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MINIMAL DOUBLY RESOLVING SETS OF ANTIPRISM GRAPHS AND MÖBIUS LADDERS

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Abstract. Consider a simple connected graph $G = (V(G), E(G))$, where $V(G)$ represents the vertex set and $E(G)$ represents the edge set respectively. A subset W of $V(G)$ is called a resolving set for a graph G if for every two distinct vertices $x, y \in V(G)$, there exist some vertex $w \in W$ such that $d(x,w) \neq d(y,w)$, where $d(u,v)$ denotes the distance between vertices u and v. A resolving set of minimal cardinality is called a metric basis for G and its cardinality is called the metric dimension of G, which is denoted by $\beta(G)$. A subset D of $V(G)$ is called a doubly resolving set of G if for every two distinct vertices x, y of G, there are two vertices $u, v \in D$ such that $d(u,x) - d(u,y) \neq d(v,x) - d(v,y)$. A doubly resolving set with minimum cardinality is called minimal doubly resolving set. This minimum cardinality is denoted by $\psi(G)$.

In this paper, we determine the minimal doubly resolving sets for antiprism graphs denoted by A_n with $n \geq 3$ and for Möbius ladders denoted by M_n, for every even positive integer $n \geq 8$. It has been proved that $\psi(A_n) = 3$ for $n \geq 3$ and

$$\psi(M_n) = \begin{cases} 3, & \text{if } n \equiv 0 \text{ or } 4 \pmod 8 \\ 4, & \text{if } n \equiv 2 \text{ or } 6 \pmod 8 \end{cases}$$

for every even positive integer $n \geq 8$.

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1. INTRODUCTION AND PRELIMINARY RESULTS

The concept of metric dimension was introduced by Slater in [14] and also independently by Harary and Melter in [6]. This concept has different applications in the diverse areas of network discovery and verification [2], robot navigation [13], and chemistry.

Consider a simple connected undirected graph $G = (V(G), E(G))$, where $|V(G)| = n$ and $|E(G)| = m$. Let $d(x,y)$ denote the distance between vertices x and y. A vertex v of graph G is said to resolve two vertices x and y of G if $d(v,x) \neq d(v,y)$. A vertex set $W = \{w_1, w_2, \ldots, w_k\}$ of G is a *resolving set* or a *locating set* of G if every two distinct vertices of G are resolved by some vertex of W. The k-tuple

$r(v,W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$ is called the vector of metric coordinates of v with respect to W. A resolving set of minimum cardinality is called a *metric basis* of G. The cardinality of metric basis, denoted by $\beta(G)$, is called the *metric dimension* of G.

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n\geq 1}$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $\beta(G_n) \leq C$ for every $n \geq 1$ then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has an unbounded metric dimension. If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with constant metric dimension. For recent results on metric dimension of certain classes of graphs, please consult [7–11].

The doubly resolving set of a graph G is a recent concept introduced by Cáceres et al.[3]. They proved that the metric dimension of the Cartesian product $G\square G$ is related in a strong way to doubly resolving sets of G with the minimum cardinality. The vertices x,y of the graph G, $n \geq 2$, are said to doubly resolve vertices u,v of G if $d(u,x) - d(u,y) \neq d(v,x) - d(v,y)$. A vertex set D of G is a doubly resolving set of G if every two distinct vertices of G are doubly resolved by some two vertices of D, i.e., if there are no two distinct vertices of G with the same difference between their corresponding metric coordinates with respect to D. The minimal doubly resolving set is a doubly resolving set with minimum cardinality. The cardinality of minimum doubly resolving set is denoted by $\psi(G)$. The problem of minimal doubly resolving set is NP-hard [12].

Note that we have $\beta(G) \leq \psi(G)$ always by the definition. Since if x,y doubly resolve u,v, then $d(u,x) - d(v,x) \neq 0$ or $d(u,y) - d(v,y) \neq 0$, and hence x or y resolve u,v, which follows that a doubly resolving set is also a resolving set. In [11], it has been proved that metric dimension of antiprism denoted by A_n, is 3. The metric dimension of Möbius ladder M_n has been discussed in [1] and it was proved that Möbius ladder M_n has bounded metric dimension.

The problem of determining the minimal cardinality of double resolving sets for prism graphs denoted by $C_n\square P_2$ was studied in [4]. It was proved that the minimal cardinality is equal to four if n is even and equal to three if n is odd. For more study on computing the doubly resolving sets of graphs, see [12].

In this paper, we extend this study to antiprism graphs denoted by A_n and to Möbius ladders denoted by M_n. We determine the cardinality of minimal doubly resolving set $\psi(A_n)$ and $\psi(M_n)$.

2. THE MINIMAL DOUBLY RESOLVING SETS FOR ANTIPRISM GRAPHS A_n

In this section, we determine the minimal doubly resolving set for antiprism graphs denoted by A_n. The antiprism A_n[5] is a 4-regular graph and for $n = 3$, it is the octahedron. The graph of the antiprism A_n for $n \geq 3$ consists of $V(A_n) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and $E(A_n) = \{x_i x_{i+1} : 1 \leq i \leq n\} \cup \{y_i y_{i+1} : 1 \leq i \leq n\} \cup \{x_i y_i : 1 \leq i \leq$

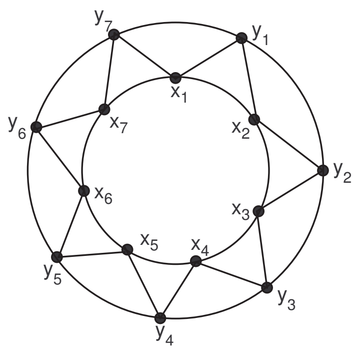

FIGURE 1. The antiprism graph A_7.

$n\} \cup \{x_{i+1}y_i : 1 \le i \le n\}$ with indices taken modulo n. As a convention, $\{y_1, y_2, \ldots, y_n\}$ forms outer cycle and $\{x_1, x_2, \ldots, x_n\}$ forms inner cycle and we have $|V(A_n)| = 2n$ and $|E(A_n)| = 4n$ for $n \ge 3$. Figure 1 displays A_7. It has been proved that metric dimension $\beta(A_n)$ is equal to 3, i.e., $\beta(A_n) = 3$ for $n \ge 3$. So we have $\psi(A_n) \ge 3$, for every $n \ge 3$, since $\psi(A_n) \ge \beta(A_n)$.

Lemma 1. *Let $S_i(y_1) = \{w \in V(A_n) : d(y_1, w) = i\}$ be the set of vertices in V at distance i from y_1. For the graph of antiprism A_n with $n \ge 8$, the sets $S_i(y_1)$ are given in Table 1 where x_{-k}, y_{-k} denote x_{n-k}, y_{n-k}, respectively.*

Proof. By the definition of $S_i(y_1)$, it is easy to see that $S_i(y_1)$ is the set of all $v \in V(A_n)$ such that $va \in E(A_n)$ for some $a \in S_{i-1}(y_1)$ and $v \notin S_l(y_1)$, $l < i-1$. Now using the definition of the antiprism A_n, it immediately follows that

$$S_1(y_1) = \{x_1, x_2, y_2, y_n\}.$$

Let us prove by induction that, for $2 \le i < \lfloor \frac{n}{2} \rfloor = t$

$$S_i(y_1) = \{x_{i+1}, x_{n-(i-2)}, y_{i+1}, y_{n-(i-1)}\}. \tag{2.1}$$

For $i = 2$, starting from $S_1(y_1)$, we obtain

$$S_2(y_1) = \{x_3, x_n, y_3, y_{n-1}\}.$$

Suppose that (2.1) holds for $2 \le i \le l < \lfloor \frac{n}{2} \rfloor - 1$. Starting from $S_l(y_1)$, we obtain the following candidates for the members of $S_{l+1}(y_1)$:

$$x_l, x_{n-l+1}, y_l, y_{n-l}, x_{l+2}, x_{n-l+3}, y_{l+2}, y_{n-l+2}.$$

The first four vertices belong to $S_{l-1}(y_1)$ and consequently cannot belong to $S_{l+1}(y_1)$. For example $x_l = x_{(l-1)+1} \in S_{l-1}(y_1)$. The set of remaining four vertices can be expressed in the form (2.1) for $i = l+1$. For $i \ge \lfloor \frac{n}{2} \rfloor$, $S_i(y_1)$ depends on n.

Case 1: If $n = 2t$, $t \geq 4$.
Starting from $S_{t-1}(y_1)$, we obtain the following candidates for $S_t(y_1)$:

$$x_{t-1}, x_{n-t+4}, y_{t-1}, y_{n-t+3}, x_{t+1}, x_{n-t+2}, y_{t+1}.$$

The first four vertices belongs to $S_{t-2}(y_1)$ and can be neglected. The set of remaining three vertices represents set $S_t(y_1)$ for $n = 2t$. Starting from $S_t(y_1)$, $S_{t+1}(y_1) = \varnothing$.

Case 2: If $n = 2t+1$, $t \geq 4$.
Starting from $S_{t-1}(y_1)$, we obtain the following candidates for $S_t(y_1)$:

$$x_{t-1}, x_{n-t+4}, y_{t-1}, y_{n-t+3}, x_{t+1}, x_{n-t+2}, y_{t+1}, y_{n-t+1}.$$

The first four vertices belongs to $S_{t-2}(y_1)$ and can be neglected. The set of remaining four vertices represents set $S_t(y_1)$ for $n = 2t+1$. Starting from $S_t(y_1)$, the candidates for $S_{t+1}(y_1)$ are:

$$x_t, x_{n-t+3}, y_t, y_{n-t+2}, x_{t+2}.$$

As $x_t, x_{n-(t-3)}, y_t, y_{n-(t-2)} \in S_{t-1}(y_1)$. So $S_{t+1}(y_1) = \{x_{t+2}\}$. It is easy to see that similar construction applied to $S_{t+1}(y_1)$ lead to $S_{t+2}(y_1) = \varnothing$. The following table displays $S_i(y_1)$. □

n	i	$S_i(y_1)$
	1	$\{x_1, x_2, y_2, y_n\}$
	$2 \leq i < \lfloor \frac{n}{2} \rfloor$	$\{x_{i+1}, x_{-(i-2)}, y_{i+1}, y_{-(i-1)}\}$
$2t$	t	$\{x_{t+1}, x_{-(t-2)}, y_{t+1}\}$
$2t+1$	t	$\{x_{t+1}, x_{-(t-2)}, y_{t+1}, y_{-(t-1)}\}$
	$t+1$	$\{x_{t+2}\}$

TABLE 1. $S_i(y_1)$ for A_n

Theorem 1. *For $n \geq 3$, $\psi(A_n) = 3$.*

Proof. There are two cases to be discussed.

Case 1: If $n = 2t$, $t \geq 4$.
Consider Table 2 which shows the vectors of metric coordinates of vertices of A_n with respect to set $D^* = \{y_1, y_3, y_{t+2}\}$ in a special way.

Starting from Table 2, note that $y_1 \in D^*$ therefore the first metric coordinate of a vector from $S_i(y_1)$ with respect to D^* is equal to i. It is easy to check that for each $i \in \{1, 2, \ldots, t\}$, there do not exist two vertices $x, y \in S_i(y_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be easily seen that for each $i, j \in \{1, 2, \ldots, t\}$, $i \neq j$, there do not exist two vertices $x \in S_i(y_1)$ and $y \in S_j(y_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$.

Case 2: If $n = 2t+1$, $t \geq 4$.
Consider Table 3 which gives the vectors of metric coordinates of vertices of A_n with respect to set $D^* = \{y_1, y_3, y_{t+2}\}$ in a special way.

i	$S_i(y_1)$	$D^* = \{y_1, y_3, y_{t+2}\}$
0	y_1	$(0,2,t-1)$
1	x_1	$(1,3,t-1)$
	x_2	$(1,2,t)$
	y_2	$(1,1,t)$
	y_n	$(1,3,t-2)$
2	x_3	$(2,1,t)$
	x_n	$(2,4,t-2)$
	y_3	$(2,0,t-1)$
	y_{n-1}	$(2,4,t-3)$
$3 \leq i \leq t-3$	x_{i+1}	$(i,i-2,t-i+2)$
$(t \geq 6)$	x_{n-i+2}	$(i,i+2,t-i)$
	y_{i+1}	$(i,i-2,t-i+1)$
	y_{n-i+1}	$(i,i+2,t-i-1)$
$t-2$	x_{t-1}	$(t-2,t-4,4)$
	x_{n-t+4}	$(t-2,t,2)$
	y_{t-1}	$(t-2,t-4,3)$
	y_{n-t+3}	$(t-2,t,1)$
$t-1$	x_t	$(t-1,t-3,3)$
	x_{n-t+3}	$(t-1,t,1)$
	y_t	$(t-1,t-3,2)$
	y_{n-t+2}	$(t-1,t-1,0)$
t	x_{n-t+1}	$(t,t-2,2)$
	x_{n-t+2}	$(t,t-1,1)$
	y_{n-t+1}	$(t,t-2,1)$

TABLE 2. Vectors of metric coordinates for A_n, $n = 2t$, $t \geq 4$

Starting from Table 3, it is easy to check that for each $i \in \{1,2,\ldots,t+1\}$, there do not exist two vertices $x, y \in S_i(y_1)$ such that all coordinates of the vector $r(x,D^*) - r(y,D^*)$ are equal to 0. Also, it can be easily seen that for each $i, j \in \{1,2,\ldots,t+1\}$, $i \neq j$, there do not exist two vertices $x \in S_i(y_1)$ and $y \in S_j(y_1)$ such that all coordinates of the vector $r(x,D^*) - r(y,D^*)$ are equal to $i-j$.

Enumeration technique shows that minimal doubly resolving set for A_3, A_4 is $D^* = \{y_1, y_2, y_3\}$, for A_5, it is $D^* = \{y_1, y_3, y_4\}$, and for A_6, A_7, it is $D^* = \{y_1, y_3, y_5\}$ respectively.

In this way, D^* is a minimal doubly resolving set for A_n, $n \geq 3$ and Theorem 1 holds. □

i	$S_i(y_1)$	$D^* = \{y_1, y_3, y_{t+2}\}$
0	y_1	$(0,2,t)$
1	x_1	$(1,3,t)$
	x_2	$(1,2,t+1)$
	y_2	$(1,1,t)$
	y_n	$(1,3,t-1)$
2	x_3	$(2,1,t)$
	x_n	$(2,4,t-1)$
	y_3	$(2,0,t-1)$
	y_{n-1}	$(2,4,t-2)$
$3 \leq i \leq t-2$	x_{i+1}	$(i,i-2,t-i+2)$
$(t \geq 5)$	x_{n-i+2}	$(i,i+2,t-i+1)$
	y_{i+1}	$(i,i-2,t-i+1)$
	y_{n-i+1}	$(i,i+2,t-i)$
$t-1$	x_t	$(t-1,t-3,3)$
	x_{n-t+3}	$(t-1,t+1,2)$
	y_t	$(t-1,t-3,2)$
	y_{n-t+2}	$(t-1,t,1)$
t	x_{n-t}	$(t,t-2,2)$
	x_{n-t+2}	$(t,t,1)$
	y_{n-t}	$(t,t-2,1)$
	y_{n-t+1}	$(t,t-1,0)$
$t+1$	x_{n-t+1}	$(t+1,t-1,1)$

TABLE 3. Vectors of metric coordinates for A_n, $n = 2t+1$, $t \geq 4$

3. THE MINIMAL DOUBLY RESOLVING SETS FOR MÖBIUS LADDERS M_n

The Möbius ladder M_n is a cubic circulant graph which consist of an even number of vertices. It can be constructed from an n-cycle by adding new edges (called ”rungs”) which connects the opposite pair of vertices in the cycle. The applications of Möbius ladders can be found in electronics, computer science, chemistry and chemical stereography. We assume that the vertices of Möbius ladder M_n are numbered $\{v_1, v_2, \dots, v_n\}$ counter clockwise in cycle C_n and opposite pairs of vertices are connected by adding edges between them, where n is an even number. In [1], it has been proved that the metric dimension $\beta(M_n) \geq 3$ for $n \geq 8$. Thus, we have $\psi(M_n) \geq \beta(M_n) \geq 3$, i.e., $\psi(M_n) \geq 3$ for every $n \geq 8$.

Lemma 2. *Let $S_i(v_1) = \{w \in V(M_n) : d(v_1, w) = i\}$ be the set of vertices in V at distance i from v_1. For M_n with $n \geq 8$, sets $S_i(v_1)$ are given in Table 4.*

Proof. There are four cases to be discussed $n \equiv 0, 2, 4$ *or* 6 $(mod\ 8)$ i.e. $n = 8k,\ 8k+2,\ 8k+4$ *or* $8k+6$ *for* $k \geq 1$. Let us consider the case when $n \equiv 0$ $(mod\ 8)$,

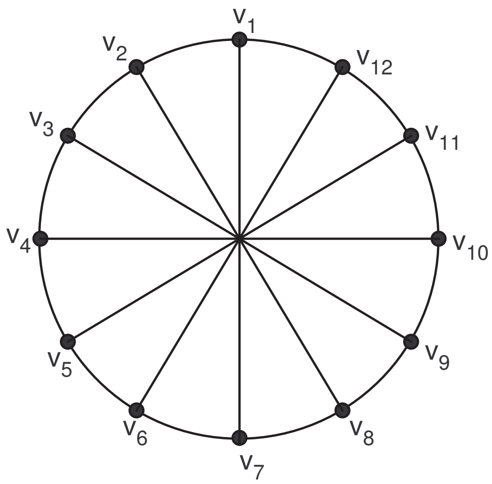

FIGURE 2. The Möbius ladder M_{12}.

so $n = 8k$ for $k \geq 1$. By definition of $S_i(v_1)$, it is easy to see that $S_i(v_1)$ is the set of all $v \in V(M_n)$ such that $va \in E(M_n)$ for some $a \in S_{i-1}(v_1)$ and $v \notin S_l(v_1)$, $l < i-1$. Now using the definition of M_n, it immediately follows that

$$S_1(v_1) = \{v_2, v_{4k+1}, v_n\}.$$

Let us prove by induction that, for $2 \leq i \leq 2k$

$$S_i(v_1) = \{v_{i+1}, v_{n-i+1}, v_{4k+2-i}, v_{4k+i}\}. \tag{3.1}$$

For $i = 2$, starting from $S_1(v_1)$, we obtain

$$S_2(v_1) = \{v_3, v_{n-1}, v_{4k}, v_{4k+2}\}.$$

Suppose that (3.1) holds for $2 \leq l < 2k$. Starting from $S_l(v_1)$, we obtain the following candidates for members of $S_{l+1}(v_1)$:

$$v_l, v_{n-l}, v_{4k+1-l}, v_{4k-1+l}, v_{l+2}, v_{n-l+2}, v_{4k+3-l}, v_{4k+1+l}.$$

The first four vertices belong to $S_{l-1}(v_1)$ and consequently cannot belong to $S_{l+1}(v_1)$. For example $v_l = v_{(l-1)+1} \in S_{l-1}(v_1)$. The set of remaining four vertices can be expressed in form (3.1) for $i = l+1$. The sets $S_i(v_1)$ for other three cases can be obtained in the same way. The following table displays $S_i(v_1)$. □

Note that the sets $S_i(v_1)$, defined in Lemma 2 can be used to determine the distance between two arbitrary vertices in $V(M_n)$ in the following way.

n	i	$S_i(v_1)$
$8k$	1	$\{v_2, v_{4k+1}, v_n\}$
	$2 \leq i \leq 2k$	$\{v_{i+1}, v_{n-i+1}, v_{4k+2-i}, v_{4k+i}\}$
$8k+2$	1	$\{v_2, v_{4k+2}, v_n\}$
	$2 \leq i \leq 2k$	$\{v_{i+1}, v_{n-i+1}, v_{4k+3-i}, v_{4k+1+i}\}$
	$2k+1$	$\{v_{2k+2}, v_{6k+2}\}$
$8k+4$	1	$\{v_2, v_{4k+3}, v_n\}$
	$2 \leq i \leq 2k+1$	$\{v_{i+1}, v_{n-i+1}, v_{4k+4-i}, v_{4k+2+i}\}$
$8k+6$	1	$\{v_2, v_{4k+4}, v_n\}$
	$2 \leq i \leq 2k+1$	$\{v_{i+1}, v_{n-i+1}, v_{4k+5-i}, v_{4k+3+i}\}$
	$2k+2$	$\{v_{2k+3}, v_{6k+5}\}$

TABLE 4. Vectors of metric coordinates for M_n, $n = 8k$, $k \geq 1$

As symmetry of M_n displays the fact that $d(v_i, v_j) = d(v_1, v_{j-i+1})$ for $j > i$. Consequently if we know the distance $d(v_1, w)$ for every $w \in V(M_n)$ then we can reconstruct the distance between every two vertices from $V(M_n)$.

Lemma 3. *For* $n \geq 8$, $\psi(M_n) = 3$ *when* $n \equiv 0$ *or* 4 $(mod\ 8)$.

Proof. **Case 1:** If $n \equiv 0$ $(mod\ 8)$.

We have $n = 8k$ for $k \geq 1$. Consider the following table which shows the vectors of metric coordinates of vertices of M_n with respect to set $D^* = \{v_1, v_{2k+1}, v_{4k+1}\}$ in the following way.

i	$S_i(v_1)$	$D^* = \{v_1, v_{2k+1}, v_{4k+1}\}$
0	v_1	$(0, 2k, 1)$
1	v_2	$(1, 2k-1, 2)$
	v_{4k+1}	$(1, 2k, 0)$
	v_n	$(1, 2k, 2)$
$2 \leq i \leq 2k-1$	v_{i+1}	$(i, 2k-i, i+1)$
	v_{n-i+1}	$(i, 2k-i+1, i+1)$
	v_{4k+2-i}	$(i, 2k-i+1, i-1)$
	v_{4k+i}	$(i, 2k-i+2, i-1)$
$2k$	v_{2k+1}	$(2k, 0, 2k)$
	v_{6k+1}	$(2k, 1, 2k)$
	v_{2k+2}	$(2k, 1, 2k-1)$
	v_{6k}	$(2k, 2, 2k-1)$

TABLE 5. Vectors of metric coordinates for M_n, $n = 8k$, $k \geq 1$

Case 2: If $n \equiv 4$ $(mod\ 8)$.

We have $n = 8k+4$ for $k \geq 1$. Consider the following table which shows the vectors of metric coordinates of vertices of M_n with respect to set $D^* = \{v_1, v_{2k+2}, v_{4k+3}\}$ in the following way.

i	$S_i(v_1)$	$D^* = \{v_1, v_{2k+2}, v_{4k+3}\}$
0	v_1	$(0, 2k+1, 1)$
1	v_2	$(1, 2k, 2)$
	v_{4k+3}	$(1, 2k+1, 0)$
	v_n	$(1, 2k+1, 2)$
$2 \leq i \leq 2k$	v_{i+1}	$(i, 2k-i+1, i+1)$
	v_{n-i+1}	$(i, 2k-i+2, i+1)$
	v_{4k+4-i}	$(i, 2k-i+2, i-1)$
	v_{4k+2+i}	$(i, 2k-i+3, i-1)$
$2k+1$	v_{2k+2}	$(2k+1, 0, 2k+1)$
	v_{6k+4}	$(2k+1, 1, 2k+1)$
	v_{2k+3}	$(2k+1, 1, 2k)$
	v_{6k+3}	$(2k+1, 2, 2k)$

TABLE 6. Vectors of metric coordinates for M_n, $n = 8k+4$, $k \geq 1$

Since $v_1 \in D^*$, Tables 5 and 6 show us that the first metric coordinate of a vector from $S_i(v_1)$ with respect to D^* is equal to i. It is easy to check that, there do not exist two vertices $x, y \in S_i(v_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be verified that there do not exist two vertices x, y such that $x \in S_i(v_1)$ and $y \in S_j(v_1)$ with $i \neq j$, so that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$.

In this way, we have found minimal doubly resolving set $D^* = \{v_1, v_{2k+1}, v_{4k+1}\}$ for M_n when $n \equiv 0 \ (mod\ 8)$ and $D^* = \{v_1, v_{2k+2}, v_{4k+3}\}$ when $n \equiv 4 \ (mod\ 8)$. This proves the statement of Lemma 3. □

Lemma 4. *$\psi(M_n) > 3$ for $n \equiv 2 (mod\ 8)$.*

Proof. We know that $\psi(M_n) \geq 3$ and thus we should prove that every subset D of vertex set $V(M_n)$ with $|D| = 3$ is not a doubly resolving set for M_n. We may assume that $v_1 \in D$. In Table 7, one can find all possible types of such set D and the corresponding non-doubly resolving pair of vertices from $V(M_n)$.

Let us prove that the vertices v_{4k+1}, v_{4k+2} are not doubly resolved by any two vertices from set $\{v_1, v_i, v_j\}$, $2 \leq i \leq 2k$, $4k+3 \leq j \leq 6k+2$. Using distances from Table 4, it follows that

(i) $d(v_1, v_{4k+1}) = 2$ and $d(v_1, v_{4k+2}) = 1$.

(ii) $d(v_i, v_{4k+1}) = d(v_1, v_{4k+2-i}) = i+1$ and $d(v_i, v_{4k+2}) = d(v_1, v_{4k+3-i}) = i$ for $2 \leq i \leq 2k-1$.

(iii) $d(v_{2k}, v_{4k+1}) = d(v_1, v_{2k+2}) = 2k+1$ and $d(v_{2k}, v_{4k+2}) = d(v_1, v_{2k+3}) = 2k$.

D	Non-doubly resolved pair
$\{v_1, v_i, v_j\},\ 2 \le i, j \le 4k+1,\ i \ne j$	$\{v_{4k+2}, v_n\}$
$\{v_1, v_i, v_j\},\ 4k+3 \le i, j \le n,\ i \ne j$	$\{v_2, v_{4k+2}\}$
$\{v_1, v_{4k+2}, v_i\},\ 2 \le i \le 2k+1$ *or* $6k+3 \le i \le n$	$\{v_{2k+2}, v_{6k+2}\}$
$\{v_1, v_{4k+2}, v_i\},\ 2k+2 \le i \le 4k+1$ *or* $4k+3 \le i \le 6k+2$	$\{v_{2k+1}, v_{6k+3}\}$
$\{v_1, v_i, v_j\},\ 2 \le i \le 2k,\ 4k+3 \le j \le 6k+2$	$\{v_{4k+1}, v_{4k+2}\}$
$\{v_1, v_i, v_j\},\ 2 \le i \le 2k,\ 6k+3 \le j \le n$	$\{v_1, v_{4k+2}\}$
$\{v_1, v_{2k+1}, v_i\},\ 4k+3 \le i \le 6k+1$	$\{v_1, v_n\}$
$\{v_1, v_{2k+1}, v_{6k+2}\}$	$\{v_{4k+2}, v_n\}$
$\{v_1, v_{2k+1}, v_i\},\ 6k+3 \le i \le n$	$\{v_{2k+2}, v_{6k+2}\}$
$\{v_1, v_{2k+2}, v_i\},\ 4k+3 \le i \le 6k+2$	$\{v_{2k+1}, v_{6k+3}\}$
$\{v_1, v_{2k+2}, v_{6k+3}\}$	$\{v_2, v_{4k+2}\}$
$\{v_1, v_{2k+2}, v_i\},\ 6k+4 \le i \le n$	$\{v_{4k+2}, v_{4k+3}\}$
$\{v_1, v_i, v_j\},\ 2k+3 \le i \le 4k+1,\ 4k+3 \le j \le 6k+2$	$\{v_{2k+1}, v_{6k+3}\}$
$\{v_1, v_i, v_j\},\ 2k+3 \le i \le 4k+1,\ 6k+3 \le j \le n$	$\{v_1, v_2\}$

TABLE 7. Non-doubly resolved pairs of M_n for $n = 8k+2,\ k \ge 1$

(iv) $d(v_j, v_{4k+1}) = d(v_1, v_{j-4k}) = j-4k-1$ and $d(v_j, v_{4k+2}) = d(v_1, v_{j-4k-1}) = j-4k-2$ for $4k+3 \le j \le 6k+2$.

From (i), (ii), (iii) and (iv), we have

$$\begin{aligned} d(v_1, v_{4k+1}) - d(v_1, v_{4k+2}) &= d(v_i, v_{4k+1}) - d(v_i, v_{4k+2}) \\ &= d(v_j, v_{4k+1}) - d(v_j, v_{4k+2}) = 1, \end{aligned}$$

i.e., $\{v_1, v_i, v_j\},\ 2 \le i \le 2k,\ 4k+3 \le j \le 6k+2$ is not a doubly resolving set of M_n. Similarly, we can consider all other types of D from Table 7 and verify their corresponding non-doubly resolved pairs of vertices. □

Lemma 5. $\psi(M_n) > 3$ *for* $n \equiv 6 (mod\ 8)$.

Proof. The proof is similar to the proof of Lemma 4. The following table displays all possible types of set D with $|D| = 3$ and the corresponding non-doubly resolving pair of vertices.

□

Lemma 6. *For every* $n \ge 8$, $\psi(M_n) = 4$ *when* $n \equiv 2$ *or* 6 $(mod\ 8)$.

Proof. From Lemmas 4 and 5, it is clear that $\psi(M_n) > 3$, therefore it is enough to show that there exist a subset of $V(M_n)$ with cardinality 4 such that it doubly resolves each pair of vertices from $V(M_n)$. Consider the following two tables:

(i) $n \equiv 2\ (mod\ 8)$.

We have $n = 8k+2$ for $k \ge 1$. Consider the following table which shows the vectors of metric coordinates of vertices of M_n with respect to set $D^* = \{v_1,\ v_{2k+1},\ v_{4k+2}, v_{6k+2}\}$ in the following way.

D	Non-doubly resolved pair
$\{v_1, v_i, v_j\},\ 2 \le i, j \le 4k+3,\ i \ne j$	$\{v_{4k+4}, v_n\}$
$\{v_1, v_i, v_j\},\ 4k+5 \le i, j \le n,\ i \ne j$	$\{v_2, v_{4k+4}\}$
$\{v_1, v_{4k+4}, v_i\},\ 2 \le i \le 2k+2$ *or* $6k+6 \le i \le n$	$\{v_{2k+3}, v_{6k+5}\}$
$\{v_1, v_{4k+4}, v_i\},\ 2k+3 \le i \le 4k+3$ *or* $4k+5 \le i \le 6k+5$	$\{v_{2k+2}, v_{6k+6}\}$
$\{v_1, v_i, v_j\},\ 2 \le i \le 2k+2,\ 4k+5 \le j \le 6k+5$	$\{v_{2k+2}, v_{2k+3}\}$
$\{v_1, v_i, v_j\},\ 2 \le i \le 2k+2,\ 6k+6 \le j \le n$	$\{v_1, v_{4k+4}\}$
$\{v_1, v_i, v_j\},\ 2k+3 \le i \le 4k+3,\ 4k+5 \le j \le 6k+5$	$\{v_{2k+2}, v_{6k+6}\}$
$\{v_1, v_i, v_j\},\ 2k+3 \le i \le 4k+3,\ 6k+6 \le j \le n$	$\{v_{6k+5}, v_{6k+6}\}$

TABLE 8. Non-doubly resolved pairs of M_n for $n = 8k+6$, $k \ge 1$

i	$S_i(v_1)$	$D^* = \{v_1, v_{2k+1}, v_{4k+2}, v_{6k+2}\}$
0	v_1	$(0, 2k, 1, 2k+1)$
1	v_2	$(1, 2k-1, 2, 2k)$
	v_{4k+2}	$(1, 2k+1, 0, 2k)$
	v_n	$(1, 2k+1, 2, 2k)$
$2 \le i \le 2k$	v_{i+1}	$(i, 2k-i, i+1, 2k+1-i)$
	v_{n-i+1}	$(i, 2k+2-i, i+1, 2k+1-i)$
	v_{4k+3-i}	$(i, 2k+2-i, i-1, 2k+3-i)$
	v_{4k+1+i}	$(i, 2k+2-i, i-1, 2k+1-i)$
$2k+1$	v_{2k+2}	$(2k+1, 1, 2k, 2)$
	v_{6k+2}	$(2k+1, 1, 2k, 0)$

TABLE 9. Vectors of metric coordinates for M_n, $n = 8k+2$, $k \ge 1$

(ii) $n \equiv 6\ (mod\ 8)$.

We have $n = 8k+6$ for $k \ge 1$. Consider the following table which shows the vectors of metric coordinates of vertices of M_n with respect to set $D^* = \{v_1, v_{2k+2}, v_{4k+4}, v_{6k+5}\}$ in the following way.

Since $v_1 \in D^*$, Tables 9 and 10 show us that the first metric coordinate of a vector from $S_i(v_1)$ with respect to D^* is equal to i. It is easy to check that, there do not exist two vertices $x, y \in S_i(v_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be verified that there do not exist two vertices x, y such that $x \in S_i(v_1)$ and $y \in S_j(v_1)$ with $i \ne j$, so that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$.

In this way, we have found minimal doubly resolving set $D^* = \{v_1, v_{2k+1}, v_{4k+2}, v_{6k+2}\}$ for M_n when $n \equiv 2\ (mod\ 8)$ and $D^* = \{v_1, v_{2k+2}, v_{4k+4}, v_{6k+5}\}$ when $n \equiv 6\ (mod\ 8)$. This proves the statement of Lemma 6. □

i	$S_i(v_1)$	$D^* = \{v_1, v_{2k+2}, v_{4k+4}, v_{6k+5}\}$
0	v_1	$(0, 2k+1, 1, 2k+2)$
1	v_2	$(1, 2k, 2, 2k+1)$
	v_{4k+4}	$(1, 2k+2, 0, 2k+1)$
	v_n	$(1, 2k+2, 2, 2k+1)$
$2 \leq i \leq 2k+1$	v_{i+1}	$(i, 2k+1-i, i+1, 2k+2-i)$
	v_{n-i+1}	$(i, 2k+3-i, i+1, 2k+2-i)$
	v_{4k+5-i}	$(i, 2k+3-i, i-1, 2k+4-i)$
	v_{4k+3+i}	$(i, 2k+3-i, i-1, 2k+2-i)$
$2k+2$	v_{2k+3}	$(2k+2, 1, 2k+1, 2)$
	v_{6k+5}	$(2k+2, 1, 2k+1, 0)$

TABLE 10. Vectors of metric coordinates for M_n, $n = 8k+6$, $k \geq 1$

Theorem 2. *For every even positive integer $n \geq 8$, we have*

$$\psi(M_n) = \begin{cases} 3, & if\ n \equiv 0\ or\ 4\ (mod\ 8) \\ 4, & if\ n \equiv 2\ or\ 6\ (mod\ 8) \end{cases}.$$

Proof. The proof follows by combining the results of Lemmas 3 and 6. □

4. CONCLUSION

In this paper, we theoretically determine the minimal cardinality $\psi(A_n)$ and $\psi(M_n)$ of doubly resolving sets for the antiprism graphs and Möbius ladders. The future work will be focused on determining some other invariants of generalized Petersen graphs related to the metric dimension and doubly resolving sets.

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