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SOME NOTES ON FIRST STRONGLY GRADED RINGS

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Abstract. Let G be a group with identity e and R be an associative ring with a nonzero unity 1. Assume that R is first strongly G-graded and $H = supp(R,G)$. For $g \in H$, define $\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}$ where $x \in C_R(R_e) = \{r \in R : rx = xr \text{ for all } x \in R_e\}$, $r_g^{(i)} \in R_g$ and $t_{g^{-1}}^{(i)} \in R_{g^{-1}}$ for all $i = 1, \ldots\ldots, n_g$ for some positive integer n_g. In this article, we study $\alpha_g(x)$ and it's properties.

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1. INTRODUCTION

Throughout this article, R is an associative ring with nonzero unity 1. For a ring R and a subset T of R, $C_R(T) = \{r \in R : rt = tr \text{ for all } t \in T\}$. For a group G, $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$. Let G be a group with identity e. Then R is said to be G-graded if there exist additive subgroups R_g of R such that $\mathbf{R} = \bigoplus_{g \in G} \mathbf{R}_g$ where $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. For $x \in R$, $x = \sum_{g \in G} x_g$ where x_g is the component of x in R_g. Also, $supp(R,G) = \{g \in G : R_g \neq 0\}$. Moreover, R_e is a subring of R and $1 \in R_e$. For more details, see [4]. Throughout this article, $H = supp(R,G)$.

First strongly graded rings have been introduced by Refai in [5]. A G-graded ring R is said to be first strongly graded if $1 \in R_g R_{g^{-1}}$ for all $g \in H$. R is first strongly G-graded if and only if H is a subgroup of G and $\mathbf{R}_g \mathbf{R}_h = \mathbf{R}_{gh}$ for all $g, h \in H$. For more details, see [5].

Definition 1 ([4])**.** Let R be a ring. Suppose that $\alpha : G \to Aut(R)$ and $\beta : G \times G \to U(R)$ where $Aut(R)$ is the group of automorphisms of R and $U(R)$ is the group of units of R. In [4], (R, G, α, β) is said to be crossed system if the following conditions hold for all $g, h, s \in G$ and $a \in R$.

(1) $\alpha_g(\alpha_h(a))\beta(g,h) = \beta(g,h)\alpha_{gh}(a)$.

(2) $\beta(g,h)\beta(gh,s) = \alpha_g(\beta(h,s))\beta(g,hs)$.
(3) $\beta(g,e) = \beta(e,g) = 1$.

In [2], a G-graded ring R is said to be crossed product over the support if $R_g \bigcap U(R) \neq \phi$ for all $g \in H$. In [1], it was shown that if R is crossed product over the support, then R is first strongly graded and then by [5], H is a subgroup of G with $\mathbf{R}_g\mathbf{R}_h = \mathbf{R}_{gh}$ for all $g,h \in H$.

Suppose that R is crossed product over the support. We may choose the family $\{u_g\}_{g\in H}$ in R such that $u_g \in R_g \bigcap U(R)$ for all $g \in H$ and assume that $u_e = 1$. So, $R_g = R_e u_g = u_g R_e$ and $\{u_g\}_{g\in H}$ is a basis for the left (right) R_e-module R. Define the map $\alpha : H \to Aut(R_e)$ by $\alpha(g) = \alpha_g$ where $\alpha_g(a) = u_g a u_g^{-1}$ for all $g \in H$ and $a \in R_e$. Also, define $\beta : H \times H \to U(R_e)$ by $\beta(g,h) = u_g u_h u_{gh}^{-1}$ for all $g,h \in H$. Then α and β satisfy the conditions (1), (2) and (3) above (see [1]). Hence, (R_e, H, α, β) is a crossed system.

Assume that R is first strongly G-graded. For $g \in H$, define $\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}$ where $x \in C_R(R_e)$, $r_g^{(i)} \in R_g$ and $t_{g^{-1}}^{(i)} \in R_{g^{-1}}$ for all $i = 1, ...,n_g$ for some positive integer n_g. In this article, we study $\alpha_g(x)$ and it's properties.

2. RESULTS

In this section, we introduce our results.

Let R be a G-graded ring and X be a non-empty subset of G. Then $R_X = \bigoplus_{g\in X} R_g$. If X is a subgroup of G, then R_X is a subring of R. For more details, see [3]. We begin our results by the following.

Theorem 1. *Consider the above crossed system (R_e, H, α, β). Suppose that X is a subgroup of H such that $X \subseteq Z(H) \bigcap Ker(\alpha)$ and $\beta(x,y) = \beta(y,x)$ for all $(x,y) \in X \times X$. If R_e is commutative, then R_X is commutative.*

Proof. Consider the family $\{u_g\}_{g\in H}$ above. Let $g,h \in X$ and $a_g, b_h \in R_e$. Then

$$(a_g u_g)(b_h u_h) = a_g \alpha_g(b_h)\beta(g,h)u_{gh} = a_g b_h \beta(g,h)u_{gh} = \\ b_h \alpha_h(a_g)\beta(h,g)u_{hg} = (b_h u_h)(a_g u_g).$$

Hence, R_X is commutative. □

Let R be a first strongly G-graded ring (not necessary to be crossed product over the support). Then $R_g R_{g^{-1}} = R_e$ for all $g \in H$. So, for every $g \in H$, there exists $n_g \in \mathbf{Z}^+$, $r_g^{(i)} \in R_g$ and $t_{g^{-1}}^{(i)} \in R_{g^{-1}}$ such that $1 = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}$ since $1 \in R_e$.

Define $\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}$ for all $x \in C_R(R_e)$.

Theorem 2. *Let R be a first strongly graded ring. Then α_g is independent of the choice of $r_g^{(i)}$'s and $t_{g^{-1}}^{(i)}$'s.*

Proof. Let $n_g, n'_g \in \mathbf{Z}^+$ and $r_g^{(i)}, s_g^{(i)} \in R_g$, $t_{g^{-1}}^{(i)}, w_{g^{-1}}^{(i)} \in R_{g^{-1}}$ such that

$$1 = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} = \sum_{j=1}^{n'_g} s_g^{(j)} w_{g^{-1}}^{(j)}.$$

Let $x \in C_R(R_e)$. Then since $w_{g^{-1}}^{(j)} r_g^{(i)} \in R_e$,

$$\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} - \sum_{j=1}^{n'_g} s_g^{(j)} x w_{g^{-1}}^{(j)} = 1.\left(\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}\right) - \left(\sum_{j=1}^{n'_g} s_g^{(j)} x w_{g^{-1}}^{(j)}\right).1 =$$
$$\sum_{j=1}^{n'_g}\sum_{i=1}^{n_g} s_g^{(j)} w_{g^{-1}}^{(j)} r_g^{(i)} x t_{g^{-1}}^{(i)} - \sum_{j=1}^{n'_g}\sum_{i=1}^{n_g} s_g^{(j)} x w_{g^{-1}}^{(j)} r_g^{(i)} t_{g^{-1}}^{(i)} = 0.$$

□

The next lemma is fundamental for our next results.

Lemma 1. *Let R be a first strongly graded ring. If $r \in R$ such that $rR_h = \{0\}$ for some $h \in H$, then $r = 0$.*

Proof. Suppose that $r \in R$ and $h \in H$ such that $rR_h = \{0\}$. Then $r = r.1 \in rR_e = rR_h R_{h^{-1}} = (rR_h) R_{h^{-1}} = \{0\}$, i.e., $r = 0$. □

Theorem 3. *Let R be a first strongly graded ring and $g \in H$. Then $\alpha_g(x)$ is the only element of R satisfies $\alpha_g(x) a_g = a_g x$ for all $a_g \in R_g$. Moreover, $\alpha_g(x) \in C_R(R_e)$ and if $x \in Z(R_e)$, then $\alpha_g(x) \in Z(R_e)$.*

Proof. Let $a_g \in R_g$. Then $t_{g^{-1}}^{(i)} a_g \in R_{g^{-1}} R_g = R_e$ and $t_{g^{-1}}^{(i)} a_g$ commutes with $x \in C_R(R_e)$ for all $i = 1, ...,n_g$. So,

$$\alpha_g(x) a_g = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} a_g = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} a_g x = a_g \left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}\right) x$$
$$= a_g.1.x = a_g x.$$

Let $x \in C_R(R_e)$ and $y \in R$ such that $r_g^{(i)} x = y r_g^{(i)}$ for all $i = 1, ...,n_g$. Then

$$\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} = \sum_{i=1}^{n_g} y r_g^{(i)} t_{g^{-1}}^{(i)} = y\left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}\right) = y.1 = y,$$

i.e., $\alpha_g(x)$ is the only element satisfies $\alpha_g(x)a_g = a_g x$ for all $a_g \in R_g$. Since R is first strongly graded, if $x \in R_e$, then $\alpha_g(x) \in R_e$. In particular, if $x \in Z(R_e)$, then $\alpha_g(x) \in Z(R_e)$. So, for $x \in Z(R_e)$ and $s \in R_e$,

$$s\alpha_g(x) = 1.s\alpha_g(x) = \sum_{i=1}^{n_g}\sum_{j=1}^{n'_g} r_g^{\prime(j)} t_{g^{-1}}^{\prime(j)} s r_g^{(i)} x t_{g^{-1}}^{(i)}.$$

Since $t_{g^{-1}}^{\prime(j)} s r_g^{(i)} \in R_e$,

$$s\alpha_g(x) = \sum_{i=1}^{n_g}\sum_{j=1}^{n'_g} r_g^{\prime(j)} x t_{g^{-1}}^{\prime(j)} s r_g^{(i)} t_{g^{-1}}^{(i)} = \left(\sum_{j=1}^{n'_g} r_g^{\prime(j)} x t_{g^{-1}}^{\prime(j)}\right).s.\left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}\right) = \alpha_g(x).s.1 = \alpha_g(x)s.$$

If $a_g \in R_g$ and $w \in R_e$, then $wa_g \in R_e R_g = R_g$ and then

$$(\alpha_g(x)w)a_g = \alpha_g(x)(wa_g) = (wa_g)x = w(a_g x) = w(\alpha_g(x)a_g)$$

$$= (w\alpha_g(x))a_g$$

Which implies that $(\alpha_g(x)w - w\alpha_g(x))R_g = \{0\}$. By Lemma 1, $\alpha_g(x)w = w\alpha_g(x)$ and hence $\alpha_g(x) \in C_R(R_e)$. □

Theorem 4. *Let R be a first strongly graded ring. Then the following hold:*

(1) $\alpha_g \circ \alpha_h = \alpha_{gh}$ *for all* $g, h \in H$.
(2) $\alpha_g^{-1} = \alpha_{g^{-1}}$ *for all* $g \in H$.
(3) $\alpha_g(xb) = \alpha_g(x)\alpha_g(b)$ *for all* $g \in H$ *and* $x, b \in C_R(R_e)$.

Proof. Since $1 \in R_e$, $x = 1.x = \alpha_e(x).1 = \alpha_e(x)$ for all $x \in C_R(R_e)$. Let $g, h \in H$, $a_g \in R_g$ and $a_h \in R_h$. Then $a_g a_h \in R_g R_h = R_{gh}$ and then

$$\alpha_{gh}(x)(a_g a_h) = (a_g a_h)x = a_g(a_h x) = a_g(\alpha_h(x)a_h) = (a_g\alpha_h(x))a_h$$

$$= (\alpha_g(x)(\alpha_h(x))a_g)a_h = \alpha_g(x)(\alpha_h(x))(a_g a_h)$$

for all $x \in C_R(R_e)$ which implies that $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ by Lemma 1 as $a_g a_h$ generates the R_e-submodule R_{gh}.

Let $g \in H$, $s_{g^{-1}}^{(j)} \in R_{g^{-1}}$ and $w_g^{(j)} \in R$, $j = 1, \ldots.., n_{g^{-1}}$ such that $1 = \sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} w_g^{(j)}$.

Then for every $x \in C_R(R_e)$,

$$\alpha_{g^{-1}}(\alpha_g(x)) = \sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} \left(\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}\right) w_g^{(j)} =$$
$$\sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} \left(\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} w_g^{(j)}\right) = \sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} \left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} w_g^{(j)} x\right) =$$
$$\sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} \left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}\right) w_g^{(j)} x = \sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} .1. w_g^{(j)} x = \left(\sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} w_g^{(j)}\right) x$$
$$= 1.x = x$$

which implies that $(\alpha_g)^{-1} = \alpha_{g^{-1}}$. For $x, b \in C_R(R_e)$ and $a_g \in R_g$,

$$\alpha_g(xb)a_g = a_g(xb) = (a_g x)b = (\alpha_g(x)a_g)b = \alpha_g(x)(a_g b) =$$
$$\alpha_g(x)(\alpha_g(b)a_g) = (\alpha_g(x)\alpha_g(b))a_g.$$

By Lemma 1, $\alpha_g(xb) = \alpha_g(x)\alpha_g(b)$.

□

Theorem 5. *Let R be a first strongly graded ring. If $x \in C_R(R_e)$ and $g \in H$, then $xR_g = R_g x$ (x centralizes R_g) if and only if $\alpha_g(x) = x$.*

Proof. Suppose that R_g is centralized by x. Then for every $a_g \in R_g$,

$$\alpha_g(x)a_g = \left(\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}\right) a_g = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} a_g.$$

Since $t_{g^{-1}}^{(i)} r_g^{(i)} \in R_e$,

$$\alpha_g(x)a_g = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} a_g x = \left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}\right) a_g x = 1.a_g x = a_g x = x a_g$$

as x centralizes R_g. By Lemma 1, $\alpha_g(x) = x$. Conversely, for every $a_g \in R_g$, $a_g x = \alpha_g(x)a_g = xa_g$, i.e., x centralizes R_g □

Corollary 1. *Let R be a first strongly graded ring. Then $Z(R) = \{x \in C_R(R_e) : \alpha_g(x) = x \text{ for all } g \in H\}$.*

Proof. Since R is first strongly graded,
$Z(R) = \bigcap_{g \in H} C_R(R_g) = \{x \in C_R(R_e) : x \in C_R(R_g) \text{ for all } g \in H\}$. Note that if $g \notin H$, then $R_g = \{0\}$ and then $C_R(R_g) = R$. By Theorem 5, $x \in C_R(R_g)$ if and only if $\alpha_g(x) = x$ and hence $Z(R) = \{x \in C_R(R_e) : \alpha_g(x) = x \text{ for all } g \in H\}$. □

Remark 1. Also, it is nice to see that If R is a first strongly graded ring, then $C_R(R_e) =$

$$\left\{x = \sum_{g\in H} x_g \in R : x_g \in R_g \text{ with } cx_g = x_g c \text{ for all } g \in H \text{ and for all } c \in R_e\right\}$$

$$= \left\{x = \sum_{g\in H} x_g \in R : x_g \in \left(R_g \bigcap C_R(R_e)\right) \text{ for all } g \in H\right\}$$

$$= \bigoplus_{g\in H} \left(R_g \bigcap C_R(R_e)\right).$$

Note that for $g \notin H$, $R_g = \{0\}$ and then $x_g = 0$.

The next result is a generalization of Corollary 1.

Theorem 6. *Let R be a first strongly graded ring and X be a subgroup of H. Then $C_R(R_X) =$*

$$\left\{t = \sum_{g\in H} t_g \in R : t_g \in C_R(R_e)\bigcap R_g, \alpha_x(t_g) = t_{xgx^{-1}} \textit{ for all } g \in H \textit{ and for all } x \in X\right\}$$

$$= \{t \in C_R(R_e) : \alpha_x(t) = t \textit{ for all } x \in X\}.$$

Proof. Let $t = \sum_{g\in H} t_g \in C_R(R_X)$ where $t_g \in R_g$. Since $R_e \subseteq R_X$, $t \in C_R(R_e)$ and then by Remark 1, $t_g \in C_R(R_e)$ for all $g \in H$. Let $x \in X$. Then for every $s_x \in R_x$,

$$s_x \sum_{g\in H} t_g = \sum_{g\in H} t_g s_x$$

since $t \in C_R(R_X)$. As $t_g \in C_R(R_e)$, by Theorem 3,

$$\sum_{g\in H} \alpha_x(t_g)s_x = \sum_{g\in H} t_g s_x.$$

Since $R_x R_g R_{x^{-1}} R_x = R_{xg}$, for all $g \in H$ and for all $x \in X$,

$$\alpha_x(t_g)s_x = t_{xgx^{-1}} s_x \text{ for all } s_x \in R_x.$$

Choose $a_x^{(i)} \in R_x$ and $b_{x^{-1}}^{(i)} \in R_{x^{-1}}$ where $i = 1, \ldots.., n_x$ for some positive integer n_x such that

$$1 = \sum_{i=1}^{n_x} a_x^{(i)} b_{x^{-1}}^{(i)}.$$

Then

$$\alpha_x(t_g) = \alpha_x(t_g).1 = \alpha_x(t_g)\sum_{i=1}^{n_x} a_x^{(i)} b_{x^{-1}}^{(i)} = \sum_{i=1}^{n_x} \alpha_x(t_g) a_x^{(i)} b_{x^{-1}}^{(i)} =$$
$$\sum_{i=1}^{n_x} t_{xgx^{-1}} a_x^{(i)} b_{x^{-1}}^{(i)} = t_{xgx^{-1}} \sum_{i=1}^{n_x} a_x^{(i)} b_{x^{-1}}^{(i)} = t_{xgx^{-1}}.$$

For the converse, Suppose that $t = \sum_{g\in H} t_g \in R$ where $t_g \in C_R(R_e) \bigcap R_g$ and $\alpha_x(t_g) = t_{xgx^{-1}}$ for all $g \in H$ and for all $x \in X$. Then for every $s_x \in R_x$,

$$s_x t = \sum_{g\in H} s_x t_g = \sum_{g\in H} \alpha_x(t_g) s_x = \sum_{g\in H} t_{xgx^{-1}} s_x = \sum_{r\in H} t_r s_x = t s_x$$

which implies that $t \in C_R(R_X)$. □

Theorem 7. *Let R be a first strongly graded ring such that R_e is commutative. If X is a subgroup of H such that $X \subseteq Z(G)$ and $\alpha_g(a) = a$ for all $g \in H$ and $a \in R_e$, then $J \bigcap C_R(R_X) \neq \{0\}$ for every non-zero two sided ideal J of R.*

Proof. Let J be a nonzero two sided ideal of R. Let $x \in X$ and $s_x \in R_x$. Define $K_{s_x} : R \to R$ by

$$K_{s_x}(t) = K_{s_x}\left(\sum_{g\in H} t_g\right) = s_x \sum_{g\in H} t_g - \sum_{g\in H} t_g s_x = \sum_{j\in H} k_j.$$

Note that $k_x = s_x t_e - t_e s_x = \alpha_x(t_e) s_x - t_e s_x = t_e s_x - t_e s_x = 0$. On the other hand, $k_{xg} = s_x t_g - t_g s_x \in R_{gx} = R_{xg}$ might be zero or nonzero. Thus the number of elements in $supp(K_{s_x}(t)$ is less than $supp(t)$. Moreover,

$$C_R(R_X) = \bigcap_{x\in X, s_x\in R_x} Ker(K_{s_x}).$$

Let $t = \sum_{g\in H} t_g \in J$ be a nonzero element. We may assume that $t_e \neq 0$. Otherwise, there exists a nonzero $t' = \sum_{g\in H} t'_g \in J$ such that $t'_e \neq 0$. So, there exists $y \in G$ such that $t_y \neq 0$. Also, there exists $b_{y^{-1}}^{(i)} \in R_{y^{-1}}$ such that $b_{y^{-1}}^{(i)} t_y \neq 0$ where $i = 1, \ldots\ldots, n_y$ for some positive integer n_y, this is because if $b_{y^{-1}}^{(i)} t_y = 0$ for all $i = 1, \ldots\ldots, n_y$, then we choose $a_y^{(i)} \in R_y$ such that

$$\sum_{i=1}^{n_y} a_y^{(i)} b_{y^{-1}}^{(i)} = 1$$

and then

$$t_y = 1.t_y = \sum_{i=1}^{n_y} a_y^{(i)} b_{y^{-1}}^{(i)} t_y = 0.$$

Hence, for every $t \in J$ there exists $b_{y^{-1}}^{(j)} t = t' = \sum_{g \in H} t'_g$ in J such that $t'_e = b_{y^{-1}}^{(j)} t_y \neq 0$ and $|supp(t)| \geq |supp(t')| \geq 1$. Now, we assumed that $t = \sum_{g \in H} t_g \in J$ such that $t_e \neq 0$. If $t \in C_R(R_X)$, then it is done. Suppose that $t \notin C_R(R_X)$. Then there exists $x \in X$ and $s_x \in R_x$ such that $K_{s_x}(t) \neq 0$. Since $K_{s_x}(t) \in J$, we find an element in J with smaller support. Keep on this procedure, we will stop since $supp(t)$ is finite. Thus, we will find an element $\xi = \sum_{g \in H} \xi_g \in J \bigcap C_R(R_X)$ such that $\xi_e \neq 0$. □

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