WALKS IN PATH GRAPH ON FOUR VERTICES AND FIBONACCI SEQUENCE

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Abstract. Using elementary knowledge of graph theory, we show that a path graph on four vertices exhibits Fibonacci structure. For arbitrary start and end vertices, the number of walks of any length is given by a Fibonacci number.

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The Fibonacci sequence is a well known mathematical object, having interesting connections to different features in nature. It is given by a simple recurrence formula.

Definition 1. A sequence \( \Phi : \mathbb{N} \to \mathbb{N} \) defined with

\[
\Phi(n + 1) = \Phi(n) + \Phi(n - 1), \quad \forall n \in \mathbb{N},
\]

with initial values \( \Phi(0) = 0, \Phi(1) = 1 \) is called the Fibonacci sequence.

In this short comment we reveal the link between Fibonacci numbers and number of walks in a path graph on four vertices, i.e. of length three, see graph \( P_3 \) in Figure 1. Note that the Fibonacci structure is linked with \( P_3 \) and cannot be generalized to \( P_N \).

![Figure 1. Path-graph of length three.](image)

To avoid any confusion, we recall the definition of a walk in a graph, see [2, p. 9]. A walk of length \( k \) in graph \( G \) is a non-empty alternating sequence \( v_0e_0v_1e_1 \cdots e_{k-1}v_k \) of vertices and edges in \( G \) such that \( e_i = \{v_i,v_{i+1}\} \) for all \( i < k \).

We denote with

\[
A := \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

\[0.2\]
the adjacency matrix of $P_3$. An elementary assertion of graph theory (see [1, Lemma 2.5]) gives the number of walks of length $k$ between vertices $i$ and $j$ as the appropriate element of $k$-th power of the adjacency matrix $B$ of the graph, i.e.

$$w_{k,i,j} := \{B^k\}_{i,j}. \tag{0.3}$$

The following lemma enables us to compute powers of adjacency matrix of a path graph (regardless the number of vertices).

**Lemma 1.** Let $B \in \mathbb{R}^{N \times N}$ be an adjacency matrix of a path graph of length $N - 1$. Then for any $k \in \mathbb{N}$ it holds that

$$\{B^{k+1}\}_{i,j} = \{B^k\}_{i-1,j} + \{B^k\}_{i+1,j}, \tag{0.4}$$

and

$$\{B^{k+1}\}_{i,j} = \{B^k\}_{i,j-1} + \{B^k\}_{i,j+1}, \tag{0.5}$$

where we set $\{B^p\}_{l,m} = 0$ for any $p \in \mathbb{N}$ and $(l,m) \notin \{1, \ldots, N\}^2$.

**Proof.** The claim immediately follows from graph theory interpretation of (0.3): Any walk of length $k + 1$ between vertices $i$ and $j$ contains a unique subwalk of length $k$ from either $i - 1$ or $i + 1$ (if such vertex exists) to $j$. Hence we can conclude (0.4), similar argument proves also (0.5). \hfill $\square$

Note that any path-graph is a bipartite graph, which implies non-existence of walk of odd length between vertices of the same parity (and walk of even length between vertices of different parity), i.e. $\{B^k\}_{i,j} = 0$ if the parity of $k$ and $i + j$ differs. Using this property we define the following split of $k$-th power of the adjacency matrix of $P_3$, to matrices of corner, edge and middle elements, as they represent three qualitatively distinct cases.

We set matrices $C_k, E_k, M_k$ for $k \in \mathbb{N}$ with

$$\{C_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i, j) \in \{1,4\} \times \{1,4\} \text{ and } (i + j + k) \text{ even}, \\ 0 & \text{otherwise}, \end{cases} \tag{0.6}$$

$$\{E_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i, j) \in (\{2,3\} \times \{1,4\}) \cup (\{1,4\} \times \{2,3\}) \\ 0 & \text{and } (i + j + k) \text{ even}, \\ \text{otherwise}, \end{cases} \tag{0.7}$$

$$\{M_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i, j) \in \{2,3\} \times \{2,3\} \text{ and } (i + j + k) \text{ even}, \\ 0 & \text{otherwise}. \end{cases} \tag{0.8}$$

They create a complete split of $A^k$, i.e. $A^k = C_k + E_k + M_k$ and maximally one of them has a non-zero contribution for each element $\{A^k\}_{i,j}$, as it is illustrated in the following scheme.
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\[ A^k = \begin{pmatrix}
0 & E_k & 0 & C_k \\
E_k & 0 & M_k & 0 \\
0 & M_k & 0 & E_k \\
C_k & 0 & E_k & 0 \\
\end{pmatrix}, \quad A' = \begin{pmatrix}
C_l & 0 & E_l & 0 \\
0 & M_l & 0 & E_l \\
E_l & 0 & M_l & 0 \\
0 & E_l & 0 & C_l \\
\end{pmatrix}, \]

for \( k = 2s - 1, l = 2s, s \in \mathbb{N}. \)

We call elements defined by first lines in (0.6), (0.7) or (0.8) structural non-zeros.

Note that \( C_1 \) is a zero matrix, the only one whose structural non-zeros attain zero value.

Matrix \( C_k \) of corner elements represents walks between vertices of the degree 1 (not necessarily distinct), \( E_k \) represents the walks connecting vertices with different degrees and finally \( M_k \) represents the walks between vertices with degree equal to 2.

We use the previously established split in the next Theorem.

**Theorem 1.** For given \( k \in \mathbb{N} \), the structural non-zeros of matrix \( C_k \) are equal to a constant, which we denote by \( c_k \). Moreover, it holds that

\[ c_k = \Phi(k - 1), \quad e_k = \Phi(k), \quad m_k = \Phi(k + 1). \]  

(0.9)

**Proof.** Applying (0.5) to all structural non-zeros of \( C_k \) we get

\[ c_{k+1} = e_k. \quad \]  

(0.10)

Applying (0.4) and (0.5) to structural non-zeros of \( E_{k+1} \) gives two relations

\[ e_{k+1} = m_k, \]  

(0.11)

\[ e_{k+1} = c_k + e_k. \]  

(0.12)

Combining (0.10) and (0.12) one gets

\[ c_{k+2} = e_{k+1} = c_k + e_k = c_{k+1} + c_k. \]

From (0.2) we deduce initial values \( c_1 = 0 \) and, with the help of (0.10), \( c_2 = e_1 = 1 \). Hence we conclude \( c_k = \Phi(k - 1) \) for any \( k \in \mathbb{N}. \) Then (0.10) and (0.11) imply \( e_k = \Phi(k) \) and \( m_k = \Phi(k + 1) \), for \( k \in \mathbb{N}. \)

We can conclude from Theorem 1, that any non-zero entry in \( A^k \) is a Fibonacci number. In other words, number of \( k \)-walks in \( P_3 \) from \( i \) to \( j \) is represented by a Fibonacci number \( \Phi(n) = \Phi(n(i, j, k)), \) where \( n(i, j, k) \in \{k - 1, k, k + 1\} \) according to (0.6)–(0.8) and Theorem 1.

**Remark 1.** The link between Fibonacci sequence and walks in path graph was revealed during the research on a generalization of bistable equation with non-smooth minima, see [3] or [4]. When substituting the double-well potential with a quadruple-well one, then the nontrivial and qualitatively distinct stationary solutions can be
represented by walks in graph $P_3$, provided additional geometrical constraints are met by the potential to ensure the transitions between minima have the equal length.

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