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QUALITATIVE PROPERTIES OF SOLUTIONS FOR MIXED TYPE FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH MAXIMA

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Abstract. In this paper we study some properties of the solutions of a second order system of functional-differential equations with maxima, of mixed type, with "boundary" conditions. We use the weakly Picard operator technique.

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1. INTRODUCTION

Differential equations with maxima are often met in the applications, for instance in the theory of automatic control. Numerous results on existence and uniqueness, asymptotic stability as well as numerical solutions have been obtained. To name a few, we refer the reader to [1–3, 11] and the references therein.

The main goal of the presented paper is to study a second order functional-differential equations with maxima, of mixed type, using the theory of weakly Picard operators. The theory of Picard operators was introduced by I. A. Rus (see [7] and [9]) to study problems related to fixed point theory. This abstract approach is used by many mathematicians and it seemed to be a very useful and powerful method in the study of integral equations and inequalities, ordinary and partial differential equations (existence, uniqueness, differentiability of the solutions), etc.

We consider the following functional-differential equation

$$-x''(t) = f(t, x(t), \max_{t-h_1 \le \xi \le t+h_2} x(\xi)),\ t \in [a,b] \tag{1.1}$$

with the "boundary" conditions

$$\begin{cases} x(t) = \varphi(t),\ t \in [a-h_1, a], \\ x(t) = \psi(t),\ t \in [b, b+h_2]. \end{cases} \tag{1.2}$$

The novelty of this paper lies in the fact that 'maxima' is taken on interval $[t-h_1, t+h_2]$, where $h_1,\ h_2 > 0$. Our results extend and improve corresponding theorems in the existing literature (see, e.g. [4–6, 8]). Also, in the end an extremal principle for the solution is given.

We suppose that:

(C_1) h_1, h_2, a and $b \in \mathbb{R}$, $a < b$, $h_1 > 0$, $h_2 > 0$;
(C_2) $f \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$;
(C_3) there exists $L_f > 0$ such that

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L_f \max_{i=1,2} |u_i - vi|,$$

for all $t \in [a,b]$ and $u_i, v_i \in \mathbb{R}, i = 1,2$;
(C_4) $\varphi \in C([a-h_1,a],\mathbb{R})$ and $\psi \in C([b,b+h_2],\mathbb{R})$.

Let G be the Green function of the following problem

$$-x'' = \chi,\ x(a) = 0,\ x(b) = 0,\ \chi \in C[a,b].$$

The problem (1.1)–(1.2), $x \in C[a-h_1,b+h_2] \cap C^2[a,b]$ is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t),\ t \in [a-h_1,a], \\ w(\varphi,\psi)(t)+ \\ \quad +\int_a^b G(t,s) f(s,x(s), \max\limits_{s-h_1 \le \xi \le s+h_2} x(\xi))ds,\ t \in [a,b], \\ \psi(t),\ t \in [b,b+h_2], \end{cases} \tag{1.3}$$

$x \in C[a-h_1,b+h_2]$, where

$$w(\varphi,\psi)(t) := \frac{t-a}{b-a}\psi(b) + \frac{b-t}{b-a}\varphi(a).$$

The equation (1.1) is equivalent with

$$x(t) = \begin{cases} x(t),\ t \in [a-h_1,a], \\ w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})(t)+ \\ \quad +\int_a^b G(t,s) f(s,x(s), \max\limits_{s-h_1 \le \xi \le s+h_2} x(\xi))ds,\ t \in [a,b], \\ x(t),\ t \in [b,b+h_2], \end{cases} \tag{1.4}$$

$x \in C[a-h_1,b+h_2]$.

In what follow we consider the operators $B_f, E_f : C[a-h_1,b+h_2] \to C[a-h_1,b+h_2]$ defined by

$$B_f(x)(t) := \begin{cases} \varphi(t),\ t \in [a-h_1,a], \\ w(\varphi,\psi)(t)+ \\ \quad +\int_a^b G(t,s) f(s,x(s), \max\limits_{s-h_1 \le \xi \le s+h_2} x(\xi))ds,\ t \in [a,b], \\ \psi(t),\ t \in [b,b+h_2], \end{cases}$$

and

$$E_f(x)(t) := \begin{cases} x(t),\ t \in [a-h_1,a], \\ w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})(t) + \\ \quad + \int_a^b G(t,s) f(s, x(s), \max_{s-h_1 \le \xi \le s+h_2} x(\xi)) ds,\ t \in [a,b], \\ x(t),\ t \in [b,b+h_2]. \end{cases}$$

Let $X := C[a-h_1, b+h_2]$ and $X_{\varphi,\psi} := \{x \in X \mid x|_{[a-h_1,a]} = \varphi,\ x|_{[b,b+h_2]} = \psi\}$. It is clear that

$$X = \bigcup_{\substack{\varphi \in C[a-h_1,a] \\ \psi \in C[b,b+h_2]}} X_{\varphi,\psi}$$

is a partition of X. We have

Lemma 1. *We suppose that the conditions* (C_1), (C_2) *and* (C_4) *are satisfied. Then*

(a) $B_f(X) \subset X_{\varphi,\psi}$ *and* $B_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$;

(b) $B_f|_{X_{\varphi,\psi}} = E_f|_{X_{\varphi,\psi}}$.

In this paper we shall prove that, if L_f is small enough, then the operator E_f is weakly Picard operator and we shall study the equation (1.1) in the terms of this operator.

2. Picard and Weakly Picard operators

In this paper we use the terminologies and notations from [7–9]. Let us recall now some essential definitions and fundamental results.

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by $A^0 = 1_X$, $A^1 = A$, $A^{n+1} := A \circ A^n, n \in \mathbb{N}$ the iterates of the operator A;

We also use the following notations: $F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \varnothing\}$ - the family of the nonempty invariant subset of A;

We begin with the definitions of a Picard and weakly Picard operator.

Definition 1. Let (X,d) be a metric space. An operator $A: X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:

(i) $F_A = \{x^*\}$;

(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2. Let (X,d) be a metric space. An operator $A: X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.

Definition 3. If A is weakly Picard operator then we consider the operator A^∞ defined by

$$A^\infty: X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 1. It is clear that $A^{\infty}(X) = F_A = \{x \in X \mid A(x) = x\}$.

The following results are very useful in the sequel.

Lemma 2. *Let $(X,d,\leq)$ be an ordered metric space and $A\colon X \to X$ an operator. We suppose that A is WPO and increasing. Then, the operator A^{∞} is increasing.*

Lemma 3. *(Abstract Gronwall lemma) Let $(X,d,\leq)$ be an ordered metric space and $A\colon X \to X$ an operator. We suppose that A is WPO and increasing. Then:*

(a) $x \leq A(x) \Rightarrow x \leq A^{\infty}(x)$;
(b) $x \geq A(x) \Rightarrow x \geq A^{\infty}(x)$.

Lemma 4. *(Abstract comparison lemma) Let $(X,d,\leq)$ an ordered metric space and $A,B,C\colon X \to X$ be such that:*

(i) *the operators A,B,C are WPOs;*
(ii) $A \leq B \leq C$;
(iii) *the operator B is increasing.*

Then $x \leq y \leq z$ implies that $A^{\infty}(x) \leq B^{\infty}(y) \leq C^{\infty}(z)$.

Theorem 1. *Let (X,d) be a complete metric space and $A,B\colon X \to X$ two operators. We suppose that*

(i) *the operator A is an α-contraction;*
(ii) $F_B \neq \varnothing$;
(iii) *there exists $\eta > 0$ such that*

$$d(A(x),B(x)) \leq \eta,\ \forall x \in X.$$

Then, if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have

$$d(x_A^*,x_B^*) \leq \frac{\eta}{1-\alpha}.$$

Another important notion is the following

Definition 4. Let A be a weakly Picard operator and $c > 0$. The operator A is c-weakly Picard operator if

$$d(x,A^{\infty}(x)) \leq c\,d(x,A(x)),\ \forall x \in X.$$

For the c-PO$_s$ and c-WPO$_s$ we have the following lemma.

Theorem 2. *Let (X,d) be a metric space and $A_i\colon X \to X$, $i = 1,2$. Suppose that*

(i) *the operator A_i is c_i-WPO$_s$, $i = 1,2$;*
(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x),A_2(x)) \leq \eta,\ \forall x \in X.$$

Then $H(F_{A_1},F_{A_2}) \leq \eta \max(c_1,c_2)$.

Example 1. Let (X,d) be a complete metric space and $A\colon X\to X$ an α-contraction. Then A is $\frac{1}{1-\alpha}$-PO.

Example 2. Let (X,d) be a complete metric space and $A:X\to X$ continuous and α-graphic contraction. Then A is $\frac{1}{1-\alpha}$-WPO.

For more details on WPOs theory see [7,9] and [10].

3. Existence and uniqueness

Our first result is the following

Theorem 3. *We suppose that:*

(a) *the conditions* (C_1)*–*(C_4) *are satisfied;*

(C_5) $\frac{L_f}{8}(b-a)^2<1$.

Then the problem (1.1)–(1.2) has a unique solution which is the uniform limit of the successive approximations.

Proof. Consider the Banach space $(C[a-h_1,b+h_2],\|\cdot\|)$ where $\|\cdot\|$ is the Chebyshev norm, $\|\cdot\|:=\max\limits_{a-h_1\le t\le b+h_2}|x(t)|$.

The problem (1.1)–(1.2) is equivalent with the fixed point equation

$$B_f(x)=x,\ x\in C[a-h_1,b+h_2].$$

From the condition (C_3) we have, for $t\in[a,b]$

$$\begin{aligned}
&\left|B_f(x)(t)-B_f(y)(t)\right|\\
&\le L_f\int_a^b G(t,s)\left[\max|x(s)-y(s)|,\left|\max_{a-h_1\le\xi\le b+h_2}x(\xi)-\max_{a-h_1\le\xi\le b+h_2}y(\xi)\right|\right]ds\\
&\le L_f\int_a^b G(t,s)\max_{a-h_1\le\xi\le b+h_2}|x(s)-y(s)|\,ds\\
&\le \frac{L_f}{8}(b-a)^2\|x-y\|.
\end{aligned}$$

This implies that B_f is an α-contraction, with $\alpha=\frac{L_f}{8}(b-a)^2$. The proof follows from the contraction principle. □

Remark 2. From the proof of Theorem 3, it follows that the operator B_f is PO. Since

$$B_f|_{X_{\varphi,\psi}}=E_f|_{X_{\varphi,\psi}}$$

and

$$X:=C[a-h_1,b+h_2]=\bigcup_{\varphi,\psi}X_{\varphi,\psi},\ E_f(X_{\varphi,\psi})\subset X_{\varphi,\psi}$$

hence, the operator E_f is WPO and

$$F_{E_f}\cap X_{\varphi,\psi}=\{x^*_{\varphi,\psi}\},\forall\varphi\in C[a-h_1,a],\ \forall\psi\in C[b,b+h_2],$$

where $x^*_{\varphi,\psi}$ is the unique solution of the problem (1.1)–(1.2).

Remark 3. E_f is α-graphic contraction, i.e.

$$\left\| E_f^2(x) - E_f(x) \right\| \le \alpha \left\| x - E_f(x) \right\|, \ \forall x \in C[a-h_1, b+h_2].$$

4. Inequalities of Čaplygin type

Now we consider the operators E_f and B_f on the ordered Banach space $(C[a-h_1, b+h_2], \|\cdot\|, \le)$. We have

Theorem 4. *We suppose that:*

(a) *the conditions* $(C_1)-(C_4)$ *are satisfied;*
(b) $\frac{L_f}{8}(b-a)^2 < 1$;
(c) $f(t,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ *is increasing,* $\forall t \in [a,b]$.

Let x be a solution of equation (1.1) and y a solution of the inequality

$$-y''(t) \le f(t, y(t), \max_{t-h_1 \le \xi \le t+h_2} y(\xi)), \ t \in [a,b].$$

Then if $y(t) \le x(t), \forall t \in [a-h_1, a] \cup [b, b+h_2]$ *we obtain that* $y \le x$.

Proof. Let us consider the operator $\widetilde{w} : C[a-h_1, b+h_2] \to C[a-h_1, b+h_2]$ defined by

$$\widetilde{w}(z)(t) := \begin{cases} z(t), \ t \in [a-h_1, a], \\ w(z|_{[a-h_1,a]}, z|_{[b,b+h_2]})(t), \ t \in [a,b], \\ z(t), \ t \in [b, b+h_2]. \end{cases}$$

First of all we remark that

$$w(y|_{[a-h_1,a]}, y|_{[b,b+h_2]}) \le w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})$$

and

$$\widetilde{w}(y) \le \widetilde{w}(x).$$

In the terms of the operator E_f, we have

$$x = E_f(x) \text{ and } y \le E_f(y).$$

From the conditions $(C_1), (C_2)$ and (C_3) follows that the operator E_f is WPO. Also, from condition (c) we have that E_f is an increasing operator. Applying Lemma 2, we have that the operator E_f^∞ is increasing. From Theorem 3 we have that $E_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$, $\forall \varphi, \psi \in \mathbb{R}$. $E_f|_{X_{\varphi,\psi}}$ is a contraction and since $\widetilde{w}(z) \in X_{\varphi,\psi}$ then

$$E_f^\infty(\widetilde{w}) = E_f^\infty(y), \ \forall y \in X_{\varphi,\psi}.$$

Let $y \le E_f(y)$, since E_f is increasing, from the Gronwall lemma (Lemma 3) we get $y \le E_f^\infty(y)$. Also $y, \widetilde{w}(y) \in X_{w(y)}$, so $E_f^\infty(y) = E_f^\infty(\widetilde{w}(y))$. But $w(y) \le w(x)$, E_f^∞ is increasing and $E_f^\infty(\widetilde{w}(x)) = E_f^\infty(x) = x$. So

$$y \le E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \le E_f^\infty(\widetilde{w}(x)) = E_f^\infty(x) = x.$$

□

5. Data dependence: monotony

In this section we study the monotony of the solution of the problem (1.1)–(1.2) with respect to φ, ψ and f.

Theorem 5. *Let $f_i \in C([a,b]\times\mathbb{R}^2,\mathbb{R}), i = 1,2,3$, be as in Theorem 3. We suppose that:*

(i) $f_1 \le f_2 \le f_3$;
(ii) $f_2(t,\cdot,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ *is monotone increasing;*

Let x_i be a solution of the equation

$$-x_i''(t) = f_i(t, x(t), \max_{t-h_1\le\xi\le t+h_2} x(\xi)),\ t \in [a,b] \text{ and } i = 1,2,3.$$

Then, $x_1(t) \le x_2(t) \le x_3(t)$, $\forall t \in [a-h_1,a]\cup[b,b+h_2]$, implies that $x_1 \le x_2 \le x_3$, i.e. the unique solution of the problem (1.1)–(1.2) is increasing with respect to f, φ and ψ.

Proof. From Theorem 3, the operators $E_{f_i}, i = 1,2,3$, are WPOs. From the condition (ii) the operator E_{f_2} is monotone increasing. From the condition (i) it follows that

$$E_{f_1} \le E_{f_2} \le E_{f_3}.$$

On the other hand we remark that

$$\widetilde{w}(x_1) \le \widetilde{w}(x_2) \le \widetilde{w}(x_3)$$

and

$$x_i = E_{f_i}^\infty(\widetilde{w}(x_i)),\ i = 1,2,3.$$

So, the proof follows from Lemma 4. □

6. Data dependence: continuity

Consider the boundary value problem (1.1)–(1.2) and suppose the conditions of the Theorem 3 are satisfied. Denote by $x^*(\cdot;\varphi,\psi,f)$, the solution of this problem. We state the following result:

Theorem 6. *Let $\varphi_i, \psi_i, f_i, i = 1,2$ be as in the Theorem 3. Furthermore, we suppose that there exists $\eta_i > 0, i = 1,2$ such that*

(i) $|\varphi_1(t) - \varphi_2(t)| \le \eta_1$, $\forall t \in [a-h_1,a]$ *and* $|\psi_1(t) - \psi_2(t)| \le \eta_1$, $\forall t \in [b, b+h_2]$;

(ii) $|f_1(t,u_1,u_2)-f_2(t,u_1,u_2)| \le \eta_2, \forall t \in C[a,b], u_1,u_2 \in \mathbb{R}$.

Then

$$\left\| x_1^*(t;\varphi_1,\psi_1,f_1) - x_2^*(t;\varphi_2,\psi_2,f_2) \right\| \le \frac{8\eta_1 + (b-a)^2\eta_2}{8 - L_f(b-a)^2},$$

where $x_i^*(t;\varphi_i,\psi_i,f_i)$ *are the solution of the problem (1.1)–(1.2) with respect to* φ_i,ψ_i,f_i, $i=1,2$, *and* $L_f = \max(L_{f_1},L_{f_2})$.

Proof. Consider the operators B_{φ_i,ψ_i,f_i}, $i=1,2$. From Theorem 3 these operators are contractions.

Additionally

$$\left\| B_{\varphi_1,\psi_1,f_1}(x) - B_{\varphi_2,\psi_2,f_2}(x) \right\| \le \eta_1 + \eta_2 \frac{(b-a)^2}{8},$$

$\forall x \in C[a-h_1,b+h_2]$.

Now the proof follows from Theorem 1, with $A := B_{\varphi_1,\psi_1,f_1}$, $B = B_{\varphi_2,\psi_2,f_2}$, $\eta = \eta_1 + \eta_2\frac{(b-a)^2}{8}$ and $\alpha := \frac{L_f}{8}(b-a)^2$ where $L_f = \max(L_{f_1},L_{f_2})$. □

In what follow we shall use the c-WPOs techniques to give some data dependence results using Theorem 2.

Theorem 7. *Let* f_1 *and* f_2 *be as in the Theorem 3. Let* $S_{E_{f_1}}, S_{E_{f_2}}$ *be the solution sets of system (1.1) corresponding to* f_1 *and* f_2. *Suppose that there exists* $\eta > 0$, *such that*

$$|f_1(t,u_1,u_2) - f_2(t,u_1,u_2)| \le \eta \tag{6.1}$$

for all $t \in [a,b]$, $u_1,u_2 \in \mathbb{R}$.

Then

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \le \frac{(b-a)^2\eta}{8 - L_f(b-a)^2},$$

where $L_f = \max(L_{f_1},L_{f_2})$ *and* $H_{\|\cdot\|_C}$ *denotes the Pompeiu-Housdorff functional with respect to* $\|\cdot\|_C$ *on* $C[a,b]$.

Proof. In the condition of Theorem 3, the operators E_{f_1} and E_{f_2} are c_1-WPO and c_2-weakly Picard operators.

Let

$$X_{\varphi,\psi} := \{x \in X \mid x|_{[a-h_1,a]} = \varphi,\ x|_{[b,b+h_2]} = \psi\}.$$

It is clear that $E_{f_1}|_{X_{\varphi,\psi}} = B_{f_1}$, $E_{f_2}|_{X_{\varphi,\psi}} = B_{f_2}$. Therefore,

$$\left| E_{f_1}^2(x) - E_{f_1}(x) \right| \le \tfrac{1}{8} L_{f_1}(b-a)^2 \left| E_{f_1}(x) - x \right|,$$

$$\left| E_{f_2}^2(x) - E_{f_2}(x) \right| \le \tfrac{1}{8} L_{f_2}(b-a)^2 \left| E_{f_2}(x) - x \right|,$$

for all $x \in C[a-h_1,b+h_2]$.

Now, choosing

$$\alpha_i = \tfrac{1}{8} L_{f_i}(b-a)^2, i = 1,2,$$

we get that E_{f_1} and E_{f_2} are c_1-weakly Picard operators and c_2-weakly Picard operators with $c_1 = (1-\alpha_1)^{-1}$ and $c_2 = (1-\alpha_2)^{-1}$. From (6.1) we obtain that

$$\left\| E_{f_1}(x) - E_{f_2}(x) \right\|_C \le (b-a)^2 \eta,$$

$\forall x \in C[a-h_1, b+h_2]$. Applying Theorem 2 we have that

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \le \frac{(b-a)^2 \eta}{8 - L_f (b-a)^2},$$

where $L_f = \max(L_{f_1}, L_{f_2})$ and $H_{\|\cdot\|_C}$ is the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on $C[a-h_1, b+h_2]$. □

7. Extremal principle

We consider the following differential equation with maxima

$$x''(t) + p(t)x(t) + q(t) \max_{t-h_1 \le \xi \le t+h_2} x(\xi) = 0 \tag{7.1}$$

where $p, q \in C([a,b], \mathbb{R})$.

Remark 4. The function $x(t)$ is an eventually negative solution of equation (7.1) if and only if $y(t) = -x(t)$ is an eventually positive solution of the equation

$$y''(t) + p(t)y(t) + q(t) \min_{t-h_1 \le \xi \le t+h_2} y(\xi) = 0. \tag{7.2}$$

From the above remark we see that the positive and the negative solution of equation (7.1) need to be discussed separately.

Theorem 8. *(Extremal principle) Let $p(t), q(t) < 0$, for all $t \in]a,b[$ and x be a solution of (7.1). Then*

(a) *if $\max\{x(t) |\ t \in [a,b]\} = x(t_0)$ and $x(t_0) > 0$ then $t_0 \in \{a,b\}$;*
(b) *if $min\{x(t) |\ t \in [a,b]\} = x(t_0)$ and $x(t_0) < 0$ then $t_0 \in \{a,b\}$.*

Proof. (a) Let $t_0 \in]a,b[$ be such that, $x(t_0) > 0$ is the maximum value of t_0 on $[a,b]$. Since $x \in C^2[a,b]$ we have that $x(t_0) > 0, x'(t_0) = 0, x''(t_0) \le 0$. From (7.1) we have

$$0 = x''(t_0) + p(t_0)x(t_0) + q(t_0) \max_{t_0-h_1 \le \xi \le t_0+h_2} x(\xi) < 0.$$

This is a contradiction, and therefore our assumption is wrong. So $t_0 \in \{a,b\}$.

(b) Let $t_0 \in]a,b[$ be such that, $x(t_0) < 0$ is the minimum value of t_0 on $[a,b]$. Since $x \in C^2[a,b]$ we have that $x(t_0) < 0, x'(t_0) = 0, x''(t_0) \ge 0$. From (7.1) we have

$$0 = x''(t_0) + p(t_0)x(t_0) + q(t_0) \max_{t_0-h_1 \le \xi \le t_0+h_2} x(\xi) > 0.$$

This is a contradiction, and therefore our assumption is wrong. So $t_0 \in \{a,b\}$.
□

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