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An interesting matrix equation

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AN INTERESTING MATRIX EQUATION

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Abstract. We study the solutions of a special matrix equation, particularly, their eigenvalues. These matrix solutions have an interesting relationship to unitary matrices.

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1. INTRODUCTION

In this note, we study the matrix equation

$$AEA^* = E, \quad (1.1)$$

where

$$E = ((-1)^r \delta_{r,n+1-j})_{r,j=1}^n, \quad n = 2k, \quad 1 \leq k < +\infty, \quad (1.2)$$

$\delta_{r,j}$ is the Kronecker delta, and A is an $n \times n$ matrix over the complex numbers \mathbb{C} . This study is motivated by the theory of differential operators where this equation plays a critical role in the characterization of self-adjoint domains [1–4].

2. EIGENVALUES

In this section we characterize the eigenvalues of matrix solutions of equation (1.1). Let $\mathbb{1}$ denote the identity matrix.

Lemma 1. *We have*

$$EE = -\mathbb{1}, \quad E^* = E^{-1} = -E, \quad \det(E) = 1. \quad (2.1)$$

If A is a solution of (1.1), then

$$\det(A) = \pm 1. \quad (2.2)$$

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Proof. The properties of E follow from a direct computation. For (2.2) note that $\det(A) = \det(A^*)$ and thus

$$\det(AEA^*) = \det(A)\det(E)\det(A^*) = [\det(A)]^2 = 1. \quad (2.3)$$

□

Remark 1. The simple examples $A = \mathbb{1}$ and $A = i\mathbb{1}$ show that both signs can occur in (2.2).

Lemma 2. *If A is a solution of (1.1), then**

$$\sigma(A^{-1}) = \sigma(A^*). \quad (2.4)$$

Proof. From (1.1) and (2.1) we get $AEA^*E = EE = -\mathbb{1}$. Hence $(AE)^{-1} = -A^*E$ and $A^* = -EA^{-1}E$, $A^{-1} = -EA^*E$. Therefore,

$$\begin{aligned} \det(A^{-1} - \lambda\mathbb{1}) &= \det(-EA^*E - \lambda E(-E)\mathbb{1}) \\ &= \det(-E)\det(A^* - \lambda\mathbb{1})\det(E) \\ &= \det(A^* - \lambda\mathbb{1}), \end{aligned}$$

and (2.4) follows. □

Remark 2. Of course (2.4) holds for any unitary matrix U , because $U^* = U^{-1}$. All eigenvalues of a unitary matrix lie on the unit circle of the complex plane. Below we derive the corresponding result for matrix solutions A of (1.1). We will see that the eigenvalues of A need not lie on the unit circle but those which do are related to each other in interesting ways and also those which are not on the unit circle have interesting relationships to each other.

Let the spectrum of A have the form

$$\sigma(A) = \{\lambda_j = r_j e^{i\theta_j} : j = 1, \dots, n\}, \quad (2.5)$$

where $r_j > 0$ and $-\pi < \theta_j \leq \pi$ for $j = 1, \dots, n$. We can now give our main result.

Theorem 1. *Let A be a solution of (1.1). Then the following properties hold:*

- (1) *If, for some j , an eigenvalue λ_j of A is not on the unit circle, i. e., $r_j \neq 1$, then $\frac{1}{r_j} e^{i\theta_j}$ is also an eigenvalue of A .*
- (2) *Let l be the number of pairs of eigenvalues of A not on the unit circle, given by (1) and counting multiplicity, then l can be any one of the numbers $0, 1, 2, \dots, k$.*
- (3) *Assume that there exist l ($0 \leq l \leq k$) pairs of eigenvalues*

$$r_1 e^{i\theta_1}, \frac{1}{r_1} e^{i\theta_1}, \dots, r_l e^{i\theta_l}, \frac{1}{r_l} e^{i\theta_l}$$

*As usual, the symbol $\sigma(A)$ stands for the spectrum of A .

of A which are not on the unit circle, where $r_j > 0$, $r_j \neq 1$, $j = 0, 1, \dots, l$ ($j = 0$ means that there are no such eigenvalues, i. e., all eigenvalues are on the unit circle), then there exist $d = 2(k - l)$ eigenvalues of A ,

$$e^{i\theta_{l+1}}, \dots, e^{i\theta_{l+d}}$$

on the unit circle and θ_j satisfy the relations

$$\sum_{j=1}^l 2\theta_j + \sum_{j=1}^d \theta_{l+j} \in \{j\pi : j = 0, \pm 2, \dots, \pm 2(k-1), 2k\} \quad (2.6)$$

if $\det(A) = 1$, and the relations

$$\sum_{j=1}^l 2\theta_j + \sum_{j=1}^d \theta_{l+j} \in \{j\pi : j = \pm 1, \pm 3, \pm 5, \dots, \pm(n-1)\} \quad (2.7)$$

if $\det(A) = -1$.

Proof. By Lemma 2, we have $\sigma(A^{-1}) = \sigma(A^*)$, i. e.,

$$\left\{ \frac{1}{\lambda_j} : j = 1, \dots, n \right\} = \{\bar{\lambda}_h : h = 1, \dots, n\}.$$

Therefore, for any $\bar{\lambda}_h \in \sigma(A^*)$, there exist $\frac{1}{\lambda_j}$ such that $\bar{\lambda}_h = \frac{1}{\lambda_j}$, i. e.,

$$r_h e^{-i\theta_h} r_j e^{i\theta_j} = 1.$$

Hence, $r_h r_j = 1$ and $\theta_j = \theta_h$ (or $\theta_j - \theta_h = 2\pi$). Therefore, if for some j , $r_j e^{i\theta_j}$ with $r_j > 0$, $-\pi < \theta_j \leq \pi$, and $r_j \neq 1$ is an eigenvalue of A , then $\frac{1}{r_j} e^{i\theta_j}$ is also an eigenvalue of A .

By (1), the eigenvalues not on the unit circle occur in pairs where each pair has the same angle and the conclusion follows.

Conditions (2.6) and (2.7) follow from the fact that $\det(A) = \prod_{j=1}^n \lambda_j = \pm 1$. \square

Remark 3. Let A satisfy (1.1). Then A may have all its eigenvalues on the unit circle or there may be none on the unit circle. In all other cases there are an even number of eigenvalues not on the unit circle and these can be paired so that any one pair has the same angle; the angles of the eigenvalues are related to each other according to (2.6) and (2.7). The examples in the next section will illustrate these points further.

3. EXAMPLES

In this section we give some examples to illustrate Theorem 1.

Example 1. Let $n = 2$. In the second order case either both eigenvalues are on the unit circle or neither one is.

CASE 1. Both eigenvalues are on the unit circle, i. e., $\sigma(A) = \{e^{i\theta_1}, e^{i\theta_2}\}$.

- (1) When $\det(A) = 1$, we have $e^{i(\theta_1+\theta_2)} = 1$ and $\theta_2 = -\theta_1$ or $\theta_1 + \theta_2 = 2\pi$. Therefore, $\sigma(A) = \{e^{i\theta}, e^{-i\theta}\}$. Note that the eigenvalues of A are symmetric with respect to the real axis. When $\lambda = 1$ or $\lambda = -1$ is an eigenvalue, then it must have multiplicity 2.
- (2) If $\det(A) = -1$, then we have $\theta_1 + \theta_2 = \pi$ and the eigenvalues of A are symmetric about the imaginary axis. When $\lambda = i$ or $\lambda = -i$ is an eigenvalue, then it must be a double eigenvalue.

CASE 2. The eigenvalues of A are not on the unit circle, i. e., $\sigma(A) = \{re^{i\theta}, \frac{1}{r}e^{i\theta}\}$, where $r > 0$ and $r \neq 1$.

- (1) When $\det(A) = 1$, we have $e^{i(2\theta)} = 1$ and $\theta = 0$ or π . Then the eigenvalues of A are r and $\frac{1}{r}$, or $-r$ and $-\frac{1}{r}$.
- (2) If $\det(A) = -1$, then we have $e^{i(2\theta)} = -1$ and $\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$. Then the eigenvalues of A are ri and $\frac{1}{r}i$, or $-ri$ and $-\frac{1}{r}i$.

Example 2. Assume that $n = 4$. Then exactly one of the following three cases must hold:

- (1) All the eigenvalues are on the unit circle, i. e., $\sigma(A) = \{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}\}$, where the numbers θ_i are such that $\sum_{j=1}^4 \theta_j = 0, \pm 2\pi, 4\pi$ if $\det(A) = 1$, and $\sum_{j=1}^4 \theta_j = \pm\pi, \pm 3\pi$ if $\det(A) = -1$.
- (2) One pair of eigenvalues is not on the unit circle and the other two eigenvalues are on the unit circle, i. e.,

$$\sigma(A) = \left\{ e^{i\theta_1}, e^{i\theta_2}, re^{i\theta}, \frac{1}{r}e^{i\theta} \right\},$$

where $r > 0$ and $r \neq 1$. In this case θ_1, θ_2 , and θ satisfy $\theta_1 + \theta_2 + 2\theta = 0, \pm 2\pi, 4\pi$ when $\det(A) = 1$, and $\theta_1 + \theta_2 + 2\theta = \pm\pi, \pm 3\pi$ when $\det(A) = -1$.

- (3) No eigenvalue of A is on the unit circle. In this case there are two pairs of eigenvalues each pair having the same angle, i. e.,

$$\sigma(A) = \left\{ r_1 e^{i\theta_1}, \frac{1}{r_1} e^{i\theta_1}, r_2 e^{i\theta_2}, \frac{1}{r_2} e^{i\theta_2} \right\},$$

where $r_j > 0$ and $r_j \neq 1$, and the numbers θ_1, θ_2 satisfy $\theta_1 + \theta_2 = 0, \pm\pi, 2\pi$ when $\det(A) = 1$, and $\theta_1 + \theta_2 = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ when $\det(A) = -1$. Here θ_1 and θ_2 are not necessarily distinct.

Example 3. Assume that $n = 6$. There are exactly four subcases:

- (1) All eigenvalues are on the unit circle, i. e.,

$$\sigma(A) = \{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}, e^{i\theta_6}\},$$

where the numbers θ_j satisfy $\sum_{j=1}^6 \theta_j = 0, \pm 2\pi, \pm 4\pi, 6\pi$ if $\det(A) = 1$, and $\sum_{j=1}^6 \theta_j = \pm\pi, \pm 3\pi, \pm 5\pi$, if $\det(A) = -1$.

- (2) One pair is not on the unit circle and the other four eigenvalues are on the unit circle, i. e.,

$$\sigma(A) = \left\{ e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, r e^{i\theta}, \frac{1}{r} e^{i\theta} \right\},$$

where $r > 0$, $r \neq 1$, and the numbers θ_j and θ satisfy

$$\sum_{j=1}^4 \theta_j + 2\theta = 0, \pm 2\pi, \pm 4\pi, 6\pi$$

when $\det(A) = 1$, and

$$\sum_{j=1}^4 \theta_j + 2\theta = \pm\pi, \pm 3\pi, \pm 5\pi$$

when $\det(A) = -1$.

- (3) Two pairs are not on the unit circle and the other two eigenvalues are on the unit circle, i. e.,

$$\sigma(A) = \left\{ e^{i\theta_1}, e^{i\theta_2}, r_1 e^{i\theta_3}, \frac{1}{r_1} e^{i\theta_3}, r_2 e^{i\theta_4}, \frac{1}{r_2} e^{i\theta_4} \right\},$$

where $r_j > 0$ and $r_j \neq 1$, and the numbers θ_j satisfy

$$\theta_1 + \theta_2 + 2\theta_3 + 2\theta_4 = 0, \pm 2\pi, \pm 4\pi, 6\pi,$$

when $\det(A) = 1$, and

$$\theta_1 + \theta_2 + 2\theta_3 + 2\theta_4 = \pm\pi, \pm 3\pi, \pm 5\pi.$$

when $\det(A) = -1$.

- (4) No eigenvalue is on the unit circle, i. e.,

$$\sigma(A) = \left\{ r_1 e^{i\theta_1}, \frac{1}{r_1} e^{i\theta_1}, r_2 e^{i\theta_2}, \frac{1}{r_2} e^{i\theta_2}, r_3 e^{i\theta_3}, \frac{1}{r_3} e^{i\theta_3} \right\},$$

where $r_j > 0$ and $r_j \neq 1$, and the numbers θ_j satisfy

$$\sum_{j=1}^3 \theta_j = 0, \pm\pi, \pm 2\pi, 3\pi \quad (3.1)$$

when $\det(A) = 1$, and

$$\sum_{j=1}^3 \theta_j = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \quad (3.2)$$

when $\det(A) = -1$.

Example 4. Let

$$A = \begin{pmatrix} r_1 e^{i\theta_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & r_2 e^{i\theta_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & r_3 e^{i\theta_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r_3} e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r_2} e^{i\theta_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{r_1} e^{i\theta_1} \end{pmatrix} \quad (3.3)$$

Then A satisfies (1.1) for any $r_j > 0$ and $\theta_j \in (-\pi, \pi]$. Choosing $r_j = 1$ for $j = 1, 2, 3$ all six eigenvalues are on the unit circle and their angles satisfy the equations (3.1) or (3.2). By choosing three distinct θ_j we get three distinct eigenvalues each with multiplicity 2, by choosing $\theta_1 = \theta_2 \neq \theta_3$ we get two distinct eigenvalues, one having multiplicity 2, the other 4. Finally by choosing $\theta_1 = \theta_2 = \theta_3$ we get one eigenvalue with multiplicity 6. Clearly this example extends to $n = 2k$ for $k > 3$ and can be used to show that the eigenvalues on the unit circle can have multiplicity m for any m , $m = 2, 4, 6, \dots, 2k$.

Remark 4. Just as in the above examples, for $n = 2k$ there are exactly $k + 1$ cases ranging from exactly j pairs not on the unit circle for $j = 1, 2, \dots, k$ to all eigenvalues on the unit circle.

Remark 5. It is interesting to compare matrices satisfying (1.1) with unitary matrices. Clearly equation (1.1) is equivalent with

$$(AE)(EA)^* = \mathbb{1} \quad (3.4)$$

which can be compared to

$$UU^* = \mathbb{1},$$

but as Theorem 1 shows there are major differences, as well as similarities, in the behavior of the eigenvalues of the matrices satisfying these two equations. Lemma 1 holds for unitary matrices since $U^{-1} = U^*$ but all the eigenvalues of a unitary matrix lie on the unit circle. The fact that one E in (3.4) is “on the wrong side” is significant.

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