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DIFFEOMORPHISM OF AFFINE CONNECTED SPACES WHICH PRESERVED RIEMANNIAN AND RICCI CURVATURE TENSORS

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Abstract. In this paper we study the preserving of Riemannian and Ricci tensors with respect to a diffeomorphism of spaces with affine connection. We consider geodesic and almost geodesic mappings of the first type. The basic equations of these maps form a closed system of Cauchy type in covariant derivatives. We determine the quantity of essential (substantial) parameters on which the general solution of this problem depends.

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1. INTRODUCTION

The theory of geodesic mappings was developed in the works of T. Thomas, H. Weyl, L.P. Eisenhart, P.A. Shirokov, K. Yano, A.Z. Petrov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, I.G. Shandra, S.E. Stepanov and others, see [10–12, 14–16, 18–20, 22, 24].

The problems raised in the study of geodesic maps were developed by V.F. Kagan, V. Vranceanu, P.K. Rashevsky, L.Ya. Shapiro, V.D. Vedenyapin, A.Z. Petrov and others. In particular, the concept of quasi-geodesic mapping was introduced by A.Z. Petrov. Close to it there are holomorphically projective mappings of Kähler spaces, considered originally by T. Otsuki and Y. Tashiro. See [15, 17, 19, 20, 23].

It natural generalization of these classes there are almost geodesic mappings introduced by N.S. Sinyukov [20]. Recently investigated almost geodesic mappings V.S. Shadny, V.S. Sobchuk, N.V. Yablonska, J. Mikeš, V.E. Berezovski, M.S. Stankovic, L.S. Velimirovic, M.L. Zlatanovic [1–9, 19, 21].

2. BASIC CONCEPTS OF THE THEORY OF DIFFEOMORPHISM OF AFFINE CONNECTED SPACES AND THEIR TYPES

We consider an n-dimensional torsion-free affine connected space in the coordinate system (x), and assume that $n \geq 2$, further on all functions are sufficiently smooth.

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We can find the basic concepts of the theory of geodesic and of almost geodesic mappings in [15, 19, 20].

Suppose, that an affine connected space A_n admits a diffeomorphism f onto an affine connected space $\bar{A}_n$ in a common coordinate system $x = (x^1, \dots, x^n)$.

Let us

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x) \tag{2.1}$$

given, where $\bar{\Gamma}^h_{ij}(x)$ and $\Gamma^h_{ij}(x)$ are components of objects of connections of affine connected spaces A_n and $\bar{A}_n$ respectively in common coordinate system (x), P^h_{ij} is a the *deformation tensor*.

Definition 1. A curve, defined in an n-dimensional affine connected space A_n, is called *geodesic*, if its tangent vector is parallel along it.

Definition 2. A curve, defined in an affine connected space A_n $(n > 2)$, is called *almost geodesic*, if exists along it a two-dimensional parallel plane containing its tangent vector.

Definition 3. Diffeomorphism f between two manifolds A_n and $\bar{A}_n$ with affine connections is called *geodesic mapping* if any geodesic curve of A_n is mapped onto a geodesic curve of $\bar{A}_n$ and vice versa.

Diffeomorphism f is geodesic mapping if and only if the deformation tensor P^h_{ij} in a common coordinate system has the following forms

$$P^h_{ij}(x) = \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i, \tag{2.2}$$

where δ^h_i is Kronecker symbol and ψ_i is a covariant vector field.

If $\psi_i \equiv 0$, then f is an *affine mapping*.

Definition 4. Diffeomorphism $f\colon\ A_n \to \bar{A}_n$ $\ (n > 2)$ is called *almost geodesic mapping*, if any geodesic curve of A_n is mapped onto an almost geodesic curve of $\bar{A}_n$.

Diffeomorphism $f\colon\ A_n \to \bar{A}_n$ is almost geodesic if and only if the deformation tensor P^h_{ij} in a common coordinate system has the following forms

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a\ P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b\ \lambda^h, \tag{2.3}$$

where

$$A^h_{ijk} = P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}, \tag{2.4}$$

$\bar{\Gamma}^h_{ij}(x)$ and $\Gamma^h_{ij}(x)$ are affine connections of spaces $A_n$and $\bar{A}_n$, λ^h is an arbitrary vector, a and b are some functions dependent on x^h and λ^h.

Here and after the symbol " , " means covariant derivation in A_n.

N.S. Sinyukov [20] defined three types of almost geodesic mappings π_1, π_2, π_3 from the basic equations. It is well known [3, 6, 8], if $n > 5$ there exist only these three types.

Almost geodesic mapping of type π_1 is characterized by the conditions

$$A^h_{(ijk)} = a_{(ij}\delta^h_{k)} + b_{(i}P^h_{jk)}, \tag{2.5}$$

where a_{ij} is some symmetric tensor, b_i is some covector, the bracket (ijk) denotes the symmetrization of the indices i, j, k (without division).

If in (2.5) the covector b_i is equal to zero, then the mapping is called *cannonical*. It is well known, that arbitrary almost geodesic mapping of type π_1 can be represented in the form of a composition of a canonical almost geodesic mapping of type π_1 and a geodesic mapping [20].

3. Condition of representation of Riemannian and Ricci tensors with respect to the diffeomorphism of the affine connected space

From equations (2.1) we obtain a relation between Riemannian tensors R^h_{ijk} and $\bar{R}^h_{ijk}$ of spaces A_n and $\bar{A}_n$ respectively:

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{i[k,j]} + P^h_{\alpha[j}P^\alpha_{k]i}, \tag{3.1}$$

where $[jk]$ denotes alternation of the indices j,k (without division).
Then the relation (3.1) can be written as

$$\bar{R}^h_{ijk} = R^h_{ijk} - A^h_{i[jk]}. \tag{3.2}$$

Therefore, we have

Theorem 1. *The Riemannian tensor is preserved under diffeomorphism $f\colon A_n \to \bar{A}_n$ if and only if*

$$A^h_{ijk} = A^h_{ikj}. \tag{3.3}$$

Contracting by indices h and k we obtain

$$\bar{R}_{ijk} = R_{ij} - A^h_{i[jk]}. \tag{3.4}$$

where R_{ij} and $\bar{R}_{ij}$ are Ricci tensors of spaces A_n and $\bar{A}_n$ respectively.
So we get

Theorem 2. *The Ricci tensor is preserved under diffeomorphism $f\colon A_n \to \bar{A}_n$ if and only if tensor A^h_{ijk} satisfies the following conditions*

$$A^\alpha_{ij\alpha} = A^\alpha_{i\alpha j}. \tag{3.5}$$

4. Preservation of the Riemann and Ricci tensors with respect to the geodesic mappings of the affine connected space

Consider the geodesic mappings which are characterized by the equations (2.2). Form such mappings tensors of the form

$$A^h_{ijk} = \delta^h_i \psi_{j,k} + \delta^h_j \psi_{i,k} + 2\delta^h_k \psi_i \psi_j + \delta^h_i \psi_j \psi_k + \delta^h_j \psi_i \psi_k. \tag{4.1}$$

On the basis of Theorem 1 Riemannian tensor R^h_{ijk} is invariant with respect to geodesic mapping, if condition (3.3) satisfies.

Substituting (4.1) into (3.3) we have

$$\delta^h_j(\psi_{i,k} - \psi_i \psi_k) + \delta^h_i(\psi_{j,k} - \psi_j \psi_k) - \delta^h_i(\psi_{k,j} - \psi_k \psi_j) - \delta^h_k(\psi_{i,j} - \psi_i \psi_j) = 0. \tag{4.2}$$

After contraction (4.2) by indices h and i we obtain

$$\psi_{j,k} - \psi_{k,j} = 0. \tag{4.3}$$

Transvecting (4.2) by indices h and j, we get

$$n(\psi_{i,k} - \psi_i \psi_k) - (\psi_{i,k} - \psi_k \psi_i) = 0. \tag{4.4}$$

From (4.3) and (4.4) follows

$$\psi_{i,k} = \psi_i \psi_k \tag{4.5}$$

Therefore

Theorem 3. *An affine connected space A_n admits geodesic mapping into an another affine connected space $\bar{A}_n$ preserving Riemmanian tensor if and only if the Cauchy type system* (4.5) *has a solution with respect to functions $\psi_i(x)$.*

On the basis of Theorem 2, the Ricci tensor R_{ij} of an affine connected space A_n is invariant with respect to geodesic mappings if satisfies condition (3.5).

Substituting (4.1) into (3.5) we get

$$\psi_{i,j} - \psi_j \psi_i = n(\psi_{i,j} - \psi_i \psi_j). \tag{4.6}$$

From (4.5) and (4.6) we obtain (4.4).

The above leads to

Theorem 4. *An affine connected space A_n admits geodesic mapping into an another affine connected space $\bar{A}_n$ preserving the Ricci tensor if and only if Cauchy type system* (4.4) *has a solution with respect to functions $\psi_i(x)$.*

Integrability conditions of equations (4.4) are the following

$$\psi_a R^h_{ijk} = 0. \tag{4.7}$$

The general solution of Cauchy type system (4.4) depends on no more than n substantial parameters.

From (2.3) and (2.4) follows

Theorem 5. *If under a geodesic mapping of an affine connected space A_n into an affine connected space $\bar{A}_n$ Ricci tensor is an invariant geometric object, then Riemannian tensor also is an invariant geometric object.*

This result can be achieved, if we consider the Weyl tensor.

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{n+1}\delta^h_i R_{[jk]} - \frac{1}{n^2-1}[(nR_{ij} + R_{ji})\delta^h_k - (nR_{ik} + R_{ki})\delta^h_j], \quad (4.8)$$

which is an invariant object under geodesic mappings.

Space A_n which admits geodesic mapping preserving Riemannian and Ricci tensors, is characterized by the following interesting property.

Theorem 6. *In an affine connected space which admits geodesic mappings preserving Riemannian and Ricci tensors, there exists a covariant constant vector field.*

Indeed, from (4.3) we have that covector ψ_i is locally gradient, that is, there exists function $\Psi(x)$ such that $\psi_i(x) = \partial_i \Psi(x)$ where $\partial_i = \partial/\partial x^i$.

We introduce function $\Phi(x) = e^{-\Psi(x)}$ and we can see on the basis of condition (4.5) satisfies $\Phi_{i,j} = 0$. Thus covector field $\varphi_i = \partial_i \Phi(x)$ is a covariant constant in space A_n that is

$$\varphi_{i,j} = 0. \quad (4.9)$$

On the other hand, if in A_n there exists a covariant constant vector field φ_i, then using of the substitution $\Psi(x) = -\ln(\Phi(x))$, $\partial_i \Phi(x) = \varphi(x)$ we can construct vector field $\psi_i(x)$, which satisfies (4.5) Thus Theorem 6 is proved.

In addition, we note that equation (4.9) forms linear system of Cauchy type in covariant derivatives with respect to unknown functions $\varphi_i(x)$. Integrability conditions of (4.9) has the form $\varphi_\alpha R^\alpha_{ijk} = 0$. There are linear algebraic equations with respect to $\varphi_i(x)$. Their differential prolongations are also linear.

The global existence of these fields was discussed in [13].

5. Preservation of Riemann and Ricci tensors with respect to the special case of canonical almost geodesic mappings of the first type of affine connected space

Consider the special case of canonical almost geodesic mappings, which is characterized by the following equation

$$P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k} = \delta^h_{(i} a_{jk)}, \quad (5.1)$$

where a_{ij} is some symmetric tensor.

Indeed, mappings which are characterized by equation (5.1), are canonical almost geodesic mappings of the first type, namely after symmetrization of equations (5.1) we obtain (2.5).

For mappings, which are characterized by equations (5.1), tensor A has the following structure

$$A^h_{ijk} = \delta^h_{(i} a_{jk)}, \tag{5.2}$$

Then, obviously, conditions (3.3) and (3.5) are fulfilled for the tensor A. Therefore, there is a

Theorem 7. *Riemannian and Ricci tensors are invariant geometrical objects under canonical almost geodesic mappings of the first type, which are characterized by* (5.1).

Because in a projective-euclidean space Riemannian tensor vanishes, and in a Ricci-flat space Ricci tensor vanishes, then by Theorem (3.2) we have

Theorem 8. *If projective euclidean or Ricci-flat space admits canonical almost geodesic mapping of the first type, which is characterized by equations* (5.1)*, into space $\bar{A}_n$, then $\bar{A}_n$is projective-euclidean or Ricci-flat space respectively.*

Considering (5.1) as a Cauchy-type system with respect to deformation tensor P^h_{ij} we can determine their condition of integrability.

For this we covariantly differentiate (5.1) on x^m, then we alternate it by indices k and m. Using of the Ricci identity we have

$$\delta^h_i a_{j[k,m]} + \delta^h_j a_{i[k,m]} + \delta^h_{[k|ij|m} = \\ - P^\alpha_{ij} R^h_{\alpha km} + P^h_{\alpha(j} R^\alpha_{i)km} + a_{j[m} P^h_{k]i} + a_{j[m} P^h_{k]j}. \tag{5.3}$$

Contracting condition of integrability (5.3) by indices h and m, we get

$$a_{jk,i} + a_{ik,j} - (n+1) a_{ij,k} = \\ - P^\alpha_{ij} R_{\alpha km} + P^\beta_{\alpha j} R^\alpha_{jk\beta} + a_{j\alpha} P^\alpha_{ki} - a_{jk} P^\alpha_{\alpha i} + a_{i\alpha} P^\alpha_{jk} - a_{ik} P^\alpha_{j\alpha}. \tag{5.4}$$

We alternate equations (5.4) by indices j and k, then we obtain

$$a_{ij,k} = a_{ik,j} + \frac{1}{(n+2)} (P^\alpha_{ij} R_{\alpha k} + P^\alpha_{ik} R_{\alpha j} - P^\beta_{\alpha j} R^\alpha_{ik\beta} + P^\beta_{\alpha k} R^\alpha_{ij\beta} \\ - P^\beta_{\alpha i} R^\alpha_{jk\beta} + P^\beta_{\alpha i} R^\alpha_{kj\beta} - a_{j\alpha} P^\alpha_{ki} + a_{k\alpha} P^\alpha_{ij} + a_{ik} P^\alpha_{j\alpha} - a_{ij} P^\alpha_{k\alpha}). \tag{5.5}$$

Observing (5.5), we can write equations in the following form

$$a_{ik,j} = \frac{1}{(n-1)(n+2)} (n(P^\alpha_{ik} R_{\alpha j} - P^\beta_{\alpha(k} R^\alpha_{i)j\beta}) + R_{\alpha(k} P^\alpha_{i)j} R^\alpha_{ij\beta} - P^\beta_{\alpha j} R^\alpha_{(ik)\beta} \\ - P^\beta_{\alpha(i} R^\alpha_{|j|k)\beta} + (n+1)(a_{j(i} P^\alpha_{k)\alpha} - a_{\alpha(i} P^\alpha_{k)j}) + 2(a_{ik} P^\alpha_{j\alpha} - a_{j\alpha} P^\alpha_{ik})). \tag{5.6}$$

Obviously, equations (5.1) and (5.6) in a given space represent a Cauchy type system with respect to functions $P^h_{ij}(x)$ and $a_{ij}(x)$ which, of course, are satisfied even algebraic conditions.

$$P^h_{ij}(x) = P^h_{ji}(x), \qquad a_{ij}(x) = a_{ji}(x). \tag{5.7}$$

So we proved

Theorem 9. *An affine connected space A_n admits almost geodesic mapping, defined by equation (5.1) onto affine connected space $\bar{A}_n$ if and only if there exist solution of mixed equation system of Cauchy type (5.1), (5.6) and (5.7) with respect to function $P^h_{ij}(x)$ and $a_{ij}(x)$.*

The general solution of Cauchy type mixed system depends on (no more than) $\frac{1}{2}n(n+1)^2$ real parameters.

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