^{\pm}_{\alpha_1\alpha_2}$ is complete.

Theorem 8. *For all values of $\alpha_1$where $\Im\alpha_1 > 0$, with the possible exception of a single value $\alpha_1 = \alpha_1^0$, and for a fixed α_2 ($\Im\alpha_2 = 0$ or $\alpha_2 = 0$), the characteristic function $\Theta^{-}_{\alpha_1\alpha_2}(\lambda)$ of the maximal dissipative operator $T^{-}_{\alpha_1\alpha_2}$ is a Blaschke product, and the spectrum of $T^{-}_{\alpha_1\alpha_2}$ is purely discrete, and lies in the open upper half plane. The operator $T^{-}_{\alpha_1\alpha_2}$ $(\alpha_1 \neq \alpha_1^0)$ has a countable number of isolated eigenvalues having finite multiplicity and limit points at infinity, and the system of all eigenfunctions and associated functions (or all root functions) of this operator is complete in the space $\mathcal{L}_r^2(J)$.*

Proof. It can be seen from the explicit formula (3.3) that $\Theta^{-}_{\alpha_1\alpha_2}(\lambda)$ is an inner function in the upper half-plane and, besides, it is meromorphic in the whole λ-plane. Therefore, we can factorize it in the following way

$$\Theta^{-}_{\alpha_1\alpha_2}(\lambda) = e^{i\lambda l(\alpha_1)} B_{\alpha_1\alpha_2}(\lambda),\ l(\alpha_1) \geq 0, \tag{3.10}$$

where $B_{\alpha_1\alpha_2}(\lambda)$ is a Blaschke product. Using (3.10), we find that

$$\left|\Theta^{-}_{\alpha_1\alpha_2}(\lambda)\right| \leq e^{-l(\alpha_1)\Im\lambda},\ \Im\lambda \geq 0. \tag{3.11}$$

Additionally, if we express $m_{\infty\alpha_2}(\lambda)$ in terms of $\Theta^{-}_{\alpha_1\alpha_2}(\lambda)$ and use (3.3), we get

$$m_{\infty\alpha_2}(\lambda) = \frac{\overline{\alpha}_1\Theta^{-}_{\alpha_1\alpha_2}(\lambda) - \alpha_1}{\Theta^{-}_{\alpha_1\alpha_2}(\lambda) - 1}. \tag{3.12}$$

For a given value α_1 ($\Im\alpha_1 > 0$), if $l(\alpha_1) > 0$ then we have $\lim_{s\to+\infty}\Theta^{-}_{\alpha_1\alpha_2}(is) = 0$ by (3.11). This, together with (3.12), results in $\lim_{s\to+\infty} m_{\infty\alpha_2}(is) = \alpha_1$. Since $m_{\infty\alpha_2}(\lambda)$ is independent of α_1, $l(\alpha_1)$ can be non-zero at not more than a single point $\alpha_1 = \alpha_1^0$ (and, further, $\alpha_1^0 = \lim_{s\to+\infty} m_{\infty\alpha_2}(is)$). Then, the theorem is proved. □

The next result can be proved in a similar manner in the proof of Theorem 8.

Theorem 9. *For all values of α_2 with $\Im\alpha_2 > 0$, with the possible exception of a single value $\alpha_2 = \alpha_2^0$, and for a fixed α_1 ($\Im\alpha_1 = 0$ or $\alpha_1 = \infty$), the characteristic function $\Theta^{+}_{\alpha_1\alpha_2}(\lambda)$ of the maximal dissipative operator $T^{+}_{\alpha_1\alpha_2}$ is a Blaschke product, and the spectrum of $T^{+}_{\alpha_1\alpha_2}$ is purely discrete, and lies in the open upper half-plane. The operator $T^{+}_{\alpha_1\alpha_2}$ ($\alpha_2 \neq \alpha_2^0$) has a countable number of isolated eigenvalues having finite multiplicity and limit points at infinity, and the system of all eigenfunctions and associated functions (or all root functions) of this operator is complete in the space $\mathcal{L}_r^2(J)$.*

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