



A SHORT NOTE ON NESTED SUMS

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Abstract. We express a set of nested sums as a 1-index weighted sum using Stars and Bars Theorem.

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1. INTRODUCTION

Nested sums appear in various fields of research. Here, we consider the following summation:

$$\sum_{a_1=0}^N \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \cdots \sum_{a_n=0}^{a_{n-1}} f(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n) \quad (1.1)$$

where $f(m)$ may be any map defined for $0 \leq m \leq 2N$. Under certain constraints on the coefficients α_i , $i \in \{1, \dots, n\}$, we simplify the expression to the form $\sum_{\gamma} f(\gamma) P(\gamma)$

where $P(\gamma)$ is the number of occurrences of γ within the sum. Surprisingly, the authors could not find any formula concerning such expressions. Consider for example the sum:

$$S = \sum_{a_1=0}^N \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \sum_{a_4=0}^{a_3} f(2a_3 - a_4),$$

applying Theorem 1 below, simplifies the expression to:

$$S = \sum_{\gamma=0}^{2N} \sum_{\phi=\phi_{min}(\gamma)}^{\phi_{max}(\gamma)} \binom{N-\gamma+\phi+2}{2} f(\gamma)$$

where, $\phi_{min}(\gamma) = \max\{0, \gamma - N\}$ and $\phi_{max}(\gamma) = \min\{N, \lfloor \gamma/2 \rfloor\}$.

2. DISCUSSION

Denote $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)^T$ and $\underline{a} = (a_1, \dots, a_n)$ and $\gamma = \underline{\alpha} \cdot \underline{a}$ we set:

$$\begin{aligned} x_1 &= N - a_1, \\ x_2 &= a_1 - a_2, \\ &\vdots \\ x_n &= a_{n-1} - a_n, \\ x_{n+1} &= a_n \end{aligned}$$

so

$$N = x_1 + x_2 + \dots + x_{n+1} \quad (2.1)$$

where $0 \leq x_i \leq N, i \in \{1, \dots, n+1\}$. This implies that:

$$\gamma = N\beta_1 - x_1\beta_1 - x_2\beta_2 \dots - x_n\beta_n$$

where $\beta_j = \sum_{i=j}^n \alpha_i$.

Substituting into Eq. 2.1 we get the *Frobenius Equations* (cf. [2]):

$$N\beta_1 - \gamma = x_1\beta_1 + \dots + x_n\beta_n \quad (2.2)$$

The number of integer solutions to this equation yields the following:

Theorem 1. *Consider the sum:*

$$S = \sum_{a_1=0}^N \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \dots \sum_{a_n=0}^{a_{n-1}} f(\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n)$$

Denote by \mathcal{J}^+ the set of indices of coefficients $\beta_j = \sum_{i=j}^n \alpha_i$ that are equal to +1, by

\mathcal{J}^0 the set of coefficients' indices that vanish and by \mathcal{J}^- the set of coefficients' indices that are equal -1. Moreover we assume that $\beta_1 = 1$. Then

$$S = \sum_{\gamma=0}^{2N} P(\gamma, \underline{\alpha}, N) f(\gamma)$$

where the occurrence function is:

$$P(\gamma, \underline{\alpha}, N) = \begin{cases} \binom{N-\gamma+\|\mathcal{J}^+\|-1}{\|\mathcal{J}^+\|-1} & \text{if } \|\mathcal{J}^-\| = 0 \\ \sum_{\varphi=\varphi_{\min}(\gamma)}^{\varphi_{\max}(\gamma)} \binom{\varphi+\|\mathcal{J}^-\|-1}{\|\mathcal{J}^-\|-1} \binom{N-\gamma+\varphi+\|\mathcal{J}^+\|-1}{\|\mathcal{J}^+\|-1} & \text{otherwise} \end{cases}$$

and

$$\varphi_{min}(\gamma) = \max\{0, \gamma - N\}, \quad \varphi_{max}(\gamma) = \min\{N\|\mathcal{J}^-\|, \lfloor \frac{\gamma}{2} \rfloor\}.$$

Proof. Recall the "Stars and Bars Theorem" by W. Feller [1]: "For any pair of positive integers n and k , the number of k -tuples of non-negative integers whose sum is n is equal to $\binom{n+k-1}{k-1}$ ".

So in the case where $\|\mathcal{J}^-\| = 0$, the number of solutions for Eq. 2.2 takes the desired form. (Here $\|\cdot\|$ denotes the cardinality of a set). In the general case we have $\|\mathcal{J}^+\|, \|\mathcal{J}^-\|, \|\mathcal{J}^0\| > 0$, and Eq. 2.2 takes the form:

$$N - \gamma + \sum_{k \in \mathcal{J}^-} x_k = \sum_{i \in \mathcal{J}^+} x_i \tag{2.3}$$

Equation $\varphi = \sum_{k \in \mathcal{J}^-} x_k$ have $\binom{\varphi + \|\mathcal{J}^-\| - 1}{\|\mathcal{J}^-\| - 1}$ roots. The number of solutions to the equation $N - \gamma + \varphi = \sum_{i \in \mathcal{J}^+} x_i$ (thinking of the left hand side as a constant value) is $\binom{N - \gamma + \varphi + \|\mathcal{J}^+\| - 1}{\|\mathcal{J}^+\| - 1}$. Summing over φ yields:

$$P(\gamma, \underline{\alpha}, N) = \sum_{\varphi = \varphi_{min}}^{\varphi_{max}} \binom{\varphi + \|\mathcal{J}^-\| - 1}{\|\mathcal{J}^-\| - 1} \binom{N - \gamma + \varphi + \|\mathcal{J}^+\| - 1}{\|\mathcal{J}^+\| - 1} \tag{2.4}$$

To produce the upper and lower bound of φ for the summation above note that $N \geq x_i \geq 0$ for all x_i 's so $N\|\mathcal{J}^-\| \geq \varphi \geq 0$. Eq. 2.3 along with the fact that $\beta_1 = 1$ (since it must be positive by definition) then implies $\varphi \leq N(\|\mathcal{J}^+\| - 1) + \gamma$. Substituting Eq. 2.1 into Eq. 2.2 yields:

$$\begin{aligned} \gamma &= (\beta_1 - \beta_2)x_2 + (\beta_1 - \beta_3)x_3 + \dots + (\beta_1 - \beta_n)x_n + \beta_1 x_{n+1} \\ &= 2 \sum_{k \in \mathcal{J}^-} x_k + \sum_{k \in \mathcal{J}^-} x_k x_j + x_{n+1} \end{aligned} \tag{2.5}$$

where we used:

$$\beta_1 - \beta_i = \begin{cases} 0 & i \in \mathcal{J}^+ \\ 1 & i \in \mathcal{J}^0 \\ 2 & i \in \mathcal{J}^- \end{cases}$$

Note that the maximal γ occurs when $\sum_{k \in \mathcal{J}^-} x_k = N$ so that $\sum_{k \in \mathcal{J}^-} x_k = 0$ (see 2.1). Taking the two last terms of Eq. 2.5 to zero, it yields $\varphi \leq \frac{\gamma}{2}$. Since $\binom{a}{b}$ vanishes when $b > a$ we also require $\varphi \geq \gamma - N$. Finally, we obtain:

$$\varphi_{min}(\gamma) = \max\{0, \gamma - N\}, \quad \varphi_{max}(\gamma) = \min\{N\|\mathcal{J}^-\|, \lfloor \frac{\gamma}{2} \rfloor\}$$

□

Example 1. Consider the case where n is odd and $\|\mathcal{J}^-\| = 0$ (e.g. γ is the alternating sum: $a_1 - a_2 + a_3 - \dots + a_n$). Let $f(k, n, p)$ be the probability mass binomial function $\binom{n}{k} p^k (1-p)^{n-k}$. Applying Theorem 1 yields :

$$\begin{aligned} \sum_{a_1=0}^N \sum_{a_2=0}^{a_1} \dots \sum_{a_n=0}^{a_{n-1}} f(\|\mathcal{J}^+\|, N - \gamma, p) \\ = \frac{p^{\|\mathcal{J}^+\| - 1}}{(\|\mathcal{J}^+\|!)^2} \left[(1-p)^{-N - \|\mathcal{J}^+\| - 1} - (1-p)^{N - \|\mathcal{J}^+\|} \right] \end{aligned}$$

where we used the identity $\binom{a-b}{b} \binom{a}{b} = (b!)^{-2}$.

REFERENCES

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