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CERTAIN FAMILIES OF MULTIVARIABLE CHAN-CHYAN-SRIVASTAVA POLYNOMIALS

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Abstract. In this paper we introduce $r-$parameter Srivastava polynomials in $r-$variable by inserting new indices. These polynomials include the Lagrange polynomials in several variables, which are also known as Chan-Chyan-Srivastava polynomials (W.-C.C. Chan, C.-J. Chyan and H.M. Srivastava, The Lagrange polynomials in several variables, *Integral Transforms and Special Functions*, 12 (2001) 139–148). We prove several two sided linear generating relations between $r-$variable and $(r-1)-$ variable Chan-Chyan-Srivastava polynomials. Some special cases of the main results are also presented.

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1. INTRODUCTION

In 1972, Srivastava [21] introduced the following family of polynomials,

$$S_n^N(z) := \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n,k} z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; N \in \mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a \in \mathbb{R}$, and $(\lambda)_\nu$, $(\lambda)_0 \equiv 1$, denotes the Pochhammer symbol defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

by means of familiar Gamma functions. After, González et al. [13] extended the Srivastava polynomials $S_n^N(z)$ as follows:

$$S_{n,m}^N(z) := \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n+m,k} z^k \qquad (m,n \in \mathbb{N}_0; N \in \mathbb{N}), \tag{1.2}$$

and investigated their properties extensively. In [3], the following family of bivariate polynomials has been introduced

$$S_n^{m,N}(x,y)=\sum_{k=0}^{[\frac{n}{N}]} A_{m+n,k}\frac{x^{n-Nk}}{(n-Nk)!}\frac{y^k}{k!} \qquad (m,n\in\mathbb{N}_0, N\in\mathbb{N}),$$

and it has shown that the polynomials $S_n^{m,N}(x,y)$ includes many well known polynomials such as Lagrange-Hermite polynomials, Lagrange polynomials and Hermite-Kampé de Feriét polynomials (see also [20]). In [23], Srivastava et al. have introduced the three-variable polynomials

$$S_n^{m,M,N}(x,y,z)=\sum_{k=0}^{[\frac{n}{N}]}\sum_{l=0}^{[\frac{k}{M}]} A_{m+n,k,l}\frac{x^l}{l!}\frac{y^{k-Ml}}{(k-Ml)!}\frac{z^{n-Nk}}{(n-Nk)!}$$
$$(m,n\in\mathbb{N}_0; M,N\in\mathbb{N}), \tag{1.3}$$

where $\{A_{m,n,k}\}$ be a triple sequence of complex numbers. Suitable choices of $\{A_{m,n,k}\}$ in equation (1) gives three variable version of well-known polynomials (see also [14]). In [15], the multivariable extension of the Srivastava polynomials in r-variable was introduced:

$$S_n^{m,N_1,N_2,\dots,N_{r-1}}(x_1,x_2,\dots,x_r) \tag{1.4}$$
$$=\sum_{k_{r-1}=0}^{\left[\frac{n}{N_{r-1}}\right]}\sum_{k_{r-2}=0}^{\left[\frac{k_{r-1}}{N_{r-2}}\right]}\cdots\sum_{k_1=0}^{\left[\frac{k_2}{N_1}\right]} A_{m+n,k_{r-1},k_1,\dots,k_{r-2}}\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2-N_1k_1}}{(k_2-N_1k_1)!}\cdots\frac{x_r^{n-N_{r-1}k_{r-1}}}{(n-N_{r-1}k_{r-1})!}$$
$$(m,n\in\mathbb{N}_0; N_1,N_2,\dots,N_{r-1}\in\mathbb{N}),$$

where $\{A_{m,k_{r-1},k_1,k_2,\dots,k_{r-2}}\}$ be a sequence of complex numbers. Recently, in [16], two-parameter one-variable Srivastava polynomials

$$S_n^{m_1,m_2}(x):=\sum_{k=0}^{n}\frac{(-n)_k}{k!}A_{m_1+m_2+n,\, m_2+k}\, x^k \tag{1.5}$$
$$(m_1,\ m_2,\ n,\ k\in\mathbb{N}_0),$$

two-parameter two-variable Srivastava polynomials

$$S_n^{m_1,m_2}(x,y)=\sum_{k=0}^{n} A_{m_1+m_2+n,m_2+k}\frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!} \tag{1.6}$$
$$(m_1,\ m_2,\ n,\ k\in\mathbb{N}_0),$$

and two-parameter three-variable Srivastava polynomials

$$S_n^{m_1,m_2,M}(x,y,z)=\sum_{k=0}^{n}\sum_{l=0}^{\left[\frac{k}{M}\right]}A_{m_1+m_2+n,m_2+k,l}\,\frac{x^l}{l!}\,\frac{y^{k-Ml}}{(k-Ml)!}\,\frac{z^{n-k}}{(n-k)!} \tag{1.7}$$
$$(m_1,\ m_2,\ n,\ k,\ l\in\mathbb{N}_0,\ M\in\mathbb{N})$$

were introduced. These polynomials include the family of polynomials which were introduced or investigated in [3, 13, 15, 17, 19, 21, 23]. In this paper we introduce $r-$parameter Srivastava polynomials in $r-$variables by inserting new indices. These polynomials include the Lagrange polynomials in several variables, which are also known as Chan-Chyan-Srivastava polynomials [4]. We prove several two sided linear generating relations and obtain various generating relations for the polynomials.

2. R-PARAMETER R-VARIABLE SRIVASTAVA POLYNOMIALS

In this section we define r-parameter r-variable Srivastava polynomials as follows:

$$S_n^{m_1,m_2,\ldots,m_{r-1},M}(x_1,x_2,\ldots,x_r) \tag{2.1}$$
$$=\sum_{k_{r-1}=0}^{n}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{\left[\frac{k_2}{M}\right]}A_{\Omega}\,\frac{x_1^{k_1}}{k_1!}\,\frac{x_2^{k_2-Mk_1}}{(k_2-Mk_1)!}\cdots\frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!},$$
$$(m_1,m_2,\ldots,m_{r-1},n\in\mathbb{N}_0;M\in\mathbb{N}),$$

where $\Omega=m_1+m_2+\ldots+m_{r-1}+n,\ m_2+\ldots+m_{r-1}+k_{r-1},\ k_1,\ m_{r-1}+k_2,\ m_{r-2}+m_{r-1}+k_3,\ldots,\ m_3+\ldots+m_{r-1}+k_{r-2}$. Note that appropriate choices of the sequence $\{A_{m_1+m_2,\,m_2,\,k_1,\,m_{r-1},\ldots,\,m_3}\}$ in (2.1) give the $r-$variable versions of the well known polynomials.

Remark 1. The Lagrange polynomials in several variables, which are known as Chan-Chyan-Srivastava polynomials [4] are defined by the generating function relation

$$\prod_{j=1}^{r}(1-x_jt)^{-\alpha_j}=\sum_{n=0}^{\infty}g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(x_1,x_2,\ldots,x_r)t^n$$
$$(\alpha_j\in\mathbb{C}\ (j=1,\ldots,r);|t|<\min\left\{|x_1|^{-1},\ldots,|x_r|^{-1}\right\}).$$

The polynomials are given explicitly by

$$g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(x_1,x_2,\ldots,x_r)$$
$$=\sum_{k_{r-1}=0}^{n}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}(\alpha_1)_{k_1}(\alpha_2)_{k_2-k_1}\cdots(\alpha_{r-1})_{k_{r-1}-k_{r-2}}(\alpha_r)_{n-k_{r-1}}$$

$$\times \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-k_1}}{(k_2-k_1)!} \cdots \frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!} \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!}.$$

Many authors have studied the properties of these polynomials. For example, the bilateral generating functions for these polynomials and miscellaneous properties are given in Liu et al. [12, 18]. In [8], the orthogonality properties and various integral representations for these polynomials are given (see also [1, 2, 5–7]). Furthermore, these polynomials are used in approximation theory. In [11], Duman et al. investigated some approximation properties of positive linear operators constructed by these polynomials (see also [9, 10]). In [22] Srivastava et al. investigated umbral calculus presentations of these polynomials.

Remark 2. If we set $M = 1$ and

$$A_{m,n,k_2,\ldots,k_{r-2},k_{r-1}} = (\alpha_1)_{k_2} (\alpha_2)_{k_3-k_2} \cdots (\alpha_{r-2})_{k_{r-1}-k_{r-2}} (\alpha_{r-1})_{n-k_{r-1}} (\alpha_r)_{m-n}$$

in equation (2.1), we find that

$$\begin{aligned}
&A_{m_1+\ldots+m_{r-1}+n,\, m_2+\ldots+m_{r-1}+k_{r-1},\, k_1,\, m_{r-1}+k_2, m_{r-2}+m_{r-1}+k_3,\ldots,\, m_3+\ldots+m_{r-1}+k_{r-2}} \\
&= (\alpha_1)_{k_1} (\alpha_2)_{m_{r-1}+k_2-k_1} (\alpha_3)_{m_{r-2}+k_3-k_2} \cdots (\alpha_{r-1})_{m_2+k_{r-1}-k_{r-2}} (\alpha_r)_{m_1+n-k_{r-1}} \\
&= (\alpha_2)_{m_{r-1}} (\alpha_3)_{m_{r-2}} \cdots (\alpha_{r-1})_{m_2} (\alpha_r)_{m_1} (\alpha_1)_{k_1} (\alpha_2+m_{r-1})_{k_2-k_1} (\alpha_3+m_{r-2})_{k_3-k_2} \\
&\times \ldots (\alpha_{r-1}+m_2)_{k_{r-1}-k_{r-2}} (\alpha_r+m_1)_{n-k_{r-1}}
\end{aligned}$$

and therefore

$$\begin{aligned}
&S_n^{m_1,m_2,\ldots,m_{r-1},1}(x_1,x_2,\ldots,x_r) = \\
&(\alpha_2)_{m_{r-1}} (\alpha_3)_{m_{r-2}} \cdots (\alpha_{r-1})_{m_2} (\alpha_r)_{m_1} g_n^{(\alpha_1,\alpha_2+m_{r-1},\alpha_3+m_{r-2},\ldots,\alpha_r+m_1)}(x_1,\ldots,x_r).
\end{aligned}$$

Now, let recall an infinite series identities which were obtained by Srivastava et al. [23].

Lemma 1 (see [23], Lemma 1)**.** *Let* $N_1, N_2, \ldots, N_{r-1} \in \mathbb{N}$, $r = \{2,3,\ldots\}$. *Then*

$$\begin{aligned}
&\sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} A(n_1,n_2,\ldots,n_r) \\
&= \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\left[\frac{n_r}{N_{r-1}}\right]} \cdots \sum_{n_1=0}^{\left[\frac{n_2}{N_1}\right]} A(n_1, n_2 - N_1 n_1, \ldots, n_r - N_{r-1} n_{r-1})
\end{aligned}$$

and

$$\sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\left[\frac{n_r}{N_{r-1}}\right]} \cdots \sum_{n_1=0}^{\left[\frac{n_2}{N_1}\right]} A(n_1,n_2,\ldots,n_r)$$

$$= \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\infty} \dots \sum_{n_1=0}^{\infty} A(n_1, n_2 + N_1 n_1, n_3 + N_2 N_1 n_1 + N_2 n_2, \dots,$$
$$n_r + \prod_{j=1}^{r-1} N_j n_1 + \prod_{j=2}^{r-1} N_j n_2 + \dots + N_{r-1} n_{r-1})$$

where $\{A(n_1, n_2, \dots, n_r)\}$ *is a bounded* $r-$*tuple sequence of real or complex numbers.*

The main result of this paper is given by the following theorem.

Theorem 1. *Let* $\{f(n)\}_{n=0}^{\infty}$ *be a bounded sequence of complex numbers. Then*

$$\sum_{m_1,\dots,m_{r-1},n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) \tag{2.2}$$
$$\times S_n^{m_1,m_2,\dots,m_{r-1},M}(x_1, x_2, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n$$
$$= \sum_{k_1,m_1,\dots,m_{r-1}=0}^{\infty} f(m_1 + \dots + m_{r-1} + M k_1)$$
$$\times A_{m_1+m_2+\dots+m_{r-1}+Mk_1,\, m_2+\dots+m_{r-1}+Mk_1,\, k_1,\, m_{r-1}+Mk_1, \dots,\, m_3+\dots+m_{r-1}+Mk_1}$$
$$\times \frac{(x_1 t^M)^{k_1}}{k_1!} \frac{(x_r t + w_1)^{m_1}}{m_1!} \frac{(x_{r-1} t + w_2)^{m_2}}{m_2!} \dots \frac{(x_2 t + w_{r-1})^{m_{r-1}}}{m_{r-1}!}$$

provided that each member of the series identity (2.2) *exists.*

Proof. Let the left hand side of (2.2) be denoted by $\Psi(x_1, \dots, x_r)$. Then using the definition of $S_n^{m_1,m_2,\dots,m_{r-1},M}(x_1, x_2, \dots, x_r)$ on the left hand side of (2.2), we have

$$\Psi(x_1, \dots, x_r)$$
$$= \sum_{m_1,\dots,m_{r-1},n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n)$$
$$\times \sum_{k_{r-1}=0}^{n} \sum_{k_{r-2}=0}^{k_{r-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{\left[\frac{k_2}{M}\right]} A_{\Omega} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2 - Mk_1}}{(k_2 - Mk_1)!} \dots \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n.$$

Let define

$$F(r-1) := Mk_1 + k_2 + k_3 + \dots + k_{r-1}.$$

By applying Lemma 1, we find

$$\Psi(x_1, \dots, x_r)$$

$$= \sum_{k_1,...,k_{r-1},m_1,...,m_{r-1},n=0}^{\infty} f(m_1+...+m_{r-1}+n+F(r-1))$$
$$\times A_{\Omega_1} \frac{x_1^{k_1}}{k_1!} ... \frac{x_{r-1}^{k_{r-1}}}{k_{r-1}!} \frac{x_r^n}{n!} \frac{w_1^{m_1}}{m_1!} ... \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^{n+Mk_1+k_2+...+k_{r-1}}$$

where $\Omega_1 = m_1+m_2+...+m_{r-1}+n+F(r-1), m_2+...+m_{r-1}+k_{r-1}+F(r-2),$
$k_1, m_{r-1}+k_2+Mk_1, ..., m_3+...+m_{r-1}+k_{r-2}+F(r-3)$.
Let $m_1 \to m_1-n, m_2 \to m_2-k_{r-1}, m_3 \to m_3-k_{r-2},$
$.., m_{r-2} \to m_{r-2}-k_3, m_{r-1} \to m_{r-1}-k_2$ then

$$\Psi(x_1,...,x_r)$$
$$= \sum_{k_1,m_1,...,m_{r-1}=0}^{\infty} f(m_1+...+m_{r-1}+Mk_1)$$
$$\times A_{m_1+...+m_{r-1}+Mk_1,\, m_2+...+m_{r-1}+Mk_1,\, k_1,\, m_{r-1}+Mk_1,...,\, m_3+...+m_{r-1}+Mk_1}$$
$$\times \frac{(x_1t^M)^{k_1}}{k_1!} \frac{\sum_{n=0}^{m_1}\binom{m_1}{n}(x_rt)^n w_1^{m_1-n}}{m_1!} \frac{\sum_{k_{r-1}=0}^{m_2}\binom{m_2}{k_{r-1}}(x_{r-1}t)^{k_{r-1}} w_2^{m_2-k_{r-1}}}{m_2!}$$
$$\times ... \frac{\sum_{k_2=0}^{m_{r-1}}\binom{m_{r-1}}{k_2}(x_2t)^{k_2} w_{r-1}^{m_{r-1}-k_2}}{m_{r-1}!}$$

and

$$\Psi(x_1,...,x_r)$$
$$= \sum_{k_1,m_1,...,m_{r-1}=0}^{\infty} f(m_1+...+m_{r-1}+Mk_1)$$
$$\times A_{m_1+m_2+...+m_{r-1}+Mk_1,\, m_2+...+m_{r-1}+Mk_1,\, k_1,\, m_{r-1}+Mk_1,...,\, m_3+...+m_{r-1}+Mk_1}$$
$$\times \frac{(x_1t^M)^{k_1}}{k_1!} \frac{(x_rt+w_1)^{m_1}}{m_1!} \frac{(x_{r-1}t+w_2)^{m_2}}{m_2!} ... \frac{(x_2t+w_{r-1})^{m_{r-1}}}{m_{r-1}!}.$$

Whence the result. □

Corollary 1. *Let the polynomials $S_n^{m_1,m_2,...,m_{r-1},M}(x_1,x_2,...,x_r)$ be defined by equation (2.1). Suppose also that $r-1-$variable polynomials $P_{m_2}^M(x_1,...,x_{r-1})$ are defined by*

$$P_{m_2}^M(x_1,...,x_{r-1}) \tag{2.3}$$
$$= \sum_{m_3=0}^{m_2} ... \sum_{m_{r-1}=0}^{m_{r-2}} \sum_{k_1=0}^{[\frac{m_{r-1}}{M}]} A_{m_1+m_2,\, m_2,\, k_1,\, m_{r-1},...,\, m_3}$$

$$\times \frac{(x_1)^{k_1}}{k_1!} \frac{(x_2)^{m_{r-1}-Mk_1}}{(m_{r-1}-Mk_1)!} \frac{(x_3)^{m_{r-2}-m_{r-1}}}{(m_{r-2}-m_{r-1})!} \cdots \frac{(x_{r-1})^{m_2-m_3}}{(m_2-m_3)!}.$$

Then, for a suitably bounded sequence $\{f(n)\}_{n=0}^{\infty}$, the following family of two-sided linear generating relations holds true between the $r-$variable polynomials $S_n^{m_1,m_2,\ldots,m_{r-1},M}(x_1,\ldots,x_r)$ and $(r-1)-$variable polynomials $P_{m_2}^{M}(x_1,\ldots,x_{r-1})$:

$$\sum_{m_1,\ldots,m_{r-1},n=0}^{\infty} f(m_1+\ldots+m_{r-1}+n) S_n^{m_1,\ldots,m_{r-1},M}(x_1,\ldots,x_r) \frac{w_1^{m_1}}{m_1!} \cdots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n \tag{2.4}$$

$$= \sum_{m_1,m_2=0}^{\infty} f(m_1+m_2) \frac{(x_r t+w_1)^{m_1}}{m_1!} P_{m_2}^{M}(x_1 t^M, x_2 t+w_{r-1},\ldots,x_{r-1}t+w_2)$$

provided that each member of the series identity (2.4) exists.

Proof. If we set $m_{r-1} \to m_{r-1}-Mk_1, m_{r-2} \to m_{r-2}-m_{r-1}, \ldots, m_2 \to m_2-m_3$ in the right side of the equation (2.2) then

$$\sum_{m_1,\ldots,m_{r-1},n=0}^{\infty} f(m_1+\ldots+m_{r-1}+n) S_n^{m_1,\ldots,m_{r-1},M}(x_1,\ldots,x_r) \frac{w_1^{m_1}}{m_1!} \cdots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n$$

$$= \sum_{m_1,m_2=0}^{\infty} f(m_1+m_2) \frac{(x_r t+w_1)^{m_1}}{m_1!}$$

$$\times \sum_{m_3=0}^{m_2} \cdots \sum_{m_{r-1}=0}^{m_{r-2}} \sum_{k_1=0}^{[\frac{m_{r-1}}{M}]} A_{m_1+m_2,\, m_2,\, k_1,\, m_{r-1},\ldots,\, m_3} \frac{(x_1 t^M)^{k_1}}{k_1!}$$

$$\times \frac{(x_2 t+w_{r-1})^{m_{r-1}-Mk_1}}{(m_{r-1}-Mk_1)!} \frac{(x_3 t+w_{r-2})^{m_{r-2}-m_{r-1}}}{(m_{r-2}-m_{r-1})!} \cdots \frac{(x_{r-1}t+w_2)^{m_2-m_3}}{(m_2-m_3)!}$$

$$= \sum_{m_1,m_2=0}^{\infty} f(m_1+m_2) \frac{(x_r t+w_1)^{m_1}}{m_1!} P_{m_2}(x_1 t^M, x_2 t+w_{r-1},\ldots,x_{r-1}t+w_2).$$

□

By setting $t = -\frac{w_1}{x_r}$ in (2.4), we get the following result.

Corollary 2. *Under the hypotheses of Corollary 1, we get the following generating relation*

$$\sum_{m_1,\ldots,m_{r-1},n=0}^{\infty} f(m_1+\ldots+m_{r-1}+n)$$

$$\times S_n^{m_1,\ldots,m_{r-1},M}(x_1,\ldots,x_r)\frac{w_1^{m_1}}{m_1!}\cdots\frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!}(-\frac{w_1}{x_r})^n$$
$$=\sum_{m_2=0}^{\infty} f(m_2)P_{m_2}^{M}(x_1(-\frac{w_1}{x_r})^M, x_2(-\frac{w_1}{x_r})+w_{r-1},\ldots,x_{r-1}(-\frac{w_1}{x_r})+w_2).$$

Remark 3. Choosing $M=1$ and

$$A_{m,n,k_2,k_3,\ldots,k_{r-1}}=(\alpha_1)_{k_2}(\alpha_2)_{k_3-k_2}\ldots(\alpha_{r-1})_{n-k_{r-1}}(\alpha_r)_{m-n} \tag{2.5}$$

in (2.3) we observe that

$$A_{m_1+m_2,\, m_2,\, k_1,\, m_{r-1},m_{r-2},\ldots,m_4,\, m_3}$$
$$=(\alpha_1)_{k_1}(\alpha_2)_{m_{r-1}-k_1}(\alpha_3)_{m_{r-2}-m_{r-1}}\ldots(\alpha_{r-1})_{m_2-m_3}(\alpha_r)_{m_1}$$

then

$$P_n^1(x_1,\ldots,x_{r-1})=(\alpha_r)_{m_1}g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_{r-1})}(x_1,x_2,\ldots,x_{r-1}) \tag{2.6}$$

where $g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_{r-1})}(x_1,x_2,\ldots,x_{r-1})$ is the Lagrange polynomials of $(r-1)$-variables.

Hence, upon setting $M=1$ and considering (2.5) and (2.7) in Corollary 2, we get

$$\sum_{n=0}^{\infty} f(n)g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_{r-1})}(x_1(-\frac{w_1}{x_r}),x_2(-\frac{w_1}{x_r})+w_{r-1},\ldots,x_{r-1}(-\frac{w_1}{x_r})+w_2) \tag{2.7}$$
$$=\sum_{m_1,\ldots,m_{r-1},n=0}^{\infty} f(m_1+\ldots+m_{r-1}+n)(\alpha_2)_{m_{r-1}}(\alpha_3)_{m_{r-2}}\ldots(\alpha_{r-1})_{m_2}(\alpha_r)_{m_1}$$
$$\times g_n^{(\alpha_1,\alpha_2+m_{r-1},\alpha_3+m_{r-2},\ldots,\alpha_r+m_1)}(x_1,x_2,\ldots,x_r)\frac{w_1^{m_1}}{m_1!}\cdots\frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!}(-\frac{w_1}{x_r})^n.$$

Let choose $f(n)=1$ in (2.7) and consider Remark 1 then we have

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_{r-1})}(x_1(-\frac{w_1}{x_r}),x_2(-\frac{w_1}{x_r})+w_{r-1},\ldots,x_{r-1}(-\frac{w_1}{x_r})+w_2) \tag{2.8}$$
$$=\sum_{m_1,\ldots,m_{r-1},n=0}^{\infty}\sum_{k_{r-1}=0}^{n}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}(\alpha_2)_{m_{r-1}}(\alpha_3)_{m_{r-2}}\ldots(\alpha_{r-1})_{m_2}(\alpha_r)_{m_1}$$
$$\times(\alpha_1)_{k_1}(\alpha_2+m_{r-1})_{k_2-k_1}\ldots(\alpha_{r-1}+m_2)_{k_{r-1}-k_{r-2}}(\alpha_r+m_1)_{n-k_{r-1}}$$
$$\times\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2-k_1}}{(k_2-k_1)!}\cdots\frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!}\frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!}\frac{w_1^{m_1}}{m_1!}\cdots\frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!}(-\frac{w_1}{x_r})^n.$$

Let Φ denote the right side of (2.8). Applying Lemma 1 into the right side of the equation (2.8) we get

$$\Phi = \sum_{k_1,\ldots,k_{r-1},m_1,\ldots,m_{r-1},n=0}^{\infty} (\alpha_1)_{k_1}(\alpha_2+m_{r-1})_{k_2}\ldots(\alpha_{r-1}+m_2)_{k_{r-1}}$$
$$\times(\alpha_r)_{m_1}(\alpha_{r-1})_{m_2}\ldots(\alpha_3)_{m_{r-2}}(\alpha_2)_{m_{r-1}}(\alpha_r+m_1)_n$$
$$\times\frac{((-\frac{w_1}{x_r})x_1)^{k_1}}{k_1!}\frac{((-\frac{w_1}{x_r})x_2)^{k_2}}{k_2!}\ldots\frac{((-\frac{w_1}{x_r})x_{r-1})^{k_{r-1}}}{k_{r-1}!}\frac{((-\frac{w_1}{x_r})x_r)^{n}}{n!}\frac{w_1^{m_1}}{m_1!}\ldots\frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!}$$

and

$$\begin{aligned}
\Phi &= \sum_{m_1,\ldots,m_{r-1}=0}^{\infty}(\alpha_r)_{m_1}(\alpha_{r-1})_{m_2}\ldots(\alpha_3)_{m_{r-2}}(\alpha_2)_{m_{r-1}}\\
&\quad\times(1+(\frac{w_1}{x_r})x_1)^{-\alpha_1}(1+(\frac{w_1}{x_r})x_2)^{-(\alpha_2+m_{r-1})}\ldots(1+(\frac{w_1}{x_r})x_{r-1})^{-(\alpha_{r-1}+m_2)}\\
&\quad\times(1+(\frac{w_1}{x_r})x_r)^{-(\alpha_r+m_1)}\frac{w_1^{m_1}}{m_1!}\ldots\frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!}\\
&= (1+(\frac{w_1}{x_r})x_1)^{-\alpha_1}(1+(\frac{w_1}{x_r})x_2)^{-\alpha_2}\ldots(1+(\frac{w_1}{x_r})x_{r-1})^{-\alpha_{r-1}}(1+(\frac{w_1}{x_r})x_r)^{-\alpha_r}\\
&\quad\times\sum_{m_1,\ldots,m_{r-1}=0}^{\infty}(\alpha_r)_{m_1}(\alpha_{r-1})_{m_2}\ldots(\alpha_3)_{m_{r-2}}(\alpha_2)_{m_{r-1}}\\
&\quad\times(1+(\frac{w_1}{x_r})x_2)^{-m_{r-1}}\ldots(1+(\frac{w_1}{x_r})x_{r-1})^{-m_2}(1+(\frac{w_1}{x_r})x_r)^{-m_1}\frac{w_1^{m_1}}{m_1!}\ldots\frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!}\\
&= (1+(\frac{w_1}{x_r})x_1)^{-\alpha_1}(1+(\frac{w_1}{x_r})x_2)^{-\alpha_2}\ldots(1+(\frac{w_1}{x_r})x_{r-1})^{-\alpha_{r-1}}(1+(\frac{w_1}{x_r})x_r)^{-\alpha_r}\\
&\quad\times(1-\frac{w_1}{1+(\frac{w_1}{x_r})x_r})^{-\alpha_r}(1-\frac{w_2}{1+(\frac{w_1}{x_r})x_{r-1}})^{-\alpha_{r-1}}\ldots(1-\frac{w_{r-1}}{1+(\frac{w_1}{x_r})x_2})^{-\alpha_2}\\
&= (1+(\frac{w_1}{x_r})x_1)^{-\alpha_1}(1+(\frac{w_1}{x_r})x_2-w_{r-1})^{-\alpha_2}\ldots(1+(\frac{w_1}{x_r})x_{r-1}-w_2)^{-\alpha_{r-1}}\\
&\quad\times(1+(\frac{w_1}{x_r})x_r-w_1)^{-\alpha_r}.
\end{aligned}$$

Corollary 3. *The generating relation*

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_{r-1})}(x_1(-\frac{w_1}{x_r}),x_2(-\frac{w_1}{x_r})+w_{r-1},\ldots,x_{r-1}(-\frac{w_1}{x_r})+w_2)$$
$$=(1+(\frac{w_1}{x_r})x_1)^{-\alpha_1}(1+(\frac{w_1}{x_r})x_2-w_{r-1})^{-\alpha_2}\ldots(1+(\frac{w_1}{x_r})x_{r-1}-w_2)^{-\alpha_{r-1}}$$

is given for $g_n^{(\alpha_1,\alpha_2,\ldots,\alpha_{r-1})}(x_1,x_2,\ldots,x_{r-1})$.

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