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REPRESENTATION NUMBERS BY SUMS OF QUADRATIC FORMS $X_1^2 + X_1X_2 + X_2^2$ IN SIXTEEN VARIABLES

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Abstract. Let $R(a_1,...,a_8;n)$ denote the number of representations of an integer n by the form $a_1(x_1^2+x_1x_2+x_2^2)+a_2(x_3^2+x_3x_4+x_4^2)+a_3(x_5^2+x_5x_6+x_6^2)+a_4(x_7^2+x_7x_8+x_8^2)+a_5(x_9^2+x_9x_{10}+x_{10}^2)+a_6(x_{11}^2+x_{11}x_{12}+x_{12}^2)+a_7(x_{13}^2+x_{13}x_{14}+x_{14}^2)$ $+a_8(x_{15}^2+x_{15}x_{16}+x_{16}^2)$. In this article, we derive formulae for $R(1,2,2,2,2,2,2,2;n)$, $R(1,1,1,2,2,2,2,2;n)$, $R(1,1,1,1,1,2,2,2;n)$ and $R(1,1,1,1,1,1,1,2;n)$. These formulae are given in terms of the function $\sigma_7(n)$ and the numbers $\tau_{8,2}(n)$ and $\tau_{8,6}(n)$.

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1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of positive integers, integers and rational numbers respectively and let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. For $a_1,...,a_8 \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we let $R(a_1,...,a_8;n)$ denote the representation number of n by the form $a_1(x_1^2+x_1x_2+x_2^2)+a_2(x_3^2+x_3x_4+x_4^2)+a_3(x_5^2+x_5x_6+x_6^2)+a_4(x_7^2+x_7x_8+x_8^2)+a_5(x_9^2+x_9x_{10}+x_{10}^2)+a_6(x_{11}^2+x_{11}x_{12}+x_{12}^2)+a_7(x_{13}^2+x_{13}x_{14}+x_{14}^2)$ $+a_8(x_{15}^2+x_{15}x_{16}+x_{16}^2)$, that is

$$R(a_1,...,a_8;n) \tag{1.1}$$
$$:= \text{card}\left\{ (x_1,...,x_{16}) \in \mathbb{Z}^{16} : n = \sum_{k=1}^{8} a_k(x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2) \right\}.$$

If l of $a_1,...,a_8$ are equal, say

$$a_i = a_{i+1} = ... = a_{i+l-1} = a \tag{1.2}$$

for convenience we indicate this in $R(a_1,...,a_8;n)$ by writing a^l for $a_i,a_{i+1},...,a_{i+l-1}$. For $k \in \mathbb{N}$ and $n \in \mathbb{Q}$ the sum of divisor function is defined by

$$\sigma_k(n) = \begin{cases} \sum\limits_{\substack{d\in\mathbb{N}\\ d|n}} d^k, & \text{if } n\in\mathbb{N}, \\ 0, & \text{if } n\in\mathbb{Q}, n\notin\mathbb{N}. \end{cases} \tag{1.3}$$

Ramakrishnan and Sahu [8] have recently proved formulae for $R(1^8;n)$ and $R(1^4,2^4;n)$ for all $n\in\mathbb{N}$. In the present paper, motivated from the work of Ramakrishnan and Sahu we derive formulae for $R(1^1,2^7;n)$, $R(1^3,2^5;n)$, $R(1^5,2^3;n)$ and $R(1^7,2^1;n)$ by using the method of Alaca, Alaca and Williams (see for example [1]). These formulae are given in terms of the function $\sigma_7(n)$ and the numbers $\tau_{8,2}(n)$ and $\tau_{8,6}(n)$ which are introduced in [8]. Köklüce [4] has derived formulae for the quadratic forms in twelve and sixteen variables which are sums of quadratic forms with discriminant -23 by using the method developed by Lomadze [7].

2. Preliminaries and Statement of the Theorem

For $a,b,r,s,n\in\mathbb{N}$ with $a\le b$ we define the convolution sum $W^{r,s}_{a,b}(n)$ by

$$W^{r,s}_{a,b}(n) := \sum_{\substack{(l,m)\in\mathbb{N}^2\\ al+bm=n}} \sigma_r(l)\sigma_s(m). \tag{2.1}$$

In recent years, some of the convolution sums have been evaluated explicitly. In this study, we require the evaluations for $W^{3,3}_{1,1}(n)$, $W^{3,3}_{1,2}(n)$, $W^{3,3}_{1,3}(n)$, $W^{3,3}_{2,3}(n)$, $W^{3,3}_{1,6}(n)$. The convolution sum

$$W^{3,3}_{1,1}(n) = \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n), \tag{2.2}$$

was evaluated explicitly by Ramanujan [9]. Formulae for the sum $W^{3,3}_{1,2}(n)$ was firstly given by Cheng and Williams [2]. Formulae for many other convolution sums of this type have also been given by the authors in that study. Ramakrishnan and Sahu [8] have recently evaluated the following four formulae containing formulae for $W^{3,3}_{1,2}(n)$ as well.

$$W^{3,3}_{1,2}(n) = -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{2}) + \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7(\frac{n}{2}) + \frac{1}{272}\tau_{8,2}(n), \tag{2.3}$$

$$W^{3,3}_{1,3}(n) = -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{3}) + \frac{1}{9840}\sigma_7(n) + \frac{81}{9840}\sigma_7(\frac{n}{3}) + \frac{1}{246}\tau_{8,3}(n), \tag{2.4}$$

$$\begin{aligned}
&W_{2,3}^{3,3}(n)\\
&=-\frac{1}{240}\sigma_3(\frac{n}{2})-\frac{1}{240}\sigma_3(\frac{n}{3})+\frac{1}{167280}\sigma_7(n)+\frac{1}{10455}\sigma_7(\frac{n}{2})+\frac{27}{55760}\sigma_7(\frac{n}{3})\\
&+\frac{27}{3485}\sigma_7(\frac{n}{6})+\frac{7}{8160}\tau_{8,2}(n)+\frac{189}{2720}\tau_{8,2}(\frac{n}{3})+\frac{1}{820}\tau_{8,3}(n)+\frac{4}{205}\tau_{8,3}(\frac{n}{2})\\
&-\frac{1}{480}\tau_{8,6}(n), \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
&W_{1,6}^{3,3}(n)\\
&=-\frac{1}{240}\sigma_3(n)-\frac{1}{240}\sigma_3(\frac{n}{6})+\frac{1}{167280}\sigma_7(n)+\frac{1}{10455}\sigma_7(\frac{n}{2})+\frac{27}{55760}\sigma_7(\frac{n}{3})\\
&+\frac{27}{3485}\sigma_7(\frac{n}{6})+\frac{7}{8160}\tau_{8,2}(n)+\frac{189}{2720}\tau_{8,2}(\frac{n}{3})+\frac{1}{820}\tau_{8,3}(n)+\frac{4}{205}\tau_{8,3}(\frac{n}{2})\\
&+\frac{1}{480}\tau_{8,6}(n), \qquad (2.6)
\end{aligned}$$

where $\tau_{8,2}(n)$, $\tau_{8,3}(n)$, $\tau_{8,6}(n)$ are n-th Fourier coefficients of the newforms $\Delta_{8,2}(n)$, $\Delta_{8,3}(n)$, $\Delta_{8,6}(n)$ given in [8]. In a recent publication Köklüce and Eser [5] have completed the evaluations of the convolution sums $W_{1,12}^{1,3}(n)$, $W_{3,4}^{1,3}(n)$, $W_{4,3}^{1,3}(n)$, and $W_{12,1}^{1,3}(n)$.

We now state our main result.

Theorem 1. *Let $n \in \mathbb{N}$ then,*

(i)

$$\begin{aligned}
R(1^1,2^7;n)&=\frac{6}{85}\sigma_7(n)-\frac{516}{85}\sigma_7(\frac{n}{2})-\frac{486}{85}\sigma_7(\frac{n}{3})+\frac{41796}{85}\sigma_7(\frac{n}{6})+\frac{9}{17}\tau_{8,2}(n)\\
&-\frac{729}{17}\tau_{8,2}(\frac{n}{3})+\frac{27}{5}\tau_{8,6}(n), \qquad (2.7)
\end{aligned}$$

(ii)

$$\begin{aligned}
R(1^3,2^5;n)&=\frac{18}{85}\sigma_7(n)-\frac{528}{85}\sigma_7(\frac{n}{2})-\frac{1458}{85}\sigma_7(\frac{n}{3})+\frac{42768}{85}\sigma_7(\frac{n}{6})+\frac{27}{17}\tau_{8,2}(n)\\
&-\frac{2187}{17}\tau_{8,2}(\frac{n}{3})+\frac{81}{5}\tau_{8,6}(n), \qquad (2.8)
\end{aligned}$$

(iii)

$$R(1^5,2^3;n)=\frac{66}{85}\sigma_7(n)-\frac{576}{85}\sigma_7(\frac{n}{2})-\frac{5346}{85}\sigma_7(\frac{n}{3})+\frac{46656}{85}\sigma_7(\frac{n}{6})-\frac{54}{17}\tau_{8,2}(n)$$

$$+\frac{4374}{17}\tau_{8,2}(\frac{n}{3})+\frac{162}{5}\tau_{8,6}(n), \quad (2.9)$$

(iv)

$$R(1^7,2^1;n)=\frac{258}{85}\sigma_7(n)-\frac{768}{85}\sigma_7(\frac{n}{2})-\frac{20898}{85}\sigma_7(\frac{n}{3})+\frac{62208}{85}\sigma_7(\frac{n}{6})-\frac{72}{17}\tau_{8,2}(n)$$
$$+\frac{5832}{17}\tau_{8,2}(\frac{n}{3})+\frac{216}{5}\tau_{8,6}(n). \quad (2.10)$$

3. Proof of Theorem 1

To prove the theorem we need the representation number formulae for the following octonary quadratic forms,

$$f_1 := x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2, \quad (3.1)$$

$$f_2 := x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+x_5^2+x_5x_6+x_6^2+2(x_7^2+x_7x_8+x_8^2) \quad (3.2)$$

and

$$f_3 := x_1^2+x_1x_2+x_2^2+2(x_3^2+x_3x_4+x_4^2)+2(x_5^2+x_5x_6+x_6^2)+2(x_7^2+x_7x_8+x_8^2). \quad (3.3)$$

For $l\in\mathbb{N}_0$ we set $r_i(l)=\text{card}\{(x_1,...,x_8)\in\mathbb{Z}^8 : l=f_i(x_1,...,x_8)\}$. Obviously $r_i(0)=1$ for $i\in\{1,2,3\}$. The representation number formula

$$r_1(l)=24\sigma_3(l)+216\sigma_3(\frac{l}{3}),\ l\in\mathbb{N}, \quad (3.4)$$

is given by Lomadze [6]. Köklüce [3] has obtained formulae for the number of representation of a positive integer l by the forms f_2 and f_3. He has proved that,

$$r_2(l)=18\sigma_3(l)-48\sigma_3(\frac{l}{2})-162\sigma_3(\frac{l}{3})+432\sigma_3(\frac{l}{6}),\ l\in\mathbb{N}, \quad (3.5)$$

$$r_3(l)=6\sigma_3(l)-36\sigma_3(\frac{l}{2})-54\sigma_3(\frac{l}{3})+324\sigma_3(\frac{l}{6}),\ l\in\mathbb{N}. \quad (3.6)$$

Proof. (i) We just prove part (i) in detail as the rest can be proved similarly. It is clear that

$$R(1^1,2^7;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ 2l+m=n}} r_1(l)r_3(m)$$

$$= r_1(0)r_3(n) + r_1(\frac{n}{2})r_3(0) + \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} r_1(l)r_3(m).$$

Thus using (3.4) and (3.6) we have

$$R(1^1,2^7;n) - (6\sigma_3(n) - 36\sigma_3(\frac{n}{2}) - 54\sigma_3(\frac{n}{3}) + 324\sigma_3(\frac{n}{6}) + 24\sigma_3(\frac{n}{2}) + 216\sigma_3(\frac{n}{6}))$$

$$= \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} (24\sigma_3(l) + 216\sigma_3(\frac{l}{3}))(6\sigma_3(m) - 36\sigma_3(\frac{m}{2}) - 54\sigma_3(\frac{m}{3}) + 324\sigma_3(\frac{m}{6}))$$

$$= 144 \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} \sigma_3(l)\sigma_3(m) - 864 \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} \sigma_3(l)\sigma_3(\frac{m}{2})$$

$$- 1296 \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} \sigma_3(l)\sigma_3(\frac{m}{3}) + 7776 \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} \sigma_3(l)\sigma_3(\frac{m}{6})$$

$$+ 1296 \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} \sigma_3(\frac{l}{3})\sigma_3(m) - 7776 \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} \sigma_3(\frac{l}{3})\sigma_3(\frac{m}{2})$$

$$- 11664 \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} \sigma_3(\frac{l}{3})\sigma_3(\frac{m}{3}) + 69984 \sum_{\substack{l,m\in\mathbb{N}\\2l+m=n}} \sigma_3(\frac{l}{3})\sigma_3(\frac{m}{6})$$

$$= 144W_{1,2}^{3,3}(n) - 864W_{1,1}^{3,3}(\frac{n}{2}) - 1296W_{2,3}^{3,3}(n) + 7776W_{1,3}^{3,3}(\frac{n}{2}) + 1296W_{1,6}^{3,3}(n)$$

$$- 7776W_{1,3}^{3,3}(\frac{n}{2}) - 11664W_{1,2}^{3,3}(\frac{n}{3}) + 69984W_{1,1}^{3,3}(\frac{n}{6}).$$

Appealing to (2.2)-(2.6) and adding $6\sigma_3(n) - 36\sigma_3(\frac{n}{2}) - 54\sigma_3(\frac{n}{3}) + 324\sigma_3(\frac{n}{6}) + 24\sigma_3(\frac{n}{2}) + 216\sigma_3(\frac{n}{6})$ to both sides of the equation we obtain the asserted formula.

(ii) It is clear that

$$R(1^3,2^5;n) = \sum_{\substack{l,m\in\mathbb{N}_0\\2l+m=n}} r_1(l)r_2(m).$$

Using equations (3.4), (3.5) and (2.1) we have

$$R(1^3,2^5;n) - (18\sigma_3(n) - 48\sigma_3(\frac{n}{2}) - 162\sigma_3(\frac{n}{3}) + 432\sigma_3(\frac{n}{6}) + 24\sigma_3(\frac{n}{2})$$

$$+ 216\sigma_3(\frac{n}{6}))$$

$$= 432W_{1,2}^{3,3}(n) - 1152W_{1,1}^{3,3}(\frac{n}{2}) - 3888W_{2,3}^{3,3}(n) + 10368W_{1,3}^{3,3}(\frac{n}{2}) + 3888W_{1,6}^{3,3}(n)$$

$-10368W_{1,3}^{3,3}(\frac{n}{2})-34992W_{1,2}^{3,3}(\frac{n}{3})+93312W_{1,1}^{3,3}(\frac{n}{6})$.

Appealing to (2.2)-(2.6) we obtain the desired result .

(iii) It is clear that

$$R(1^5,2^3;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ l+m=n}} r_1(l)r_3(m).$$

Using equations (3.4), (3.6) and (2.1) we have

$$R(1^5,2^3;n)-(6\sigma_3(n)-36\sigma_3(\frac{n}{2})-54\sigma_3(\frac{n}{3})+324\sigma_3(\frac{n}{6})+24\sigma_3(n)$$
$$+216\sigma_3(\frac{n}{3}))$$
$$=144W_{1,1}^{3,3}(n)-864W_{1,2}^{3,3}(n)-1296W_{1,3}^{3,3}(n)+7776W_{1,6}^{3,3}(n)+1296W_{1,3}^{3,3}(n)$$
$$-7776W_{2,3}^{3,3}(n)-11664W_{1,1}^{3,3}(\frac{n}{3})+69984W_{1,2}^{3,3}(\frac{n}{3}).$$

Appealing to (2.2)-(2.6) we obtain the desired result.

(iv) Clearly

$$R(1^7,2^1;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ l+m=n}} r_1(l)r_2(m).$$

Using equations (3.4), (3.5) and (2.1) we have

$$R(1^7,2^1;n)-(18\sigma_3(n)-48\sigma_3(\frac{n}{2})-162\sigma_3(\frac{n}{3})+432\sigma_3(\frac{n}{6})+24\sigma_3(n)$$
$$+216\sigma_3(\frac{n}{3}))$$
$$=432W_{1,1}^{3,3}(n)-1152W_{1,2}^{3,3}(n)-3888W_{1,3}^{3,3}(n)+10368W_{1,6}^{3,3}(n)+3888W_{1,3}^{3,3}(n)$$
$$-10368W_{2,3}^{3,3}(n)-34992W_{1,1}^{3,3}(\frac{n}{3})+93312W_{1,2}^{3,3}(\frac{n}{3}).$$

Appealing to (2.2)-(2.6) we obtain the desired result. □

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