 SOME NOTES ON FIRST STRONGLY GRADED RINGS

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Abstract. Let $G$ be a group with identity $e$ and $R$ be an associative ring with a nonzero unity 1. Assume that $R$ is first strongly $G$-graded and $H = \text{supp}(R,G)$. For $g \in H$, define $\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x_{g^{-1}}^{(i)}$ where $x \in C_R(R_e) = \{r \in R : rx = xr \text{ for all } x \in R_e \}$, $r_g^{(i)} \in R_g$ and $t_g^{(i)} \in R_{g^{-1}}$ for all $i = 1, \ldots, n_g$ for some positive integer $n_g$. In this article, we study $\alpha_g(x)$ and its properties.

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1. INTRODUCTION

Throughout this article, $R$ is a ring with unity. For a ring $R$ and a subset $T$ of $R$, $C_R(T) = \{r \in R : rt = tr \text{ for all } t \in T \}$. For a group $G$, $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G \}$. Let $G$ be a group with identity $e$. Then $R$ is said to be $G$-graded if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g \in G} R_g$ where $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. For $x \in R$, $x = \sum_{g \in G} x_g$ where $x_g$ is the component of $x$ in $R_g$. Also, $\text{supp}(R,G) = \{g \in G : R_g \neq 0 \}$. Moreover, $R_e$ is a subring of $R$ and $1 \in R_e$. For more details, see [4]. Throughout this article, $H = \text{supp}(R,G)$.

First strongly graded rings have been introduced by Refai in [5]. A $G$-graded ring $R$ is said to be first strongly graded if $1 \in R_g R_{g^{-1}}$ for all $g \in H$. $R$ is first strongly $G$-graded if and only if $H$ is a subgroup of $G$ and $R_g R_h = R_{gh}$ for all $g, h \in H$. For more details, see [5].

Definition 1 ([4]). Let $R$ be a ring. Suppose that $\alpha : G \to \text{Aut}(R)$ and $\beta : G \times G \to U(R)$ where $\text{Aut}(R)$ is the group of automorphisms of $R$ and $U(R)$ is the group of units of $R$. In [4], $(R,G,\alpha, \beta)$ is said to be a crossed system if the following conditions hold for all $g, h, s \in G$ and $a \in R$.

1) $\alpha_g(\alpha_h(a))\beta(g, h) = \beta(g, h)\alpha_g(ha)$. 

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(2) \( \beta(g,h)\beta(gh,s) = \alpha_g(\beta(h,s))\beta(g,h) \).
(3) \( \beta(g,e) = \beta(e,g) = 1 \).

In [2], a \( G \)-graded ring \( R \) is said to be crossed product over the support if \( R_g \cap U(R) \neq \emptyset \) for all \( g \in H \). In [1], it was shown that if \( R \) is crossed product over the support, then \( R \) is first strongly graded and then by [5], \( H \) is a subgroup of \( G \) with \( R_g R_h = R_{gh} \) for all \( g,h \in H \).

Suppose that \( R \) is crossed product over the support. We may choose the family \( \{u_g\}_{g \in H} \) in \( R \) such that \( u_g \in R_g \cap U(R) \) for all \( g \in H \) and assume that \( u_e = 1 \). So, \( R_g = R_e u_g = u_g R_e \) and \( \{u_g\}_{g \in H} \) is a basis for the left (right) \( R_e \)-module \( R \).

Define the map \( \alpha : H \to \text{Aut}(R_e) \) by \( \alpha(g) = \alpha_g \) where \( \alpha_g(a) = u_g a u_g^{-1} \) for all \( g \in H \) and \( a \in R_e \). Also, define \( \beta : H \times H \to U(R_e) \) by \( \beta(g,h) = u_g u_h u_g^{-1} \) for all \( g,h \in H \). Then \( \alpha \) and \( \beta \) satisfy the conditions (1), (2) and (3) above (see [1]). Hence, \( (R_e, H, \alpha, \beta) \) is a crossed system.

Assume that \( R \) is first strongly \( G \)-graded. For \( g \in H \), define \( \alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_g^{(i)} \\
\) where \( x \in C_R(R_e), r_g^{(i)} \in R_g \) and \( t_g^{(i)} \in R_{g^{-1}} \) for all \( i = 1, \ldots, n_g \) for some positive integer \( n_g \). In this article, we study \( \alpha_g(x) \) and it’s properties.

2. Results

In this section, we introduce our results.

Let \( R \) be a \( G \)-graded ring and \( X \) be a non-empty subset of \( G \). Then \( R_X = \bigoplus_{g \in X} R_g \).

If \( X \) is a subgroup of \( G \), then \( R_X \) is a subring of \( R \). For more details, see [3]. We begin our results by the following.

**Theorem 1.** Consider the above crossed system \( (R_e, H, \alpha, \beta) \). Suppose that \( X \) is a subgroup of \( H \) such that \( X \subseteq Z(H) \cap \text{Ker}(\alpha) \) and \( \beta(x,y) = \beta(y,x) \) for all \( (x,y) \in X \times X \). If \( R_e \) is commutative, then \( R_X \) is commutative.

**Proof.** Consider the family \( \{u_g\}_{g \in H} \) above. Let \( g,h \in X \) and \( a_g, b_h \in R_e \). Then
\[
(a_g u_g)(b_h u_h) = a_g a_h (b_h) \beta(g,h) u_{gh} = a_g b_h \beta(g,h) u_{gh} = b_h a_h (a_g) \beta(h,g) u_{hg} = (b_h u_h)(a_g u_g).
\]

Hence, \( R_X \) is commutative. \( \square \)

Let \( R \) be a first strongly \( G \)-graded ring (not necessary to be crossed product over the support). Then \( R_g R_{g^{-1}} = R_e \) for all \( g \in H \). So, for every \( g \in H \), there exists \( n_g \in \mathbb{Z}^+ \), \( r_g^{(i)} \in R_g \) and \( t_{g^{-1}}^{(i)} \in R_{g^{-1}} \) such that \( l = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} \), since \( 1 \in R_e \).
Define $\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_g^{(i)} g_{i-1}$ for all $x \in C_R(R_e)$.

**Theorem 2.** Let $R$ be a first strongly graded ring. Then $\alpha_g$ is independent of the choice of $r_g^{(i)}$'s and $t_g^{(i)}$'s.

**Proof.** Let $n_g, n'_g \in \mathbb{Z}^+$ and $r_g^{(i)}, s_g^{(j)} \in R_g, t_g^{(i)} \in R_{g^{-1}}$, $w_g^{(i)} \in R_{g^{-1}}$ such that

$$1 = \sum_{i=1}^{n_g} r_g^{(i)} t_g^{(i)} g_{i-1} = \sum_{j=1}^{n'_g} s_g^{(j)} w_g^{(j)} g_{i-1}.$$

Let $x \in C_R(R_e)$. Then since $w_g^{(j)} t_g^{(i)} g_{i-1} \in R_e$,

$$\sum_{i=1}^{n_g} r_g^{(i)} x t_g^{(i)} g_{i-1} - \sum_{j=1}^{n'_g} s_g^{(j)} x w_g^{(j)} g_{i-1} = 1 - \left( \sum_{i=1}^{n_g} r_g^{(i)} x t_g^{(i)} g_{i-1} \right) - \left( \sum_{j=1}^{n'_g} s_g^{(j)} x w_g^{(j)} g_{i-1} \right) = 0.$$

The next lemma is fundamental for our next results.

**Lemma 1.** Let $R$ be a first strongly graded ring. If $r \in R$ such that $r R_h = \{0\}$ for some $h \in H$, then $r = 0$.

**Proof.** Suppose that $r \in R$ and $h \in H$ such that $r R_h = \{0\}$. Then $r = r \cdot 1 \in r R_e = r R_h R_{h^{-1}} = (r R_h) R_{h^{-1}} = \{0\}$, i.e., $r = 0$. □

**Theorem 3.** Let $R$ be a first strongly graded ring and $g \in H$. Then $\alpha_g(x)$ is the only element of $R$ satisfying $\alpha_g(x) a_g = a_g x$ for all $a_g \in R_g$. Moreover, $\alpha_g(x) \in C_R(R_e)$ if and only if $x \in Z(R_e)$.

**Proof.** Let $a_g \in R_g$. Then $t_{g^{-1}}^{(i)} a_g \in R_{g^{-1}} R_g = R_e$ and $t_{g^{-1}}^{(i)} a_g$ commutes with $x \in C_R(R_e)$ for all $i = 1, \ldots, n_g$. So,

$$\alpha_g(x) a_g = \sum_{i=1}^{n_g} r_g^{(i)} x t_g^{(i)} g_{i-1} a_g = \sum_{i=1}^{n_g} r_g^{(i)} t_g^{(i)} g_{i-1} a_g x = a_g \left( \sum_{i=1}^{n_g} r_g^{(i)} t_g^{(i)} g_{i-1} \right) x$$

$$= a_g . 1 . x = a_g x.$$

Let $x \in C_R(R_e)$ and $y \in R$ such that $r_g^{(i)} x = y r_g^{(i)}$ for all $i = 1, \ldots, n_g$. Then
\[ \alpha_g(x) = \sum_{i=1}^{n_g} r^{(i)}_g x t^{(i)}_{g-1} = \sum_{i=1}^{n_g} y r^{(i)}_g t^{(i)}_{g-1} = y \left( \sum_{i=1}^{n_g} r^{(i)}_g t^{(i)}_{g-1} \right) = y \cdot 1 = y, \]

i.e., \( \alpha_g(x) \) is the only element satisfies \( \alpha_g(x)a_g = a_gx \) for all \( a_g \in R_g \). Since \( R \) is first strongly graded, if \( x \in R_e \), then \( \alpha_g(x) \in R_e \). In particular, if \( x \in Z(R_e) \), then \( \alpha_g(x) \in Z(R_e) \). So, for \( x \in Z(R_e) \) and \( s \in R_e \),

\[ s \alpha_g(x) = 1.s \alpha_g(x) = \sum_{i=1}^{n_g} \sum_{j=1}^{n_g} r^{(i)}_g x t^{(i)}_{g-1} = \sum_{j=1}^{n_g} \sum_{i=1}^{n_g} r^{(i)}_g x t^{(i)}_{g-1} = \alpha_g(x).s.1 = \alpha_g(x)s. \]

If \( a_g \in R_g \) and \( w \in R_e \), then \( w a_g \in R_e R_g = R_g \) and then

\[ (\alpha_g(x)w)a_g = \alpha_g(x)(wa_g) = (wa_g)x = w(a_gx) = w(\alpha_g(x)a_g) = (w \alpha_g(x))a_g \]

Which implies that \( (\alpha_g(x)w - w \alpha_g(x))R_g = \{0\} \). By Lemma 1, \( \alpha_g(x)w = w \alpha_g(x) \) and hence \( \alpha_g(x) \in C_R(R_e) \).

**Theorem 4.** Let \( R \) be a first strongly graded ring. Then the following hold:

1. \( \alpha_g \circ \alpha_h = \alpha_{gh} \) for all \( g, h \in H \).
2. \( \alpha_g^{-1} = \alpha_{g^{-1}} \) for all \( g \in H \).
3. \( \alpha_g(xb) = \alpha_g(x)\alpha_g(b) \) for all \( g \in H \) and \( x, b \in C_R(R_e) \).

**Proof.** Since \( 1 \in R_e \), \( x = 1.x = \alpha_e(x).1 = \alpha_e(x) \) for all \( x \in C_R(R_e) \). Let \( g, h \in H \), \( a_g \in R_g \) and \( a_h \in R_h \). Then \( a_ga_h \in R_g R_h = R_{gh} \) and then

\[ \alpha_{gh}(x)(a_ga_h) = a_g(a_hx) = a_g(\alpha_h(x)a_h) = (a_g\alpha_h(x))a_h \]

\[ = (\alpha_g(x)(\alpha_h(x))a_h = \alpha_g(x)(\alpha_h(x))(a_ga_h) \]

for all \( x \in C_R(R_e) \) which implies that \( \alpha_g(\alpha_h(x)) = \alpha_{gh}(x) \) by Lemma 1 as \( a_ga_h \) generates the \( R_e \)-submodule \( R_{gh} \).

Let \( g \in H \), \( s^{(j)}_{g-1} \in R_{g-1} \) and \( w^{(j)}_g \in R \), \( j = 1, \ldots, n_g-1 \) such that \( 1 = \sum_{j=1}^{n_g-1} s^{(j)}_{g-1} w^{(j)}_g \).

Then for every \( x \in C_R(R_e) \),

\[ \alpha_g(x) = \sum_{i=1}^{n_g} r^{(i)}_g x t^{(i)}_{g-1} = \sum_{i=1}^{n_g} y r^{(i)}_g t^{(i)}_{g-1} = y \left( \sum_{i=1}^{n_g} r^{(i)}_g t^{(i)}_{g-1} \right) = y \cdot 1 = y, \]
\[ \alpha_g^{-1}(\alpha_g(x)) = \sum_{j=1}^{n_g-1} s^{(j)} g^{-1} \left( \sum_{i=1}^{n_g} r^{(i)} g_{g^{-1}} x t^{(i)} g_{g^{-1}} \right) w^{(j)} g = \]

\[ \sum_{j=1}^{n_g-1} s^{(j)} g^{-1} \left( \sum_{i=1}^{n_g} r^{(i)} g_{g^{-1}} x t^{(i)} g_{g^{-1}} w^{(j)} g \right) = \sum_{j=1}^{n_g-1} s^{(j)} g^{-1} \left( \sum_{i=1}^{n_g} r^{(i)} g_{g^{-1}} w^{(j)} g x \right) = \]

\[ = 1.x = x \]

which implies that \((\alpha_g)^{-1} = \alpha_g^{-1}\). For \(x, b \in C_R(R_e)\) and \(a_g \in R_g\),

\[ \alpha_g(xb)a_g = \alpha_g(xb) = (a_g x)b = (a_g(x)a_g)b = \alpha_g(x)(a_g b) = \alpha_g(x)(\alpha_g(b)a_g) = (\alpha_g(x)\alpha_g(b))a_g. \]

By Lemma 1, \(\alpha_g(xb) = \alpha_g(x)\alpha_g(b)\).

**Theorem 5.** Let \(R\) be a first strongly graded ring. If \(x \in C_R(R_e)\) and \(g \in H\), then \(x R_g = R_g x\) (\(x\) centralizes \(R_g\)) if and only if \(\alpha_g(x) = x\).

**Proof.** Suppose that \(R_g\) is centralized by \(x\). Then for every \(a_g \in R_g\),

\[ \alpha_g(x)a_g = \left( \sum_{i=1}^{n_g} r^{(i)} g_{g^{-1}} x t^{(i)} g_{g^{-1}} \right) a_g = \sum_{i=1}^{n_g} r^{(i)} g_{g^{-1}} x t^{(i)} g_{g^{-1}} a_g. \]

Since \(t^{(i)} g_{g^{-1}} x t^{(i)} g_{g^{-1}} \in R_e\),

\[ \alpha_g(x)a_g = \sum_{i=1}^{n_g} r^{(i)} g_{g^{-1}} x t^{(i)} g_{g^{-1}} a_g x = \left( \sum_{i=1}^{n_g} r^{(i)} g_{g^{-1}} \right) a_g x = 1.a_g x = a_g x = xa_g \]

as \(x\) centralizes \(R_g\). By Lemma 1, \(\alpha_g(x) = x\). Conversely, for every \(a_g \in R_g\),

\[ a_g x = a_g(x)a_g = xa_g, \text{ i.e., } x \text{ centralizes } R_g. \]

**Corollary 1.** Let \(R\) be a first strongly graded ring. Then \(Z(R) = \{x \in C_R(R_e) : \alpha_g(x) = x \text{ for all } g \in H\}\).

**Proof.** Since \(R\) is first strongly graded,

\[ Z(R) = \bigcap_{g \in H} C_R(R_g) = \{x \in C_R(R_e) : x \in C_R(R_g) \text{ for all } g \in H\}. \]

Note that if \(g \notin H\), then \(R_g = \{0\}\) and then \(C_R(R_g) = R\). By Theorem 5, \(x \in C_R(R_g)\) if and only if \(\alpha_g(x) = x\) and hence \(Z(R) = \{x \in C_R(R_e) : \alpha_g(x) = x \text{ for all } g \in H\}. \]
Remark 1. Also, it is nice to see that if $R$ is a first strongly graded ring, then $CR(R_e) = \left\{ x = \sum_{g \in H} x_g \in R : x_g \in R_g \text{ with } c x_g = x_g c \text{ for all } g \in H \text{ and for all } c \in R_e \right\}$

\[
= \left\{ x = \sum_{g \in H} x_g \in R : x_g \in \left( R_g \cap CR(R_e) \right) \text{ for all } g \in H \right\} \\
= \bigoplus_{g \in H} \left( R_g \cap CR(R_e) \right).
\]

Note that for $g \in H$, $R_g = \{0\}$ and then $x_g = 0$.

The next result is a generalization of Corollary 1.

Theorem 6. Let $R$ be a first strongly graded ring and $X$ be a subgroup of $H$. Then $CR(R_X) = \left\{ t = \sum_{g \in H} t_g \in R : t_g \in CR(R_e) \bigcap R_g \right\}$

\[
= \left\{ t \in CR(R_e) : \alpha_x(t) = t_{xg^{-1}} \text{ for all } g \in H \text{ and all } x \in X \right\}.
\]

Proof. Let $t = \sum_{g \in H} t_g \in CR(R_X)$ where $t_g \in R_g$. Since $R_e \subseteq R_X$, $t \in CR(R_e)$ and then by Remark 1, $t_g \in CR(R_e)$ for all $g \in H$. Let $x \in X$. Then for every $s_x \in R_x$,

\[
s_x \sum_{g \in H} t_g = \sum_{g \in H} t_g s_x
\]

since $t \in CR(R_X)$. As $t_g \in CR(R_e)$, by Theorem 3,

\[
\sum_{g \in H} \alpha_x(t_g) s_x = \sum_{g \in H} t_g s_x.
\]

Since $R_x R_g R_{x^{-1}} R_x = R_{xg}$, for all $g \in H$ and for all $x \in X$,

\[
\alpha_x(t_g) s_x = t_{xg^{-1}} s_x
\]

Choose $a^{(i)}_x \in R_x$ and $b^{(i)}_{x^{-1}} \in R_{x^{-1}}$ where $i = 1, \ldots, n_x$ for some positive integer $n_x$ such that

\[
1 = \sum_{i=1}^{n_x} a^{(i)}_x b^{(i)}_{x^{-1}}.
\]

Then
we choose for some positive integer \( n \) \( \alpha_x(t_g) = \alpha_x(t_g).1 = \frac{\alpha_x(t_g)}{\alpha_x(t_g)} \sum_{i=1}^{n_x} \alpha_x(t_g) a_x^{(i)} b_x^{(i)} x^{-1} = \sum_{i=1}^{n_x} \alpha_x(t_g) a_x^{(i)} b_x^{(i)} x^{-1} = t x g x^{-1}. \)

For the converse, Suppose that \( t = \sum_{g \in H} t_g \in R \) where \( t_g \in C_R(R_e) \cap R_g \) and \( \alpha_x(t_g) = t x g x^{-1} \) for all \( g \in H \) and for all \( x \in X \). Then for every \( s_x \in R_x \),

\[
 s_x t = \sum_{g \in H} s_x t_g = \sum_{g \in H} \alpha_x(t_g) s_x = \sum_{g \in H} t x g x^{-1} s_x = \sum_{r \in H} t_r s_x = t s_x
\]

which implies that \( t \in C_R(R_X) \). \( \square \)

**Theorem 7.** Let \( R \) be a first strongly graded ring such that \( R_e \) is commutative. If \( X \) is a subgroup of \( H \) such that \( X \subseteq Z(G) \) and \( \alpha_g(a) = a \) for all \( g \in H \) and \( a \in R_e \), then \( \bigcap_{X \subseteq Z(G)} \ker(K_{s_x}) \neq \{0\} \) for every non-zero two sided ideal \( J \) of \( R \).

**Proof.** Let \( J \) be a nonzero two sided ideal of \( R \). Let \( x \in X \) and \( s_x \in R_x \). Define \( K_{s_x}: R \to R \) by

\[
 K_{s_x}(t) = K_{s_x} \left( \sum_{g \in H} t_g \right) = s_x \sum_{g \in H} t_g - \sum_{g \in H} t_g s_x = \sum_{j \in H} k_j.
\]

Note that \( k_x = s_x t_e - t_e s_x = \alpha_x(t_e) s_x - t_e s_x = t_e s_x - t_e s_x = 0 \). On the other hand, \( k_{x g} = s_x t_g - t_g s_x \in R_{x g} = R x g \) might be zero or nonzero. Thus the number of elements in \( \text{supp}(K_{s_x}(t)) \) is less than \( \text{supp}(t) \). Moreover,

\[
 C_R(R_X) = \bigcap_{x \in X, s_x \in R_x} \ker(K_{s_x}).
\]

Let \( t = \sum_{g \in H} t_g \in J \) be a nonzero element. We may assume that \( t_e \neq 0 \). Otherwise, there exists a nonzero \( t' = \sum_{g \in H} t'_g \in J \) such that \( t'_e \neq 0 \). So, there exists \( y \in G \) such that \( t_y \neq 0 \). Also, there exists \( b_{y-1}^{(i)} \in R_{y-1} \) such that \( b_{y-1}^{(i)} t_y \neq 0 \) where \( i = 1, \ldots, n_y \) for some positive integer \( n_y \), this is because if \( b_{y-1}^{(i)} t_y = 0 \) for all \( i = 1, \ldots, n_y \), then we choose \( a_y^{(i)} \in R_y \) such that

\[
 \sum_{i=1}^{n_y} a_y^{(i)} b_{y-1}^{(i)} = 1
\]
and then
\[ t_y = 1 \cdot t_y = \sum_{i=1}^{n_y} a_{y}^{(i)} b_{y-1}^{(i)} t_y = 0. \]

Hence, for every \( t \in J \) there exists \( b_{y-1}^{(i)} t = t' = \sum_{g \in H} t'_g \) in \( J \) such that \( t'_e = b_{y-1}^{(i)} t_y \neq 0 \) and \(|supp(t)| \geq |supp(t')| \geq 1\). Now, we assumed that \( t = \sum_{g \in H} t_g \in J \) such that \( t_e \neq 0 \). If \( t \in C_R(R_X) \), then it is done. Suppose that \( t \notin C_R(R_X) \). Then there exists \( x \in X \) and \( s_x \in R_X \) such that \( K_{s_x}(t) \neq 0 \). Since \( K_{s_x}(t) \in J \), we find an element in \( J \) with smaller support. Keep on this procedure, we will stop since \( supp(t) \) is finite. Thus, we will find an element \( \xi = \sum_{g \in H} \xi_g \in J \bigcap C_R(R_X) \) such that \( \xi_e \neq 0 \). \( \square \)

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